Grammars

Grammars

Grammars express languages

Example: the English language

$$\langle sentence \rangle \rightarrow \langle noun_phrase \rangle \ \langle predicate \rangle$$

$$\langle noun_phrase \rangle \rightarrow \langle article \rangle \ \langle noun \rangle$$

$$\langle predicate \rangle \rightarrow \langle verb \rangle$$

$$\langle article \rangle \rightarrow a$$

$$\langle article \rangle \rightarrow the$$

$$\langle noun \rangle \rightarrow boy$$

$$\langle noun \rangle \rightarrow dog$$

$$\langle verb \rangle \rightarrow runs$$

$$\langle verb \rangle \rightarrow walks$$

A derivation of "the boy walks":

$$\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \ boy \ \langle verb \rangle \\
&\Rightarrow the \ \ boy \ \ walks
\end{aligned}$$

A derivation of "a dog runs":

$$\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ dog \ \langle verb \rangle \\
&\Rightarrow a \ dog \ runs
\end{aligned}$$

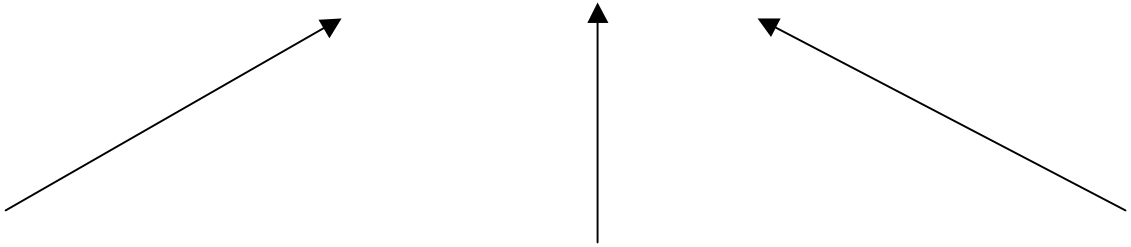
Language of the grammar:

L = { "a boy runs",
"a boy walks",
"the boy runs",
"the boy walks",
"a dog runs",
"a dog walks",
"the dog runs",
"the dog walks" }

Notation

$$\langle noun \rangle \rightarrow boy$$

$$\langle noun \rangle \rightarrow dog$$



Variable or Non-terminal

Production rule

Terminal

Another Example

Grammar:

$$S \to aSb$$

$$S \to \lambda$$

Derivation of sentence ab:

$$S \Rightarrow aSb \Rightarrow ab$$

$S \to aSb$ $\qquad$ $S \to \lambda$

Derivation of sentence $aabb$:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$$S \to aSb \qquad\qquad S \to \lambda$$

Other derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

Language of the grammar

$$S \to aSb$$

$$S \to \lambda$$

$$L = \left\{ a^n b^n : n \geq 0 \right\}$$

More Notation

Grammar $G = (V, T, S, P)$

$V =$ Set of variables

$T =$ Set of terminal symbols

$S =$ Start variable

$P =$ Set of Production rules

Example

Grammar G :

$$S \to aSb$$

$$S \to \lambda$$

$$G = (V, T, S, P)$$

$$V = \{S\}$$

$$T = \{a, b\}$$

$$P = \{S \to aSb,\ \ S \to \lambda\}$$

More Notation

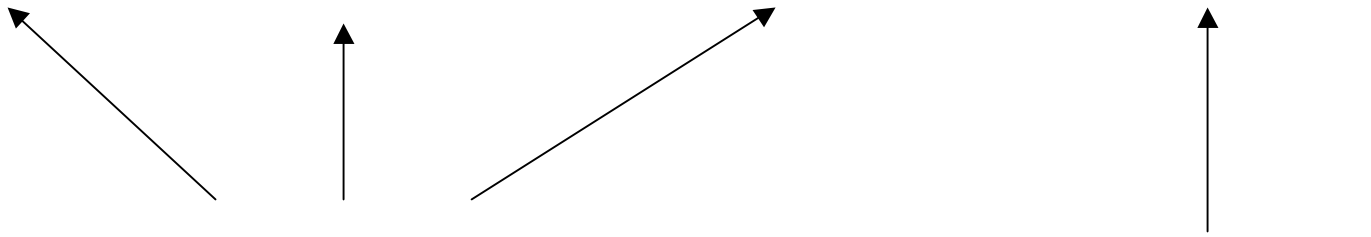
Sentential Form:

A sentence that contains

Variables and terminals

Example:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$



Sentential Forms sentence

We write:

$$S \overset{*}{\Rightarrow} aaabbb$$

Instead of:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

In general we write:

$$w_1 \overset{*}{\Rightarrow} w_n$$

If:

$$w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \cdots \Rightarrow w_n$$

By default:

$$w \overset{*}{\Longrightarrow} w$$

Examples

Grammar: $S \to aSb$

$S \to \lambda$

$S \overset{*}{\Rightarrow} \lambda$

$S \overset{*}{\Rightarrow} ab$

$S \overset{*}{\Rightarrow} aabb$

$S \overset{*}{\Rightarrow} aaabbb$

$S \overset{*}{\Rightarrow} aaSbb$

$aaSbb \overset{*}{\Rightarrow} aaaaaSbbbbb$

Another Example

Grammar G:

$$S \to Ab$$

$$A \to aAb$$

$$A \to \lambda$$

Derivations:

$$S \Rightarrow Ab \Rightarrow b$$

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow abb$$

$$S \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aabbb$$

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aaaAbbbb$$
$$\Rightarrow aaaaAbbbbb \Rightarrow aaaabbbbb$$

$$S \overset{*}{\Rightarrow} aaaabbbbb$$

$$S \overset{*}{\Rightarrow} aaaaaabbbbbbb$$

$$S \overset{*}{\Rightarrow} a^n b^n b$$

Language of a Grammar

For a grammar G

with start variable S

$$L(G) = \{w: \quad S \overset{*}{\Rightarrow} w\}$$

Example

For grammar G: $S \to Ab$

$A \to aAb$

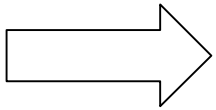
$A \to \lambda$

$$L(G) = \{a^n b^n b : n \geq 0\}$$

Since: $S \overset{*}{\Rightarrow} a^n b^n b$

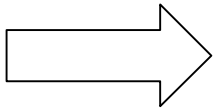
A Convenient Notation

$$A \to aAb$$
$$A \to \lambda$$



$$A \to aAb \mid \lambda$$

$$\langle article \rangle \to a$$
$$\langle article \rangle \to the$$



$$\langle article \rangle \to a \mid the$$

Linear Grammars

A Linear Grammar

Grammars with
at most one variable on the right side
of a production

Examples:

$$S \to aSb \qquad S \to Ab$$

$$S \to \lambda \qquad A \to aAb$$

$$A \to \lambda$$

A Non-Linear Grammar

Grammar G:

$$S \to SS$$

$$S \to \lambda$$

$$S \to aSb$$

$$S \to bSa$$

$$L(G) = \{w: \quad n_a(w) = n_b(w)\}$$

Another Linear Grammar

Grammar G:

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

Right-Linear Grammars

All productions have the form:

$$A \to xB$$

or

$$A \to x$$

Example:

$$S \to abS$$

$$S \to a$$

Left-Linear Grammars

All productions have form:

$$A \to Bx$$

or

$$A \to x$$

Example:

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

Regular Grammars

Regular Grammars

Definition:

A regular grammar

is any

right-linear or left-linear grammar

Examples:

$$S \to abS$$

$$S \to a$$

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

Observation

Regular grammars generate regular languages

Examples:

Grammar G:

$S \to abS$

$S \to a$

$L(G) = (ab)^* a$

Grammar G:

$S \to Aab$

$A \to Aab \mid B$

$B \to a$

$L(G) = aab(ab)^*$

Theorem

A Language L is regular

if and only if

there is a grammar G such that $L = L(G)$