# Resolving Loads with Positive Interior Stresses (Extended Abstract)

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## 1 Problem Setting

Let $P = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ be a point set in the plane and let $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ be a set of forces (2d vectors) such that $\mathbf{f}_i$ acts on $\mathbf{p}_i$. We call the set $F$ *load*. Let $G(P, E)$ denote a framework that is based on $P$ with edge set $E$. A stress $\omega$ is a (symmetric) assignment of scalars to the edges of a framework. A framework *resolves a load* if

$$\forall \mathbf{p}_i : \sum_{(i,j)\in E} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{f}_i. \tag{1}$$

If the load $F$ produces a linear or angular momentum it cannot be resolved by any framework. Therefore we restrict ourselves to loads with $\sum_{\mathbf{p}_i} \mathbf{f}_i = \mathbf{0}$ and $\sum_{\mathbf{p}_i} \langle \mathbf{p}_i^{\perp}, \mathbf{f}_i \rangle = 0$. (The vector $\mathbf{p}_i^{\perp}$ denotes $\mathbf{p}_i$ rotated by 90 degrees.) If a framework can resolve any load it is called statically rigid. Static rigidity is an equivalent condition for infinitesimal rigidity (see [3]). In general a statically rigid framework cannot avoid negative stresses on interior edges for some loads.

We are interested in a framework that fullfils (1) with a stress that is positive on every interior edge. We look at special candidates for these frameworks, the so-called *pointed pseudo-triangulations*. A pseudo-triangulation of $P$ is a partition of the convex hull of $P$ into polygons with three corners such that every $\mathbf{p}_i$ is part of some polygon. We call a pseudo-triangulation *pointed* if every point is incident to an angle greater 180 degrees. A pointed pseudo-triangulation is the embedding of a minimal rigid graph, and hence statically rigid. For a comprehensive discussion on pseudo-triangulations we direct the reader to the survey by Rote, Santos and Streinu [7].

## 2 LP Formulation

There exists a high-dimensional polytope whose corners correspond to the pointed pseudo-triangulations a point set can have [6]. This polytope is called PPT-polytope and has the following description in the unknowns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (each of the $\mathbf{v}_i$ is a 2d vector):

$$\begin{aligned} \langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{p}_i - \mathbf{p}_j \rangle &\geq \langle \mathbf{p}_i, \mathbf{p}_j^{\perp} \rangle^2 \quad \forall i, j \leq n, \\ \langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{p}_i - \mathbf{p}_j \rangle &= \langle \mathbf{p}_i, \mathbf{p}_j^{\perp} \rangle^2 \quad (i, j) \in conv(P), \end{aligned} \tag{2}$$

$$\begin{aligned} \textstyle\sum_i^n \mathbf{v}_i &= \mathbf{0}, \\ \textstyle\sum_i^n \langle \mathbf{v}_i, \mathbf{p}_j^{\perp} \rangle &= 0. \end{aligned} \tag{3}$$

The PPT-polytope is a simple polytope with dimension $2n - 3$. Hence, a corner of it is specified by $2n - 3$ tight inequalities. (We consider the equation of the convex hull edges as tight inequalities.) The pointed pseudo-triangulation that is associated with a specific corner is given by the edges induced by the tight inequalities.

We study the minimization of the function $\sum_i^n \langle \mathbf{v}_i, \mathbf{f}_i \rangle$ over the PPT-polytope given by (2) and (3) (in the following considered as primal program). Let us assume that the primal program has a unique solution. The constraints of the corresponding dual program have the following form:

$$1 \leq i \leq n : \sum_{j=1}^{n} u_{ij}(\mathbf{p}_i - \mathbf{p}_j) + \mathbf{t} + r\mathbf{p}_i^{\perp} = \mathbf{f}_i. \tag{4}$$

The variables $\mathbf{t}$ and $r$ are a result of the equations (3). For every possible interior edge we obtain by LP duality the dual constraint

$$u_{ij} \geq 0.$$

By complementary slackness we can argue that if in the solution of the primal a constraint is not tight (there is no edge defined by this inequality) then in the dual the corresponding dual variable $u_{ij}$ is zero. Thus, a (non-zero) $u_{ij}$ appears only on the edges of the primal solution.

We observe that the dual variables that come from the conditions (3) are the only difference between (4) and (1). Fortunately, we can show that under our assumptions the variables $\mathbf{t}$ and $r$ can only be zero.

**Theorem 2.1** *If the load in the primal program has no linear and angular momentum then we have for the dual variables* $\mathbf{t} = \mathbf{0}$ *and* $r = 0$.

As a consequence of Theorem 2.1 the dual variables $u_{ij}$ define a stress that resolves the load $F$ and is positive on every interior edge. Hence, the solution of the primal program computes a framework with the desired

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property. Since we assumed that the primal solution is unique there is for any load exactly one pointed pseudo-triangulation that resolves it with positive interior stress.

Instead of considering all pointed pseudo-triangulation of $P$ we can study these pointed pseudo-triangulations that contain a prescribed set of edges $E_c$. We modify the PPT-polytope and make all inequalities that refer to edges in $E_c$ tight by turning them into equations. As long as $E_c$ is non-crossing and leaves an angle greater 180 degrees at every vertex we can complete $E_c$ to a pointed pseudo-triangulation [8]. Hence, the modified PPT-polytope represents a non-empty facet of the original PPT-polytope.

Forcing edges to appear in the framework has the following consequences for our LP approach: Since the inequalities of $E_c$ are now equations we have no information about the sign of the corresponding dual variables $u_{ij}$. In other words, the edges $E_c$ behave like convex hull edges. This is the only difference – for all other interior edges we still have $u_{ij} \geq 0$.

# 3 Applications

**Constructing Regular Meshes.** Let $G = (P, E)$ be a non-crossing straight-line embedding in the plane called mesh. If there exists some stress for $G$ that resolves the everywhere-zero load with positive interior edge weights then we call $G$ regular. Usually regularity is defined by convex liftings, but due to the Maxwell-Cremona correspondence our definition is equivalent (see [4, 10]).

A regular mesh is necessary for the discretization of a process described by the Laplace equation [9]. It is easy to construct a regular mesh for a given point set $P$: the Delaunay triangulation is always regular. On the other hand the underlying graph might have a completely different combinatorial structure and is therefore not a good model for the discretization.

We sketch a method how to construct a regular embedding by switching some of the edges. We first compute a stress on $G$ that uses only a few negative edges (we omit the details here). We delete the edges with negative weights and obtain a stressed embedding $G' = (P, E')$. In $G$ we had $\sum_{(i,j)\in E} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0}$, but in $G'$ the stressed edges sum up to a possible non-zero vector $\mathbf{f}'_i := \sum_{(i,j)\in E'} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)$. It can be observed that the load given by $\mathbf{f}'_i$ has no linear or angular momentum. We apply our load resolving method and construct a pointed pseudo-triangulation $PT$ that resolves the $\mathbf{f}'_i$s with a stress that is positive on the interior edges. Clearly, the union of $PT$ and $G'$ gives a regular mesh that contains many edges of $G$. The mesh might contain crossings but applying Bow's trick [5] gives a planarized embedding at the expense of new vertices.

**Optimal Pointed Pseudo-Triangulations of Polygons.** Let $S$ be a simple polygon with point set $P$ and $h$ be a height assignment for $P$. Actually, the heights $h_i$ are considered as upper bounds of the height the point $\mathbf{p}_i$ can realize. The maximal surface that respects these upper bounds and that is convex on the interior of $S$ projects vertically down to a pseudo-triangulation $\mathcal{PT}_h$ [1]. $\mathcal{PT}_h$ can be constructed by an $O(n^2)$ flip sequence – each flip might require $O(n^3)$ time [2].

Assume that all corners of $S$ lie on the convex hull of $P$. In this scenario the load resolving method can be applied to compute $\mathcal{PT}_h$ faster than in $O(n^5)$. We apply the dualism between 3d polyhedra and stresses on planar embeddings that resolve the everywhere-zero load (Maxwell-Cremona [4]). We built a 3d polyhedron that consists of a lower part (a pyramid that spans the lifted points of the convex hull of $P$) and an upper part that refers to the lifting of $\mathcal{PT}_h$. By the results of [1] the heights realized on the corners of $S$ are exactly $h_i$. We compute a stress on the pyramid edges that (1) induces a load that is zero on the apex and (2) induces a lifting that gives the heights $h_i$ for the corners of $S$. We apply the load resolving method (with edge constraints given by the polygon edges) to compute the upper part of the polyhedron. Due to the positive interior stress we know that the upper half is convex.

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