Convex Partitions with 2-Edge Connected Dual Graphs

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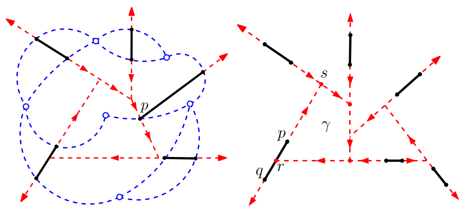
Abstract

In the paper we show that for every set of n disjoint line segments in the plane, there is a partition (more may exist) of the plane into $n+1$ open convex cells whose dual graph is 2-edge connected. Questions about the dual graph of a convex partition are motivated by the still unresolved conjecture about compatible geometric matchings.

For a set S of disjoint line segments in the plane, a convex partition of the plane is a set C of open convex *cell*s such that the cells are pairwise disjoint, they are disjoint from any segment, and their closures cover the entire plane. Note that every segment endpoint must be incident on at least two cells. Let σ be an assignment of every segment endpoint to two adjacent convex cells. The convex partition C and the assignment σ defines a *dual graph* $D(C,\sigma)$: the cells in C correspond to the nodes of the dual graph, and every segment endpoint p corresponds to an edge between the two cells assigned to p. For n disjoint segments, the dual graph has $2n$ edges, with possible double edges (see Fig. 1(a)).



(a) (b)

Figure 1: Dual Graph and Extension Trees

Theorem 1. *For every set of n disjoint segments in the plane, there is a convex partition with $n+1$ cells and an assignment σ such that the dual graph $D(C,\sigma)$ is 2-edge connected.*

It is straightforward to construct *a* convex partition for a set of n segments as follows. Let π be a permutation on the $2n$ segment endpoints. Process the segment endpoints in the order π. For an endpoint p, extend the incident segment beyond the endpoint until until the extension hits another segment, a previous extension, or infinity. If no three segment endpoints are collinear, then this algorithm produces $n+1$ convex cells for n segments, and every segment endpoint is incident on exactly two cells. We call this a STRAIGHT-FORWARD convex partition, C_π. Aichholzer *et al.* [1] conjectured that there is a permutation π such that the dual graph $D(\pi)$ is the union of two spanning trees. The conjecture would immediately imply that such a dual graph is 2-edge connected. We present a counterexample to this conjecture (see Fig. 2).

Theorem 2. *For every $n \geq 15$, there are n disjoint segments in the plane such that the dual graph $D(\pi)$ has a bridge (cut edge) for any permutation π.*

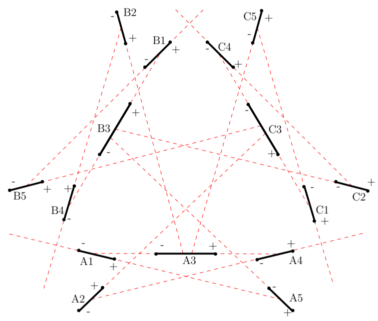


Figure 2: Counterexample with 15 segments

Constructing a convex partition

In a STRAIGHT-FORWARD convex partition (defined above), whenever two extensions meet, one of them stops and the other one continues in its original direction. Here, however, we let the two extensions merge and continue in any direction in the closed wedge bounded by the two extensions; every resulting cell is still convex.

In particular, we use the concept of *extension trees* introduced by Bose *et al.* [2]. A set S of n disjoint line segments in the plane is a matching M with $2n$ vertices. We augment M with *directed edges* (including *directed rays* going to infinity) and the Steiner vertices they induce, such that the directed edges and the input segments jointly partition the plane into convex cells.

We require that: (1) every segment endpoint emits exactly one outgoing edge; (2) every Steiner point disjoint from M is incident on exactly one outgoing edge; (3) no Steiner point on an input segment is incident on any outgoing edge; and (4) the directed edges do not form a cycle.

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The *extended-path* of a segment endpoint p is a directed path along directed edges starting from p and ending on an input segment or at infinity. An *extension tree* is the union of all extended-paths that have a common endpoint, which is called the *root* of the extension tree.

In our construction, we let σ assign a segment endpoint p to the two cells adjacent to the unique outgoing edge incident on p. Thus the 2-edge connected dual graphs $D(C, \sigma)$ can be characterized by a forbidden pattern (see Fig. 1(b)).

Lemma 1. *A dual graph $D(C, \sigma)$ is 2-edge connected if and only if no extended-path of any segment endpoint hits the same segment.*

Algorithm. We are given a set S of n disjoint line segments. First we draw extension trees that go to infinity as follows. Draw successively any straight-line extension that goes to infinity or hits a previously drawn extension. After this initial step, the straight-line extension of any remaining segment endpoint hits another segment. We repeat the subroutine ADDTREE for adding new maximal extension trees until all extended-paths have been drawn.

Subroutine ADDTREE.

1. Find a segment endpoint whose extended-path has not been drawn but its straight-line extension hits a segment pq such that at least one of the extended-paths of pq has already been drawn (refer to Lemma 2).
2. Draw the straight-line extension of the unextended endpoint and successively draw any new straight-line extension that hits a previous extension. We have created a maximal extension tree T whose root r is incident on the segment pq. Since at least one extended-path of pq is part of a previous extension tree, at most one extended-path of pq can be part of T.
3. If an extended-path of pq is a forbidden path, we modify T with subroutine FLEXTREE(T) below.

In subroutine FLEXTREE (see Fig. 3), we will modify T while maintaining two invariants: (a) the interior of P expands; (b) points s and r remain vertices of P;

Subroutine FLEXTREE(T)

1. Loop until a new segment endpoint appears on the extended-path γ. Let $\overrightarrow{xy}$ be the last edge of the (flexed) polygon P along γ such that the vertices x and y are Steiner, and x is convex. Let $\overrightarrow{yz}$ be the other edge incident on y along γ. Let $\overrightarrow{wy}$ be a third directed edge incident on y, in the exterior of P such that $\overrightarrow{yz}$ and $\overrightarrow{wy}$ are consecutive in among the edges incident on y.
   Rotate the edge $\overrightarrow{xy}$ so as to expand the polygon P. If $\angle xyz > 180°$, then also move $\overrightarrow{yz}$. Rotating the edge $\overrightarrow{xy}$ will result in one of the following four cases:
   (a) y arrives at w and $w = (p')$ is a segment endpoint.
   (b) y arrives at w and w is a Steiner point disjoint from input segments (iterate).
   (c) $\overrightarrow{xy}$ or $\overrightarrow{yz}$ meets an endpoint p' whose extension is not in the boundary of the polygon P. Notice this is the case where the segment endpoint p' is incident on at least three cells.
   (d) $\overrightarrow{xy}$ becomes collinear with the previous edge of P (iterate).
2. Split the tree T by calling the recursive subroutine SPLITTREE(T, p').

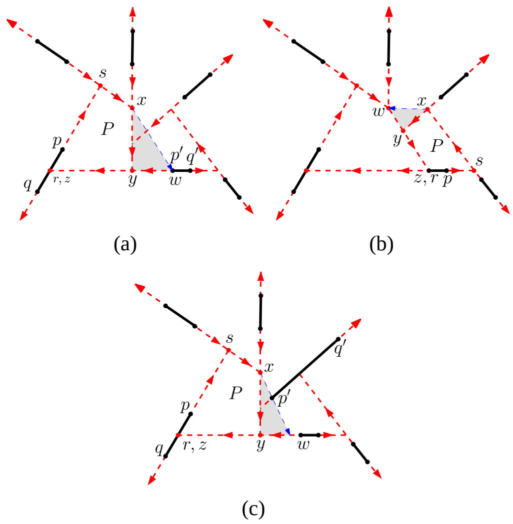

Figure 3: FlexTree Operation (a), (b) and (c)

Subroutine SPLITTREE(T, p')

1. Split tree T into two extension trees T_1 and T_2. The tree T_1 consists of the extended-paths that terminate at r, and T_2 consists of the extended-paths that now terminate at p'.
2. If tree T_2 contains a forbidden path, then call subroutine FLEXTREE(T_2).

Lemma 2. *If some extended-paths have been drawn and the straight-line extension from every unextended endpoint would hit another segment (rather than a previous extension or infinity), then we can draw a new straight-line extension that hits another segment which has at least one already extended-path.*

Lemma 3. *Subroutine* FLEXTREE(T) *modifies an input extension tree T with a forbidden extended-path γ until a segment endpoint p' appears along γ, where $p' \neq p$ and the extended-path of p' is part of T. During this modification, the set of extended-paths in T remains the same, and T remains disjoint from all previously drawn extension trees.*

References

[1] O. Aichholzer, S. Bereg, A. Dumitrescu, A. Garca, C. Huemer, F. Hurtado, M. Kano, A. Mrquez, D. Rappaport, S. Smorodinsky, D. Souvaine, J. Urrutia, and D. R. Wood, Compatible geometric matchings, *Comput. Geom. Theory Appl.* (2008).

[2] P. Bose, M. E. Houle, and G. T. Toussaint, Every set of disjoint line segments admits a binary tree, *Discrete Comput. Geom.* **26** (3) (2001), 387–410.