Relative Convexity and the Medial Cover

Jason Williams*

***Abstract*—We define the medial cover of a set A consisting of one or more simple polygons contained inside a simple polygon B. The medial cover belongs to the set T of all sets that both contain A and have connected components that locally behave like the convex hull of A relative to B. As normally defined, the convex hull of A relative to B is the largest set in T, and each of its connected components contains the maximum possible number of components of A. By contrast, the medial cover groups components of A by proximity. We describe how algorithms for computing the medial axis of a polygon and for path planning in a simple polygon can be used to compute the medial cover in $O(n \log n)$ time, which is optimal.**

I. Relative convexity

If A and B are simple polygons in the Euclidean plane such that A is contained in B and the boundaries of A and B are disjoint, the boundary of the convex hull of A relative to B can be compared to a rubber band: it is the unique minimum-length simple closed curve that separates the interior of A from the complement of B [1]. If we instead allow A to have multiple components, each of which is a simple polygon, there are at least two definitions of relative convexity that, although similar, are not equivalent. We call the first geodesic convexity: $H \subseteq B$ is geodesically convex relative to B if and only if for any two points contained in H, the shortest path in B connecting them is contained in H [2]. We call the second visible convexity: H is visibly convex relative to B if and only if for any two points in H, the line segment connecting them is contained in H if it is contained in B [3]. Both definitions satisfy convexity axioms that allow us to define the convex hull of A relative to B as the intersection of all sets convex relative to B that contain A with the assurance that the result contains A, is convex relative to B, and is the smallest of all sets with those two properties in the sense that it is a subset of all of them [4]. A geodesic convex hull of A relative to B is both connected and contains the visible convex hull of A with respect to B, which may consist of multiple connected components (Figure 1.)

II. Relatively convex covers

Although there is a unique relative convex hull for a given definition of relative convexity, the presence of multiple connected components in the relative visible convex hull suggests the possibility that there are multiple ways of separating the interior of A from the complement of B with disjoint rubber bands in tension. We formalize this by defining a convex cover of A relative to B as a set K containing A such that if κ is any of the connected components of K and L is the union of the connected components of A that intersect κ, κ is the convex hull of L relative to B (Figure 2, left.) Relatively convex covers are similar to r-tightenings [5]. We show that the convex hull of A relative to B is the largest element in the set T of convex covers of A relative to B when T is ordered by set inclusion. We then define a set as locally convex relative to B if and only if each of its connected components is convex relative to B (Figure 2, middle) and show that the corresponding local relative convex hull - the intersection of all sets containing A that are locally convex relative to B - is the smallest element in T (Figure 2, right.) In an analogy to clustering where a connected component of a convex cover of A relative to B defines a cluster containing the connected components of A it intersects, the relative convex hull of A with respect to B contains the largest possible clusters, while the local relative convex hull contains the smallest.

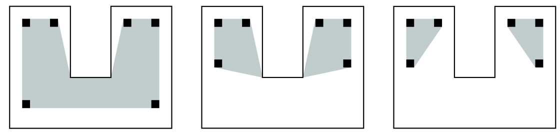

Figure 1. In each of the subfigures, A is the union of the black squares, B is the non-convex simple polygon outlined in black, and the relative convex hull of A with respect to B is gray. (left) The geodesic and visible relative convex hulls are identical in this example. (middle) The geodesic hull shown here is connected but has a nonmanifold boundary. (right) This visible hull has two components, each bounded by a simple closed curve. The visible hull is a subset of the geodesic hull.

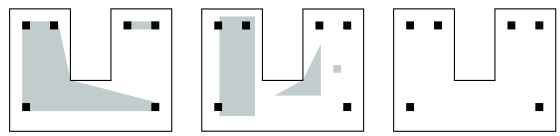

Figure 2. (left) A convex cover of A relative to B. (middle) A set that is locally convex relative to B. (right) The local relative convex hull of A is equal to A (and so is not visible) because the connected components of A are convex.

III. Medial cover

The ordering of T motivates our definition of the medial cover, which is itself a convex cover of A relative to B. Rather than possessing a particular position in the ordering of T, the medial cover groups components of A by proximity (Figure 3.) Metaphorically, we can produce the boundary of the medial cover by snapping rubber bands from an initial configuration where each band is superimposed on a connected component of the subset of the medial axis of the closure of $B - A$ whose points are equidistant from the boundaries of A and B; under a general position assumption each such component is a simple closed curve (Figure 4.) By adapting algorithms for computing the medial axis of a polygon with holes and for point-to-point

*College of Computing, Georgia Institute of Technology, jasonw@cc.gatech.edu.

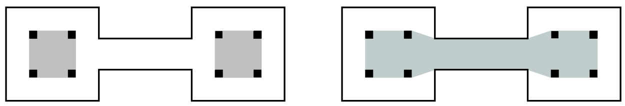

Figure 3. (left) The medial cover of A with respect to B, shown in gray, groups nearby components of A. (right) The relative convex hull may group distant components of A.

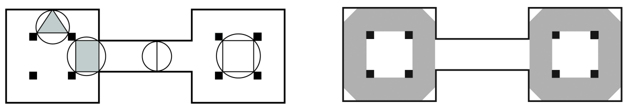

Figure 4. (left) Let the *contact set* of each maximal disk of the closure of $B - A$ be the set of points where its boundary intersects the boundary of $B - A$. Selected maximal disks and the boundaries of the convex hulls of their contact sets are shown. If a contact set contains points of the boundaries of both A and B, its convex hull is part of the *free space*, and is shaded gray. (right) The total free space is shown in gray, and each connected component of the free space is, topologically, an annulus. We show there is a unique noncontractible simple closed curve of minimum length in each annulus. The union of all such curves is the boundary of the medial cover.

path planning in a simple polygon, we show the medial cover can be computed in $O(n \log n)$ time. Combining this with the fact that a medial cover algorithm outputs the convex hull of A if the boundary of B is far from A, we conclude that the time complexity of computing a medial cover is $\Theta(n \log n)$.

IV. Acknowledgments

This work was conducted under the guidance of Jarek Rossignac (College of Computing, Georgia Institute of Technology) with advice on early drafts from John McCuan (School of Mathematics, Georgia Institute of Technology) and Jack Snoeyink (Department of Computer Science, University of North Carolina at Chapel Hill.)

References

[1] J. Sklansky and D. Kibler. A theory of nonuniformly digitized pictures. IEEE Transactions on Systems, Man, and Cybernetics, 6(9):637-647, 1976.
[2] G. Toussaint. Computing geodesic properties in a simple polygon. Invited paper, Revue d'Intelligence Artificielle, 3(2)9-42, 1989.
[3] F. Sloboda and B. Zatko. On approximation of Jordan surfaces in 3D. In G. Bertrand, A. Imiya and R. Klette (eds.) Lecture Notes in Computer Science; Vol. 2243: Digital and Image Geometry. Springer, New York, 2001.
[4] M. Van de Vel. Theory of Convex Structures. Elsevier, New York, 1993.
[5] J. Williams and R. Rossignac. Tightening: Morphological simplification. International Journal of Computational Geometry and Applications, 17(5):487-503, 2007.