Routing Multi-Class Traffic Flows in the Plane is Hard

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Problem Formulation The input to the *Red/Blue paths problem* problem is a polygonal domain P, consisting of an outer polygon and polygonal obstacles (holes). Two edges of the outer polygon are designated as the *source* and the *sink*. A *thick path* is the Minkowski sum of a usual (thin) source-sink path and a unit-diameter disk centered at the origin. The holes in the domain, as well as the paths sought, are of one of 2 *types*: red and blue. Red (resp., blue) paths must avoid red (resp., blue) holes, but may pass freely through the blue (resp., red) holes. The question is: Given two numbers, r and b, is it possible to route r red and b blue thick paths so that no two paths overlap.

Motivation Our problem statement is suggested by the specifics of path planning for air traffic management. The aircraft differ in their capabilities of going through certain regions of airspace (e.g., hazardous weather constraints). In particular, one weather system can serve as an obstacle for one class of aircraft while being safely passable by another class of better equipped (or larger) aircraft. A good route planner must take this into account by possibly permitting "stronger" aircraft to fly through certain weather conditions, which serve as obstacles to "weaker" aircraft.

In this abstract we present the following result:

Theorem 1 *The Red/Blue paths problem is NP-hard.*

We reduce from the INDEPENDENT SET. Let G be a graph with n vertices. In the independent set problem, the question is: Given an integer k, do there exist k vertices of G no two of which are connected by an edge? Starting from G, we construct an instance of the Red/Blue paths problem as follows.

For each vertex of G we create a *vertex gadget* (Figure 1(a)). We stack the n vertex gadgets one on top of another, forming the *vertex part* of the construction (Fig. 1(b)).

For each edge of G we create an *edge gadget*. To build the gadget, we first create an $8n$-by-$8n$ square with top and bottom sides being red segments; we put n equally spaced blue segments of height 4 along the right side of the square (Fig. 1(c)). If edge e is incident to a vertex i, we add length 1 to the top and the bottom of the ith obstacle in the edge gadget corresponding to e (thus, there are exactly two stretched obstacles in each edge gadget). Finally, the top and the bottom boundary of each edge gadget are shifted by 1 up and down (Fig. 1(d)).

To finish the construction we put the vertex part and the m edge gadgets side by side from left to right; we align the obstacles in the edge gadgets with the rightmost obstacles in the vertex part (Figure 2). We claim that there exists an independent set of size k in G if and only if $8(n-k)$ red and $4k$ blue paths can be routed all the way from the left of the construction to the right.

First, suppose there is an independent set of size k in G. Route four blue paths through each vertex gadget that corresponds to a vertex in the independent set; route eight red paths through the other gadgets. We claim that the paths may pass through all edge gadgets. Indeed, consider one edge gadget. If the gadget did not have obstacles with additional length, the paths could go through the gadget just in the same way they came out of the vertex gadgets. Adding height 1 to the top and bottom of an obstacle in the gadget may have a pair of blue paths shifted up and down, causing also shifting of the other paths. But since the blue paths go through vertex gadgets that collectively correspond to an independent set, there is at most one pair of shifted blue paths within one edge gadget. Thus, the shifted paths will fit into the extra space at the top and the bottom. This proves that if there is an independent set of size k in G, there exist $8(n-k)$ red and $4k$ blue paths in our instance of the Red/Blue paths problem.

On the other hand, in order to route $8(n-k)$ red and $4k$ blue paths, the blue paths have to go, in quadruples, through some k vertex gadgets. Due to the stretched obstacles, the vertex gadgets must correspond to an independent set in G.

Open Problem Finding an approximate solution to the Red/Blue paths problem is open.

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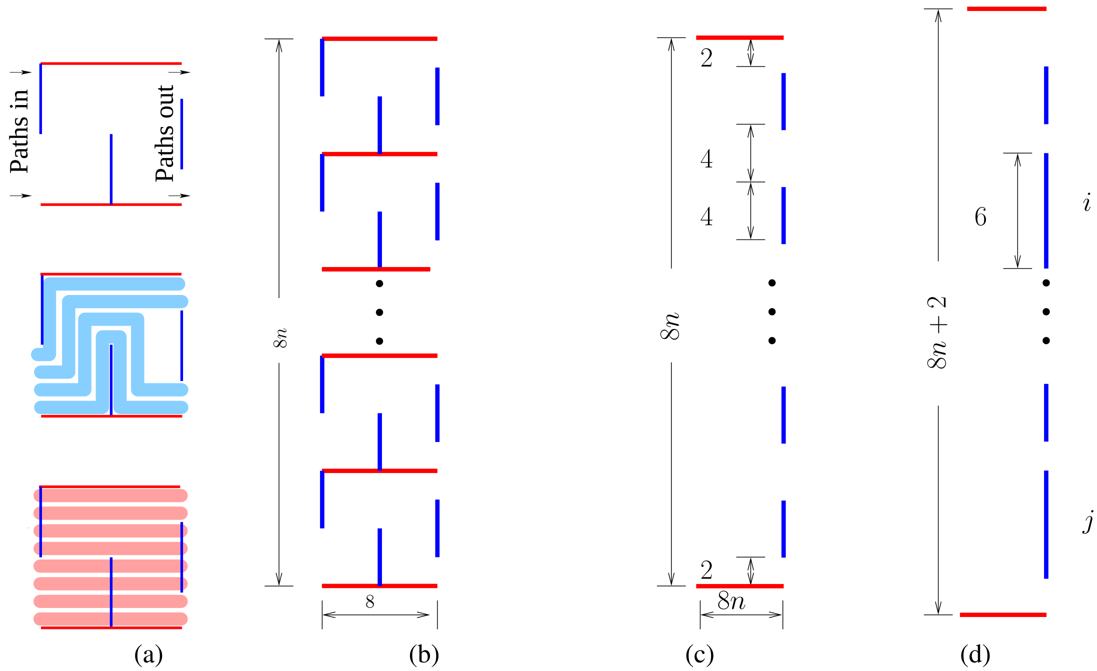

Figure 1: (a) Top: the vertex gadget is an 8-by-8 square with top and bottom sides being red segments; there are three blue obstacles inside the gadget, each is a vertical segment of height 4. (a) Middle and bottom: if the paths, going through the gadget are of one color, they are either (at most) four blue paths or (at most) eight red paths. (b): the vertex part — n stacked vertex gadgets. (c): each edge gadget is built from an $8n$-by-$8n$ square with n blue obstacles, each being a height-4 blue segment. (d): in the gadget for an edge (i, j), we stretch ith and jth obstacles by adding length 1 to each of them, both from above and from below; also the top and the bottom sides of each edge gadget are shifted by 1 up and down.

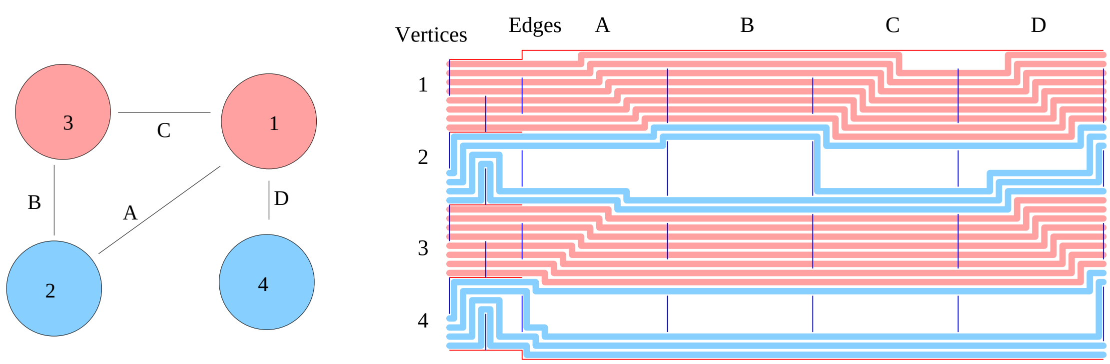

Figure 2: The vertex part and the edge gadgets are put one after another. This example shows the construction for the graph on the left. Stretching the obstacles shifts the paths by 1 up and down; the shifted paths fit fine into the gadgets because the top and the bottom of the gadgets were shifted by 1 too.