Robust Shape Clustering: Computing 1-medians on Riemannian Manifolds

P. Thomas Fletcher   Suresh Venkatasubramanian   Sarang Joshi

Introduction[1]

Propelled by advances in medical imaging technology, clinicians and researchers are generating vast amounts of complex 2D and 3D shape data describing the brain, the heart, and other critical organs in both diseased and normal patients. Such shape data is often modelled as points on manifolds. Thus, characterizing such data requires us to extend statistical notions of centrality to manifold-valued data. In previous work [2, 6, 13], the notion of centrality of empirical data was defined via the Fréchet mean [7], which was first developed for manifold-valued data by Karcher [8]. Although the mean is an obvious central representative, one of its major drawbacks is its lack of robustness, i.e. sensitivity to outliers.

Our Contributions One of the most common robust estimators of centrality in Euclidean spaces is the *geometric median*, or the 1-median. In this paper we extend the notion of geometric median to general Riemannian manifolds, thus providing a robust statistical estimator of centrality for manifold-valued data. We prove some basic properties of the generalization and exemplify its robustness for data on common manifolds encountered in computer vision. Although the methods presented herein are quite general, for concreteness we will focus on the following explicit examples: i) the space of 3D rotations, ii) the space of positive-definite tensors, and iii) the space of planar shapes.

The Riemannian Geometric Median

Let M be a Riemannian manifold. Given points $x_1, \ldots, x_N \in M$ and corresponding positive real weights $w_i, \ldots, w_N$, with $\sum_i w_i = 1$, define the weighted sum-of-distances function $f(x) = \sum_i w_i d(x, x_i)$, where d is the Riemannian distance function on M.

We define the *weighted geometric median*, m, as the minimizer of f, i.e., $m = \arg\min_{x \in M} \sum_i^N w_i d(x, x_i)$ When all the weights are equal, $w_i = 1/N$, we call m simply the *geometric median*.

Theorem 1. *The weighted geometric median defined above exists and is unique if (a) the sectional curvatures of M are nonpositive, or if (b) the sectional curvatures of M are bounded above by $\Delta > 0$ and* $\operatorname{diam}(U) < \pi/(2\sqrt{\Delta})$.

The Weiszfeld Algorithm for Manifolds

For Euclidean data the geometric median can be computed by an algorithm introduced by Weiszfeld [14] and later improved by Kuhn and Kuenne [10] and Ostresh [12]. We adapt this to general Riemannian manifolds. The gradient of the Riemannian sum-of-distances function is given by

$$\nabla f(x) = -\sum_{i=1}^{N} w_i \operatorname{Log}_x(x_i)/d(x, x_i), \qquad (0.1)$$

where we require that x is not one of the data points x_i. This leads to a natural steepest descent iteration-median:

$$m_{k+1} = \operatorname{Exp}_{m_k}(\alpha v_k),$$
$$v_k = \sum_{i \in I_k} \frac{w_i \operatorname{Log}_{m_k}(x_i)}{d(m_k, x_i)} \cdot \left(\sum_{i \in I_k} \frac{w_i}{d(m_k, x_i)} \right)^{-1}. \qquad (0.2)$$

Theorem 2. *If the sectional curvatures of M are nonnegative and the conditions (b) of Theorem 1 are satisfied, then* $\lim_{k \to \infty} m_k = m$ *for* $0 \le \alpha \le 2$.

[1] This research is partially supported by Award Number R01EB007688 from the National Institute Of Biomedical Imaging And Bioengineering

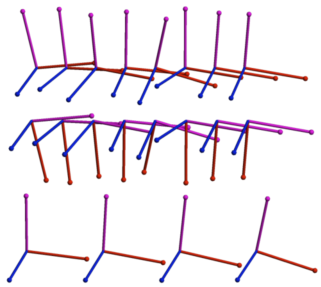

Figure 1: Computation of the geometric median for 3D rotations with example rotations (top) and outliers (bottom). The geometric median results with 0, 5, 10, and 15 outliers.

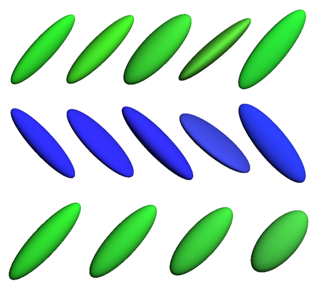

Figure 2: Computation of the geometric median for 3D tensors with example tensors (top) and outliers (bottom). The geometric median results with 0, 5, 10, and 15 outliers.

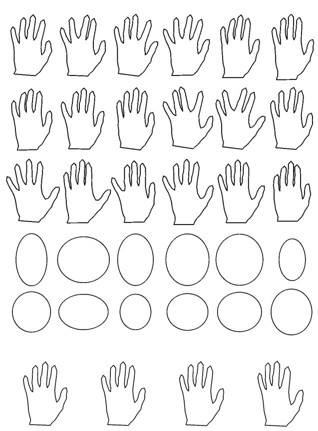

Figure 3: Geometric median computation for data from the hands database[4] with example hands (top) and outlier ellipses (bottom). Results are shown with 0, 5, 10, 15 outliers

References

[1] Bose, P., Maheshwari, A., and Morin, P. Fast approximations for sums of distances, clustering and the fermat–weber problem. *Comput. Geom. Theory Appl. 24*, 3 (2003), 135–146.

[2] Buss, S. R., and Fillmore, J. P. Spherical averages and applications to spherical splines and interpolation. *ACM Transactions on Graphics 20*, 2 (2001), 95–126.

[3] Chandrasekaran, R., and Tamir, A. Algebraic optimization: The Fermat-Weber problem. *Mathematical Programming 46* (1990), 219–224.

[4] Cootes, T. F., Taylor, C. J., Cooper, D. H., and Graham, J. Active shape models – their training and application. *Comp. Vision and Image Understanding 61*, 1 (1995), 38–59.

[5] Eckhardt, U. Weber's problem and Weiszfeld's algorithm in general spaces. *Mathematical Programming 18* (1980), 186–196.

[6] Fletcher, P. T., Lu, C., and Joshi, S. Statistics of shape via principal geodesic analysis on Lie groups. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition* (2003), pp. 95–101.

[7] Fréchet, M. Les éléments aléatoires de nature quelconque dans un espace distancié. *Ann. Inst. H. Poincaré 10*, 3 (1948), 215–310.

[8] Karcher, H. Riemannian center of mass and mollifier smoothing. *Comm. on Pure and Appl. Math. 30* (1977), 509–541.

[9] Kuhn, H. W. A note on Fermat's problem. *Mathematical Programming 4* (1973), 98–107.

[10] Kuhn, H. W., and Kuenne, R. E. An efficient algorithm for the numerical solution of the generalized Weber problem in spatial economics. *J. Regional Sci. 4* (1962), 21–34.

[11] Lopuhaa, H. P., and Rousseeuw, P. J. Breakdown points of affine equivariant estimators of multivariate location and covariance matrices. *The Annals of Statistics 19*, 1 (1991), 229–248.

[12] Ostresh, L. M. On the convergence of a class of iterative methods for solving the Weber location problem. *Operations Research 26* (1978), 597–609.

[13] Pennec, X. Intrinsic statistics on Riemannian manifolds: basic tools for geometric measurements. *Journal of Mathematical Imaging and Vision 25*, 1 (2006).

[14] Weiszfeld, E. Sur le point pour lequel la somme des distances de n points donnés est minimum. *Tohoku Math. J. 43* (1937), 355–386.