The Fréchet Distance Problem in Weighted Regions

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I. Introduction

Measuring similarity between curves is a fundamental problem that appears in various applications, including computer graphics and computer vision, pattern recognition, robotics, and structural biology.

A natural choice for measuring the similarity between curves is the *Fréchet distance*. The Fréchet distance for two parametric curves P,Q: $[0,1] \rightarrow \mathcal{R}^d$ is defined as

$$\delta_F(P,Q) = \inf_{\alpha,\beta:[0,1]\rightarrow[0,1]} \sup_{t\in[0,1]} d'(P(\alpha(t)), Q(\beta(t)))$$

where α and β range over all continuous non-decreasing functions with $\alpha(0) = \beta(0) = 0$ and $\alpha(1) = \beta(1) = 1$, and d' is a distance metric between points. The functions α and β are also refereed to as reparametrizations. We call (α, β) a *matching* between P and Q.

The Fréchet distance is described intuitively by a man walking a dog on a leash. The man follows a curve (path), and the dog follows another path. Both can control their speed but backtracking is not allowed. The Fréchet distance between the curves is the length of the shortest leash that is sufficient for the man and the dog to walk their paths from start to end.

In this paper, we study the Fréchet distance problem in weighted regions in $\mathcal{R}^2$. Given a weighted subdivision $R = \{R_1, R_2, \ldots, R_m\}$ of the plane with a total of n edges and two parameterized polygonal chains P and Q, approximate the Fréchet distance between P and Q, where the (weighted) distance between two points $P(s)$ and $Q(t)$ on P and Q, respectively, is defined as $S(P(s)Q(t)) = \sum_{i=1}^{m} w_i * R_i(P(s)Q(t))$, with $R_i(P(s)Q(t))$ the length of link $P(s)Q(t)$ within region R_i, and w_i a positive weight associated with R_i. For simplicity, we assume R is triangulated and P, Q lie on the boundaries of the weighted regions. Fig. 1 depicts an example of this problem.

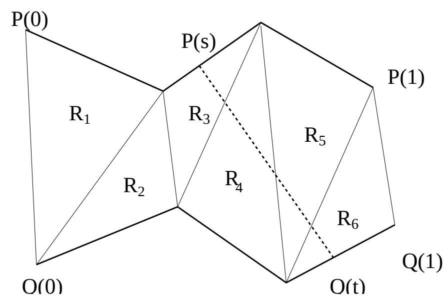

Fig. 1. An example of the Fréchet distance problem in weighted regions.

II. Discretization Using Steiner Points

In this section we show how to apply a discretization scheme used to approximate shortest path in weighted regions, e.g. [1], [5].

Let E be the set of all edges in R. Let V be the set of vertices in R. For any point v on an edge in E, let $E(v)$ be the set of edges incident to v and let $d(v)$ be the minimum distance between v and edges in $E \setminus E(v)$. For each edge $e \in E$, let $d(e) = \sup\{d(v)|v \in e\}$ and let v_e be the point on e so that $d(v_e) = d(e)$. For each $v \in V$, the vertex radius for v is defined as $r(v) = \frac{\epsilon D}{n w_{max}(v)}$, where ϵ is an positive real number defining the quality of the approximation, D is the lower bound of $\delta_F(P,Q)$, and $w_{max}(v)$ is the maximum weight among all weighted regions incident to v. The disk of radius $r(v)$ centered at v defines the vertex-vicinity of v. D can be computed in $O(pq \log(pq))$ time using the standard (unweighted case) continuous Fréchet distance algorithm described in [3], where p and q are the number edges in P and Q, respectively.

For each edge $e = v_1 v_2$ in E, we place Steiner points $v_{i,1}, v_{i,2}, \ldots, v_{i,k_i}$ outside of the vertex-vicinities, for $i = 1, 2$, such that $|v_i v_{i,1}| = r(v_i)$, $|v_{i,j} v_{i,j+1}| = \epsilon d(v_{i,j})$, for $j = 1, 2, \ldots, k_i - 1$, and $v_{i,k_i} = v_e$. It has been shown in [1] that the number of Steiner points placed on an edge is $O(C(e) 1/\epsilon \log 1/\epsilon)$, where $C(e) = O(\frac{|e|}{d(e)} \log \frac{|e|}{\sqrt{r(v_1) r(v_2)}})$. We refer to the line segment bounded by two consecutive Steiner points as a *Steiner edge*. We refer to the quadrilateral formed by two Steiner edges as a *Steiner strip*.

The set of line segments bounded by two edges of P and Q and intersecting the same sequence of edges of R describe an hourglass. Let H be the hourglass defined by a sequence of Steiner edges $\{e_1, e_2, \ldots, e_k\}$, where $e_1 \in P$ and $e_2 \in Q$.

Lemma 1: Let l and l' be two segments in H, Then, $S(l) \leq (1 + 2\epsilon) S(l') + 2\epsilon D$.

Proof: Let $R_{i_1}, R_{i_2}, \ldots, R_{i_{k-1}}$ be the weighted regions in H, such that R_{i_j} is between e_j and e_{j+1}. Then,

$$S(l) = \sum_{j=1}^{k-1} w_{i_j} R_{i_j}(l) \leq \sum_{j=1}^{k-1} w_{i_j} (R_{i_j}(l') + e_j + e_{j+1})$$

If a Steiner edge e_j is outside of any vertex-vicinity, $|e_j| \leq R_{i_{j-1}}(l')$ and $|e_j| \leq R_{i_j}(l')$. If e_j is incident to a vertex v, then $|e_j| = r(v) = (\epsilon D)\ (n w_{max}(v))$. One segment can intersect at most n Steiner edges inside vertex vicinities. The result follows. ∎

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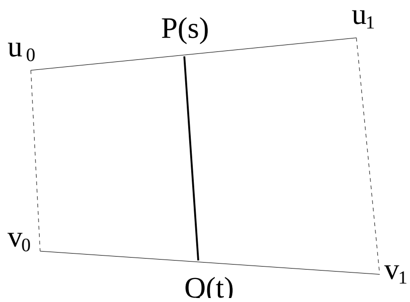

Fig. 2. A special case of the Fréchet distance problem.

Let l_H be an arbitrary segment in H. Let α and β be two reparametrizations that defines a matching between P and Q. Let $J = \{H_1, H_2, \ldots, H_k\}$ be the set of hourglasses that are traversed by the link $P(\alpha(t))Q(\beta(t))$. For an hourglass $H \in J$, let $I_H = \{e | e = P(\alpha(t))Q(\beta(t)), t \in [0,1], e \in H, \}$. Let $H(\alpha, \beta)$ be the segment in I_H with the largest weighted length. That is, $S(H(\alpha,\beta)) = \max_{e \in I_H} S(e)$.

Lemma 2: $|\max_{H \in J} S(l_H) - \delta(\alpha,\beta)| \leq 4\epsilon\delta(\alpha,\beta)$, where $\delta(\alpha,\beta) = \sup_{t\in[0,1]} S(P(\alpha(t))Q(\beta(t)))$.

Proof: Applying $Lemma$. 1, we have $S(l_H) \leq (1 + 2\epsilon)S(H(\alpha,\beta)) + 2\epsilon D$, and $S(H(\alpha,\beta)) \leq (1 + 2\epsilon)S(l_H) + 2\epsilon D$. Then $\max_{H\in J} S(l_H) \geq \frac{1-2\epsilon}{1+2\epsilon}\delta(\alpha,\beta) \geq (1 - 4\epsilon)\delta(\alpha,\beta)$, $\max_{H\in J} S(l_H) \leq (1+2\epsilon)\max_{H\in J} S(H(\alpha,\beta)) + 2\epsilon D$ and $\max_{H\in J} S(l_H) \leq (1+4\epsilon)\delta(\alpha,\beta)$. ■

Let $\delta(J) = \max_{H\in J} S(l_H)$, which give us a $4\epsilon - approximation$ of $\delta_(\alpha,\beta)$. Given a sequence of hourglasses $J = \{H_1, H_2, \ldots, H_k\}$, we call J *legal* if and only if there exist two reparmetrizations α, β that define a "leash" traversing the same sequence of hourglasses as J.

Lemma 3: We can find a $4\epsilon - approximation$ $\delta'_F(P,Q)$ such that $|\delta'_F(P,Q) - \delta_F(P,Q)| \leq 4\epsilon\delta_F(P,Q)$.

Proof: We can approximate $\delta_F(P,Q)$ by minimizing $\delta(J)$ over all legal sequences J We need to show that $(1-4\epsilon)\delta_F(P,Q) \leq \inf_{J\ is\ legal} \delta(J) \leq (1+4\epsilon)\delta_F(P,Q)$. First, we prove there exists a legal sequence of hourglasses J that give use an approximation less than $(1+4\epsilon)\delta_F(P,Q)$. Let $\hat{\alpha}$,$\hat{\beta}$ be the optimal reparametrizations that gives the Fréchet distance between P and Q. Let J be the sequence of hourglasses traversed by the "leash" defined by α and β. Obviously, $\delta(J) \leq (1+4\epsilon)\delta_F(P,Q)$. Next, suppose there are two reparametrizations α', β' and a corresponding sequence J', such that $\delta(J') \leq (1-4\epsilon)\delta_F(P,Q)$. This leads to $(1-4\epsilon)\delta(\alpha',\beta') \leq \delta(J') \leq (1-4\epsilon)\delta_F(P,Q)$, i.e. $\delta(\alpha',\beta') \leq \delta_F(P,Q)$. The assumption contradicts the fact that $\delta_F(P,Q)$ is the Fréchet distance between P and Q. ■

III. A Special Case

Here, we study a special case of the problem where each curve consists of one line segment and $u_0u_1v_0v_1$ is a quadrilateral, with $u_0 = P(0)$, $u_1 = P(1)$, $v_0 = Q(1)$, $v_1 = Q(1)$ the endpoints of P and Q. See Fig. 2 for an illustration.

We apply the approach in [3] to construct a modified free space diagram D. A link $P(s)Q(t)$ in the primal space is associated to a point (s,t) in D. There is a one-to-one correspondence between all possible matchings of P and Q and all s,t-monotone paths in D, from its bottom-left corner to its top-right corner.

Lemma 4: All links passing though a Steiner point v correspond to a curve in the free space diagram D, with equation $C_v : st + c_1s + c_2t + c_3 = 0$, where c_1, c_2, and c_3 are constants.

We call C_v the *dual curve* of v. A point to the left of C_v in D corresponds to a link to the left of v in the primal space. We can partition D by the dual curves of the Steiner points such that each cell in D corresponds to an hourglass in the primal plane. Using the algorithm in [2], the partition can be computed in $O(N \log N + k)$ time and $O(N + k)$ space, where N is the total number of Steiner points and k is the number cells, which is $O(N^2)$ in worst case.

The Fréchet distance between P and Q can then be approximated as follows:
1. Place Steiner points as described previously.
2. Partition D by dual curves of all Steiner points.
3. For each cell Z, choose an arbitrary link l and assign $S(l)$ as the weight of the cell.
4. Find a monotone path T in D, from its left bottom corner to its top right corner such that the cost of T is minimum, where the cost of a path is defined as the maximum weight of the cells traversed by the path.

Since the sequence of cells traversed by a monotone path in D corresponds to a legal sequence of hourglasses, by Lemma 3 T is a 4ϵ-approximation of the Fréchet distance between P and Q.

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