Morse-Smale Decomposition, Cut Locus and Applications in Sensor Networks

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I. Introduction

We apply Morse theory in the study of sensor networks and distributed sensor data. Sensor nodes are deployed in a 2D region $\mathcal{M}$ with boundaries and possibly interior holes, and the sensor data samples a continuous real function f. We are interested in both the topology of the discrete sensor field in terms of the sensing holes (voids without sufficient sensors deployed), as well as the topology of the signal field in terms of its critical points. Towards this end, we extend the construction of the Morse-Smale complex in the setting of a domain with boundaries and develop distributed algorithms to construct the Morse-Smale decomposition. The sensor field is decomposed into *simply-connected* pieces, inside each of which the function is *homogeneous*, i.e., the integral lines flow uniformly from a local maximum to a local minimum. This compact structure captures the essential topological features of the signal field sampled by distributed sensors, and has numerous applications in sensor data aggregation and distributed data-guided navigation in the network.

A major component of the result in this paper is to establish the equivalence of the Morse-Smale decomposition with the 'cut locus' of the flow, defined as the points through which the flow has different homotopy types (get around the holes in different ways), or different limit endpoints, with the flows in their neighborhood. Since the cut locus can be detected locally in a discrete network, this connection turns out to be the key in the robust detection of the saddle points and the Morse-Smale decomposition of the sensor field.

II. Flow, Decomposition and Dual Complex

Observe that existing approaches to adapting Morse theory to discrete domains([1]–[3]) do not apply to sensor networks due to unavailability of a triangulated mesh. We therefore devise a method from scratch that handles a discrete network, and presence of boundaries without explicitly detecting the boundary of the network.

Handling boundaries requires reconsidering the differential structure starting from the definition of gradients. We provide only a short description here, the rigorous presentation is available online [4]. The major obstruction in defining a Morse-Smale decomposition of a manifold with boundaries is that the gradient vectors do not imply smooth integral lines the way they do in manifolds without boundaries. To obtain meaningful substitute constructs that can be used in a dense sensor network, we redefine gradients to be the maximum of the projection of the true gradient over directions that locally points to the manifold interior. This keeps the vectors in the interior unchanged, but modifies those at the boundary. Figure 1 shows the idea.

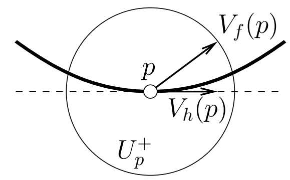


Fig. 1. Definition of gradient vector of function h on $\mathcal{M}$. V_f is the original gradient.

A flow can now be defined in the usual manner. *An Orbit* is defined for each point as the trajectory obtained by following the gradient vector starting at that point. A stable manifold of a critical point c is defined as the the set of points whose orbits converge to c. The *unstable manifold* is defined as the stable manifold under the gradient vector field of the negative function.

Definition 2.1. ***Morse Smale Decomposition.*** *The decomposition obtained by removing boundaries of the stable and unstable manifolds.*

Computing these boundaries in a sensor network is still a challenge. We therefore interpret them in terms of *cut locus* generated by the orbits.

Definition 2.2. ***Ordinary point, cut point, cut locus of h.*** *A point p is said to be ordinary if $\forall\varepsilon$ there is a δ such that for any q in a δ-neighborhood orbits of p and q are within a distance ε. A point is a* cut point *if it is not an ordinary point. The set of all cut points is the cut locus of h.*

We show that the cut locus w.r.t h and $-h$ give exactly the boundaries in the definition above. In addition, the several other properties can be proven:

Theorem 2.3. *Each Morse-Smale cell is simply connected.*

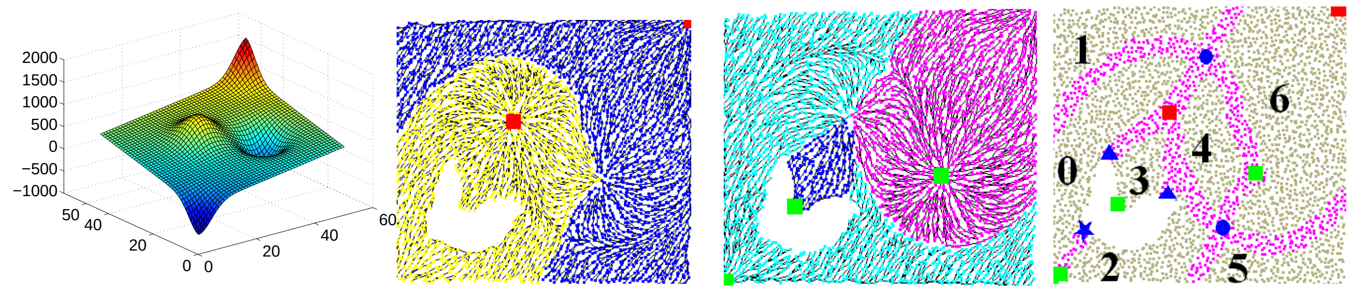
Theorem 2.4. *The flow inside each Morse-Smale cell is homogeneous.*

Based on the properties above, we describe a *Dual*. The construction of it is similar to Cech dual, except that we insert a simplex for each connected component of intersections of cells. Then the following can be shown:

Theorem 2.5. *The dual complex is homotopy equivalent to $\mathcal{M}$.*

III. Applications in Sensor Networks

To construct the decomposition in sensor networks, we let each sensor compute the gradient in the direction of each neighbor, and set the gradient vector variable to point to the neighbor of highest gradient. If locations are not available, this is simply the neighbor with the highest reading. Then techniques similar to [6] can be used to compute the cut locus. The resulting decompositions are shown in fig. 2. For details of this method, and the applications described below, see the full version [7].



(i) (ii) (iii) (iv)

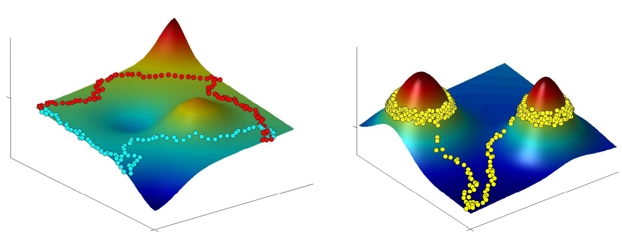
Fig. 2. (i) Original signal field. The network has a hole. (ii) Stable manifolds. (iii) Unstable manifolds. (iv) Morse-Smale decomposition with each cell labeled. Red square: *max*; green square: *min*; blue disk: *regular saddle*; blue star: *max-saddle*; blue triangle: *min-saddle*.

A. Data centric routing

We are interested in two types of data dependent queries :

- **Value restricted routing.** Find a path from a source node s to a destination node t with all values on the path within a user-specified range.
- **Iso-contour query.** From a query node q, find the iso-contours at value v, or count or report iso-contour components at the given value or range. This involves local determination of contour components and routing to each of these components.

To perform the routing, we make use of the dual constructed in the last section as a high level routing structure, that determines a path over the adjacency graph of the cell decomposition. To perform routing within a cell, we make use of iso-contours and gradients of the signal. By the properties of Morse-Smale decomposition, these form a simple and easy to use coordinate system that can be used to navigate within the cell or to a neighboring cell. In addition, we use randomization in selecting path in the dual as well as in routing within a cell to get better load balancing.



(i) (ii)

Fig. 3. (i) Depending on the range restriction applied, the algorithm constructs different paths. The red path is in response to a request for a path in a *high* range, the blue path on request for a *low* path. (ii) Iso-contours and routes to the different iso-contours found by the algorithm.

B. Data aggregation by sweeps

The idea of using a sweep over a sensor network to perform data aggregation was proposed in [5], where this method was shown to be more efficient than standard aggregation tree based methods.

The intuition behind performing a sweep is to schedule the transmissions in a way that reduces collisions and energy usage. Sweeping each cell of the decomposition individually is intuitively even more appealing than a regular sweep - since each cell is homogeneous, and does not contain any saddle, the sweep proceeds without pauses. The decomposition also lets the sweep proceed simultaneously in different components, allowing further parallelism. Our simulations show that this is indeed the case; sweeping by the decomposition reduces collisions, the sweeps execute faster, and nodes need to keep awake for smaller intervals.

Type	Total Time	Avg up-time	Max up-time	#collisions
Normal	17182	984.5	12147	10013
X coord	11748	579.1	8475	9156
Decomposed	7788	386.1	2056	5766

TABLE I. The sweep time for the network with 2 holes. The table shows results for sweeping the network by the input signal itself, sweeping by a coordinate of the geographic location, and the sweep by the cell decomposition. See [7] for explanation of these results.

IV. Conclusion

We have presented here the theory of Morse-Smale decomposition for bounded subsets of $\mathbb{R}^2$. The resulting decomposition and structural information about the network obtained in the process is shown to facilitate several applications in sensor networks. Note that executing the scheme does not require detecting the boundary or any other topological or geometric features of the network. The topology of the network is extracted from the information implicit in the *flow* of the signal gradient. Encoded as the dual, this information permits the routing mechanisms. The precise guarantees of the decomposition in the network case is under investigation.

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