Body-and-cad geometric constraint systems

Kirk Haller
SolidWorks Corporation
Concord, MA 01742
khaller@solidworks.com

Audrey Lee-St. John*
Mount Holyoke College
South Hadley, MA 01075
astjohn@mtholyoke.edu

Meera Sitharam †
University of Florida
Gainesville, FL 32611
sitharam@cise.ufl.edu

Ileana Streinu‡
Smith College
Northampton, MA 01063
istreinu@smith.edu

Neil White
University of Florida
Gainesville, FL 32611
white@math.ufl.edu

ABSTRACT

Motivated by constraint-based CAD software, we introduce a new, very general, rigidity model: the *body-and-cad structure*, composed of rigid bodies in 3D constrained by pairwise **c**oincidence, **a**ngle and **d**istance constraints. We have identified 21 relevant geometric constraints and a new, necessary, but not sufficient, counting condition for *minimal rigidity* of body-and-cad structures: *nested sparsity*. We remark that the classical body-and-bar rigidity model can be viewed as a body-and-cad structure that uses only one constraint from this new set of constraints.

1. INTRODUCTION

This paper and accompanying poster introduce *body-and-cad structures*, a class of 3D geometric frameworks with specific coincidence, angle and distance constraints between rigid bodies. To the best of our knowledge, these constraints have not been studied before from this perspective.

Motivation. Popular computer aided design (CAD) software based on geometric constraint solvers allow users to design complex 3D systems by placing geometric constraints among sets of rigid body building blocks. The constraints are specified by identifying *primitive geometries* (points, lines, planes, or splines) on participating rigid bodies. Analyzing all of these simultaneously is a very difficult problem. *In this paper, we focus on a subset of these constraints that are amenable to a rigidity-theoretical investigation.*

We define a *body-and-cad structure* to be composed of rigid bodies connected by **pairwise c***oincidence,* **a***ngle* (parallel, perpendicular, or arbitrary fixed angle) and **d***istance* constraints. These may only be placed on the primitive geometries of points, lines or planes. In an accompanying paper [1], we develop the pattern of the **rigidity matrix** and identify a necessary combinatorial counting property called *nested sparsity*, which is the counterpart of the well-known Maxwell condition for fixed length rigidity. We also show that this condition is *not* sufficient.

Related work. Classical rigidity theory focuses on distance constraints between points [2] or rigid bodies [8, 9]. *Direction* constraints are well-understood and arise from parallel redrawing applications [10]; 2D systems with both length and direction constraints are characterized in [6]. Angle constraints in the plane have been studied in [11] and [5]. Combinatorial *sparsity* conditions [4, 7] are intimately tied with rigidity theory, appearing often as necessary conditions (as for 3D bar-and-joint rigidity) and sometimes even as complete characterizations (as for 2D bar-and-joint, body-and-bar in arbitrary dimension) [2, 8].

2. BODY-AND-CAD STRUCTURES

Geometric constraints. Besides the well-studied distance constraint between points (as in body-and-bar structures), we identify 20 new pairwise coincidence, distance and angle constraints between points, lines and planes. We label constraints by the geometries involved, e.g., a line-plane perpendicular constraint between bodies A and B indicates that a line on A is perpendicular to a plane on B. Here is the full set of body-and-cad constraints that we study:

- **Plane-plane constraints.** Parallel, perpendicular, fixed angle, coincidence, distance.
- **Plane-line constraints.** Parallel, perpendicular, fixed angle, coincidence, distance.
- **Plane-point constraints.** Coincidence, distance.
- **Line-line constraints.** Parallel, perpendicular, fixed angle, coincidence, distance.
- **Line-point constraints.** Coincidence, distance.
- **Point-point constraints.** Coincidence, distance.

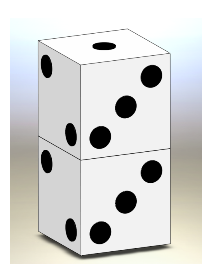


Figure 1: Two dice rigidly stacked; die A is above B. Faces are labeled by the number of dots, and face 6 lies at the bottom (opposite 1). The length of an edge is 1.

*Partially supported by NSF CCF-0728783.
†Partially supported by a Research Grant from SolidWorks 2007
‡Partially supported by NSF CCF-0728783.

Body-and-cad rigidity. A body-and-cad structure is *rigid* if the only motions respecting the constraints are the *trivial* 3D motions (rotation and translation); otherwise, it is *flexible*. It is *infinitesimally rigid* if the only infinitesimal motions are trivial. Infinitesimal rigidity is the linearized version of rigidity.

Body-and-cad minimal rigidity. The concept of *minimal rigidity* is usually defined as follows: a rigid structure is minimally rigid if the removal of any constraint results in a flexible structure. However, in our case, geometric constraints may correspond to more than one "primitive" constraint. Formally, a *primitive* constraint yields only one row in the rigidity matrix, while the body-and-cad constraints may yield several rows. In our setting, we define minimal rigidity as above, but referring to the removal of primitive constraints only.

The **example** in Figure 1 illustrates the subtleties of this concept. Let A and B be two dice rigidly stacked with the following constraints: (i) (**Plane-plane parallel**) A's Face 1 is parallel to B's Face 1, (ii) (**Plane-plane perpendicular**) A's Face 2 is perpendicular to B's Face 3, (iii) (**Plane-line distance**) The distance between A's Face 1 and B's Line 12 (intersection of Faces 1 and 2) is 1, and (iv) (**Point-point coincidence**) A's Corner 236 (the point defined by Faces 2, 3 and 6) is coincident to B's Corner 123. This structure is *rigid*. We say the structure is *overconstrained* since it remains rigid even after removal of constraint (iii). The resulting structure is now minimally rigid; constraints (i), (ii) and (iv) correspond to 6 primitive constraints. Thus, the removal of any primitive constraint results in a flexible structure.

Now consider stacking the dice with the following two constraints: (i) (**Line-line coincidence**) A's Line 26 is coincident to B's Line 12 and (ii) (**Line-line coincidence**) A's Line 36 is coincident to B's Line 13. This structure is still rigid. While it becomes flexible after the removal of either constraint (i) or (ii), it is *not* minimally rigid. Since a line-line coincidence constraint corresponds to 4 primitive constraints, this structure has 8 primitive constraints and is overconstrained. To give some intuition, note that a structure composed of 2 rigid bodies has 12 degrees of freedom. Of these, 6 are trivial, so we fix body A to factor them out. Now consider constraint (i); the structure is left with 2 degrees of freedom, as B may slide along the line and rotate about it. This indicates that a line-line coincidence constraint is somehow "killing" 4 degrees of freedom.

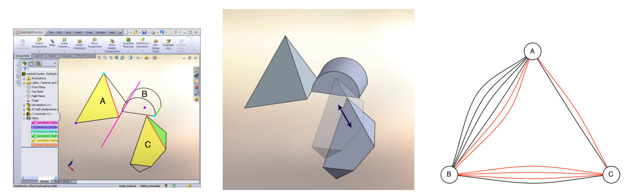
3. NESTED SPARSITY

We introduce a combinatorial condition called *nested sparsity* that is derived naturally from the body-and-cad rigidity matrix. We have shown that nested sparsity is necessary for generic rigidity of body-and-cad structures and provide an counterexample to show that it is not sufficient [3].

A graph on n vertices is (k, ℓ)*-sparse* if every subset of n' vertices spans at most $kn' - \ell$ edges; it is *tight* if, in addition, it spans $kn - \ell$ total edges. Let $G = (V, R \cup B)$ be a graph with its edge set colored into red and black edges, corresponding to R and B, respectively. We say G is $(k_1, \ell_1, k_2, \ell_2)$*-nested sparse* if G is (k_1, ℓ_1)-sparse and $G_1 = (V, R)$ is (k_2, ℓ_2)-sparse; the graph is $(k_1, \ell_1, k_2, \ell_2)$*-tight* if G is (k_1, ℓ_1)-tight.

Given a body-and-cad structure, let $G = (V, R \cup B)$ be the graph obtained by assigning vertices to bodies and constraints to disjoint edge sets R and B, corresponding respectively to primitive angular and blind constraints. In [1], we show that $(6, 6, 3, 3)$-nested sparsity is a necessary condition for generic minimal body-and-cad rigidity. We provide the counterexample that shows it is not sufficient.

Figure 2 depicts a *flexible* structure whose associated graph is $(6, 6, 3, 3)$-nested sparse. It is composed of 3 bodies A, B and C; Figure 2b colors the constraints. A and B have 2 point-point distance constraints (cyan and purple) and a line-line coincidence constraint (pink); A and C have a line-line angle constraint (orange) and a plane-plane coincidence constraint (yellow); B and C have a plane-line coincidence constraint (green).



(a) Constraint structure in SolidWorks. (b) The structure is flexible with one degree of freedom. (c) Corresponding graph is $(6, 6, 3, 3)$-nested tight.

Figure 2: Counterexample shows nested sparsity condition is not sufficient.

4. CONCLUSIONS AND FUTURE DIRECTIONS

Motivated by CAD applications, we have initiated the study of body-and-cad rigidity. Constraint-based CAD software contains a rich set of geometric constraints. As a first step towards understanding these, we have identified a class of constraints amenable to rigidity-theoretical investigation. We are hopeful that the study of all or some of the body-and-cad constraints introduced here will prove to be more tractable than classical 3D bar-and-joint rigidity.

References

[1] K. Haller, A. Lee-St. John, M. Sitharam, I. Streinu, and N. White. Body-and-cad geometric constraint systems. *Submitted to 24th Annual ACM Symposium on Applied Computing, Technical Track on Geometric Constraints and Reasoning GCR'09*, 2009.

[2] G. Laman. On graphs and rigidity of plane skeletal structures. *Journal of Engineering Mathematics*, 4:331–340, 1970.

[3] A. Lee. *Geometric Constraint Systems with Applications in CAD and Biology.* PhD thesis, University of Massachusetts Amherst, May 2008.

[4] A. Lee and I. Streinu. Pebble game algorithms and sparse graphs. *Discrete Mathematics*, 2007.

[5] F. Saliola and W. Whiteley. Constraining plane configurations in cad: Circles, lines, and angles in the plane. *SIAM Journal on Discrete Mathematics*, 18(2):246–271, 2004.

[6] B. Servatius and W. Whiteley. Constraining plane configurations in cad: Combinatorics of directions and lengths. *SIAM Journal on Discrete Mathematics,*, 12(1):136–153, 1999.

[7] I. Streinu and L. Theran. Sparse hypergraphs and pebble game algorithms. *European Journal of Combinatorics*, to appear. http://arxiv.org/abs/math/0703921.

[8] T.-S. Tay. Rigidity of multigraphs I: linking rigid bodies in n-space. *Journal of Combinatorial Theory Series*, B 26:95–112, 1984.

[9] N. White and W. Whiteley. The algebraic geometry of motions of bar-and-body frameworks. *SIAM Journal of Algebraic Discrete Methods*, 8:1–32, 1987.

[10] W. Whiteley. A matroid on hypergraphs, with applications in scene analysis and geometry. *Discrete and Computational Geometry*, 4:75–95, 1989.

[11] Y. Zhou. *Combinatorial Decomposition, Generic Independence and Algebraic Complexity of Geometric Constraints Systems: Applications in Biology and Engineering.* PhD thesis, University of Florida, 2006.