On the number of simple arrangements of five double pseudolines

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1 Introduction

Pseudoline arrangements (or in higher dimension, pseudo-hyperplane arrangements) have been extensively studied in the last decades as a useful combinatorial abstraction of configurations of points [2, 3, 7]. Recently, Habert and Pocchiola [6] introduced double pseudoline arrangements as a combinatorial abstraction of configurations of disjoint convex bodies in the plane.

In order to carry out computer experiments and to develop the understanding of pseudoline arrangements, several algorithms have been implemented to enumerate them [4, 5, 1]. In this paper, we present the implementation of an incremental algorithm to enumerate "mixed" arrangements, that is, with both pseudolines and double pseudolines.

2 Preliminaries

Mixed arrangements We denote the *projective plane* by $\mathcal{P}$ and represent it as a disk with antipodal boundary points identified.

A simple closed curve of $\mathcal{P}$ is a *(simple) pseudoline* if it is not contractible, and a *double pseudoline* otherwise (Fig. 1). The complement of a double pseudoline ℓ has two connected components: a Möbius strip M_ℓ and a topological disk D_ℓ.

An *arrangement of simple and double pseudolines* is a finite set of pseudolines and double pseudolines such that (1) any two pseudolines have a unique intersection point; (2) a pseudoline and a double pseudoline have exactly two intersection points and cross transversally at these points; and (3) any two double pseudolines have exactly four intersection points, cross transversally at these points, and induce a cell decomposition of $\mathcal{P}$ (Fig. 1). Throughout this paper, we only consider *simple* arrangements, that is, where no three curves meet at the same point.

Two arrangements A and B are *isomorphic* if there is an homeomorphism of the projective plane that sends A on B (or equivalently if there is an isotopy joining A to B). In this paper, we are interested in enumerating isomorphism classes of simple arrangements.

Mutations A *mutation* is a local transformation of an arrangement L that only inverts a triangular face of L. More precisely, it is a homotopy of arrangements during which only one curve ℓ moves, sweeping a single vertex of the remaining arrangement $L \setminus \{\ell\}$ (Fig. 1).

The graph of mutations on arrangements is known to be connected [6]: any two arrangements (with the same numbers of simple and double pseudolines) are homotopic via a finite sequence of mutations followed by an isotopy.

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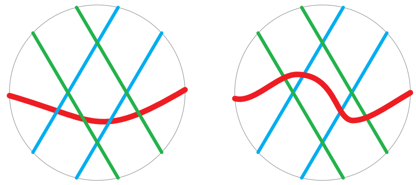


Figure 1: A mutation of the pseudoline in an arrangement with one pseudoline and two double pseudolines.

From this result we may derive a simple enumeration algorithm consisting of exploring the graph of mutations. This first algorithm is sufficient for the enumeration of small cases but already fails (because of RAM memory limitations) for arrangements of five double pseudolines. In order to go a little bit further (and particularly, to enumerate arrangements of five double pseudolines), we use an incremental version of this algorithm, developed in [8], based on the following result:

Theorem 1 ([8]) *Any two arrangements containing a subarrangement L (and with the same numbers of simple and double pseudolines) are homotopic via a finite sequence of mutations where L remains fixed, followed by an isotopy.*

3 The incremental algorithm

Description For any integers n and m, let $\mathcal{A}_{n,m}$ denote the set of isomorphism classes of arrangements of n pseudolines and m double pseudolines, and $p_{n,m}$ denote its cardinality.

Our algorithm enumerates the set $\mathcal{A}_{n,m}$ from the set $\mathcal{A}_{n,m-1} = \{a_1, \dots, a_{p_{n,m-1}}\}$, by mutating an added double pseudoline. For each $i \in \{1, \dots, p_{n,m-1}\}$, the algorithm

1. adds a double pseudoline α to the arrangement a_i;
2. performs mutations of α to enumerate the set S_i of arrangements of $\mathcal{A}_{n,m}$ containing a_i, modulo isomorphism preserving a_i;
3. selects from this set S_i the subset R_i of arrangements with no subarrangements in $\{a_1, \dots, a_{i-1}\}$;
4. computes the set T_i of isomorphism classes of arrangements of R_i (not preserving a_i).

In other words, T_i is the set of isomorphism classes of arrangements whose first subarrangement among $\{a_1, \dots, a_{p_{n,m-1}}\}$ is a_i. Thus, it is clear that

$$\mathcal{A}_{n,m} = \bigsqcup_{i=1}^{p_{n,m-1}} T_i \quad \text{and} \quad p_{n,m} = \sum_{i=1}^{p_{n,m-1}} |T_i|.$$

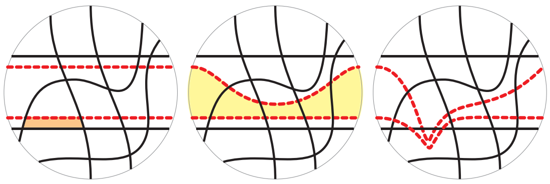
A similar algorithm enumerates $\mathcal{A}_{n,m}$ from $\mathcal{A}_{n-1,m}$ by mutating an added pseudoline.

Adding a simple or double pseudoline One of the important steps of the incremental method is to add a simple or double pseudoline to an initial arrangement. It is easy to achieve when the initial arrangement contains a pseudoline, but turns out to be harder when we have only double pseudolines. Our method uses three steps (see Fig. 2):

1. *duplicate a double pseudoline*: we choose one arbitrary double pseudoline ℓ and duplicate it, drawing a new double pseudoline ℓ' completely included in the Möbius strip M_ℓ. If M_ℓ contains a vertex of the arrangement, we choose a triangle inside M_ℓ and adjacent to ℓ and denote R the rectangle delimited by ℓ' and the three sides of our triangle. Otherwise, we denote R any rectangle delimited by ℓ and ℓ'.

2. *flatten*: we pump the double pseudoline ℓ' (using an enhanced version [8] of the *Pumping Lemma* of Habert and Pocchiola [6]) such that no vertex of the arrangement lies in the Möbius strip $M_{\ell'}$. During this process, we do not touch the rectangle R.

3. *add four crossings*: we replace the rectangle R by four crossings between ℓ and ℓ'.



(a) (b) (c)

Figure 2: Three steps to insert a double pseudoline in a double pseudoline arrangement: duplicate a double pseudoline (a), flatten it (b) and add four crossings (c).

4 Results

A C++ implementation of this algorithm is available at `http://www.di.ens.fr/~pilaud/recherche/dpl/`.

This implementation provided us with the following complete enumeration of all arrangements of n pseudolines and m double pseudolines with $n + m \leq 5$:

$n \backslash m$	0	1	2	3	4	5
0	1	1	1	13	6 570	181 403 533
1	1	1	4	626	4 822 394	
2	1	2	48	86 715		
3	1	5	1 329			
4	1	25				
5	1					

In particular, the number c_n of mixed arrangements with n curves is:

n	1	2	3	4	5
c_n	2	3	20	7 250	186 313 997

Let us briefly comment on running time. Observe first that our algorithm can be parallelized very easily (separating each enumeration of S_i, for $i \in \{1, \ldots, p_{n,m-1}\}$). In order to obtain the last column, we used four processors of 2GHz for almost 3 weeks.

5 Further developments

The following questions and developments may be treated in a subsequent paper:

1. *Developing further implementation*: we will soon extend the implementation to the enumeration of arrangements in the Möbius strip, to indexed arrangements, and to non-simple arrangements.

2. *Drawing an arrangement*: we have seen how to add a pseudoline in an arrangement. Combined with a planar-graph-drawing algorithm, this provides an algorithm to draw an arrangement in the unit disk. For example, $p_{1,m}$ can be interpreted as the number of drawings ot the arrangements of $\mathcal{A}_{0,m}$.

3. *Axiomatization*: pseudoline arrangements admit simple axiomatizations [2, 7], with few axioms dealing with configurations of at most five pseudolines. The *Axiomatization Theorem* of [6] affirms that the complete list of arrangements of at most five simple and double pseudolines is an axiomatization of mixed arrangements. Is there any simpler axiomatization? Is it possible to algorithmically reduce this axiomatization?

4. *Realizability*: it is well-known that certain pseudoline arrangements are not realizable in the Euclidean plane. Inflating pseudolines into thin double pseudolines in such an arrangement give rise to non-realizable double pseudoline arrangements. Are there smaller examples?

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