Generalized Ham-Sandwich Cuts

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Abstract

Bárány, Hubard, and Jerónimo recently showed that for given well separated convex bodies $S_1, \ldots, S_d$ in $\mathbb{R}^d$ and constants $\beta_i \in [0,1]$, there exists a unique hyperplane h with the property that $\mathrm{Vol}(h^+ \cap S_i) = \beta_i \cdot \mathrm{Vol}(S_i)$; h^+ is the closed positive transversal halfspace of h, and h is a "generalized ham-sandwich cut". They prove an analogous result for other well-separted sets in R^d that support suitable measures. Here we give alternative, simple proofs that were discovered while developing an algorithm to compute a cut for *discrete* sets of points. We also strengthen the original result by showing that the conditions assuring existence and uniqueness of generalized cuts are also necessary.

1 Introduction.

Given d sets $S_1, S_2, ..., S_d \in \mathbb{R}^d$, a ham-sandwich cut is a hyperplane h that simultaneously bisects each S_i. "Bisect" means that $\mu(S_i \cap h^+) = \mu(S_i \cap h^-) < \infty$, h^+, h^- the closed halfspaces defined by h and μ a suitable, "nice" measure on Borel sets in $\mathbb{R}^d$, e.g., the volume. The well known ham-sandwich theorem guarantees the existence of such a cut. As with other consequences of the Borsuk-Ulam theorem [6] there is a discrete version that applies to sets $P_1, \ldots, P_d$ of points in general position in $\mathbb{R}^d$, under counting measure. This can be proved using a standard argument that takes the average of n probability measures, one centered at each data point. The variance of the measures is decreased to zero, and one argues about the limit of the cuts (see [4] or [6]). However Lo et. al [5] gave a direct combinatorial proof for the discrete ham-sandwich theorem. An important consequence was that this particular proof became the basis for an efficient algorithm to compute ham-sandwich cuts for sets of points.

In a similar vein, Bereg [3] studied a discrete version of a result of Bárány and Matoušek [2] that showed the existence of wedges that simultaneously equipartition three measures on $\mathbb{R}^2$ (they are called equitable two-fans). By seeking a *direct, combinatorial proof of a discrete version* (for counting measure on points sets in $\mathbb{R}^2$) he (i) was able to strengthen the original result and (ii) also obtained a beautiful, nearly optimal algorithm to construct an equitable two-fan.

The present paper is in the same spirit. The starting point is a recent, interesting generalization of the ham-sandwich theorem. Bárány, Hubard, and Jerónimo [1] pursued the possibility of hyperplanes

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that cut off a given fraction (not necessarily 1/2) of each set. They focused on *well-separated sets* (a family $S_1, \dots, S_k$ of $k \leq d+1$ connected sets in $\mathbb{R}^d$ is well-separated if, for every choice of $x_i \in S_i$, the affine hull of $x_1, \dots, x_k$ is a $(k-1)$-dimensional flat in $\mathbb{R}^d$) and proved

Proposition 1 Let $K_1, ..., K_d$ be well separated convex bodies in R^d and $\beta_1, \dots, \beta_d$ given constants with $0 \leq \beta_i \leq 1$. Then there is a unique hyperplane $h \subset \mathbb{R}^d$ with the property that $\mathrm{Vol}(K_i \cap h^+) = \beta_i \cdot \mathrm{Vol}(K_i)$, $i = 1, \dots, d$.

Here h^+ denotes the closed, positive transversal halfspace defined by h: that is the halfspace where, if Q is an interior point of h^+ and $z_i \in K_i \cap h$, the d-simplex $\Delta(z_1, ..., z_d, Q)$ is *negatively* oriented [1]. Specifying this choice of halfspaces is what allows h to be uniquely determined. Bárány et. al. gave analogous results for such *generalized ham-sandwich cuts* for other kinds of well separated sets that support suitable measures.

We were interested in a version of Proposition 1 for n points partitioned into d sets in $\mathbb{R}^d$; i.e., points in $S = P_1 \cup \cdots \cup P_d$, $P_i \cap P_j = \phi, i \neq j$, $|S| = n$. For this context we use the definition that *point sets* $P_1, ..., P_d$ *are well separated if their convex hulls,* $\mathrm{Conv}(P_1), \dots, \mathrm{Conv}(P_n)$, *are well separated.* and in [7] we gave a direct combinatorial proof of a discrete version of Proposition 1.

One benefit of that proof was the $O(n(\log n)^{d-3})$ algorithm that it inspired (see [7]). In the present paper we exhibit two others. The ideas in that proof lead to an alternative proof of Proposition 1, simpler than the original one. In addition we can obtain a stronger result by showing that the conditions for generalized cuts are also necessary:

Corollary 2 Let $K_1, ..., K_d$ be convex bodies in R^d. If, for every choice of $\beta_i \in [0, 1]$, $i = 1, \dots, d$, there is a unique hyperplane $h \subset \mathbb{R}^d$ with the property that $\mathrm{Vol}(K_i \cap h^+) = \beta_i \cdot \mathrm{Vol}(K_i)$, $i = 1, \dots, d$, then the sets are well-separated.

We prove the Corollary, and give the new proof of Proposition 1, both for convex bodies in R^d, and for other connected sets that support suitably nice measures.

References

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