A tight lower bound on the average distance from the Fermat-Weber center of a planar convex body

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Abstract

The Fermat-Weber center of a planar body Q is a point in the plane from which the average distance to the points in Q is minimal. We show that for any convex body Q in the plane, the average distance from the Fermat-Weber center of Q to the points of Q is larger than $\Delta(Q)/6$, where $\Delta(Q)$ is the diameter of Q. This proves a conjecture of Paz Carmi, Sariel Har-Peled and Matthew Katz.

1 Introduction

The Fermat-Weber center of a measurable planar set Q with positive area is a point in the plane that minimizes the average distance to the points in Q. Such a point is the ideal location for a base station (e.g., fire station or a supply station) serving the region Q, assuming the region has uniform density. Given a measurable set Q with positive area and a point p in the plane, let $\mu_Q(p)$ be the average distance between p and the points in Q, namely,

$$\mu_Q(p) = \frac{\int_{q \in Q} \text{dist}(p, q)\, \mathrm{d}q}{\text{area}(Q)},$$

where $\text{dist}(p, q)$ is the Euclidean distance between p and q. Let FW_Q be the Fermat-Weber center of Q, and write $\mu_Q^* = \min\{\mu_Q(p) : p \in \mathbb{R}^2\} = \mu_Q(FW_Q)$.

Carmi, Har-Peled and Katz [3] showed that there exists a constant $c > 0$ such that $\mu_Q^*/\Delta(Q) \geq c$ for *any* convex body Q, where $\Delta(Q)$ denotes the diameter of Q. It is easy to construct nonconvex regions where the average distance from the Fermat-Weber center is arbitrarily small compared to the diameter. Let c_1 denote the infimum of $\mu_Q^*/\Delta(Q)$ over all convex bodies Q in the plane. They also showed that $1/7 \leq c_1 \leq 1/6$ and conjectured that $c_1 = 1/6$. The inequality $c_1 \leq 1/6$ is given by an infinite sequence of rhombi, P_ε, where one diagonal has some fixed length, say 2, and the other diagonal tends to zero; see Fig. 1. By symmetry, the Fermat-Weber center of a rhombus is its center of symmetry, and one can verify that $\mu_{P_\varepsilon}^*/\Delta(P_\varepsilon)$ tends to $1/6$.

The lower bound for c_1 has been recently further improved by Abu-Affash and Katz from $1/7$ to $4/25$ [1]. Here we establish that $c_1 = 1/6$ and thereby confirm the above conjecture of Carmi, Har-Peled and Katz.

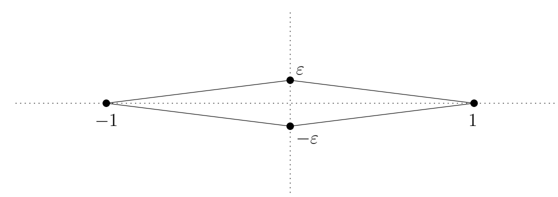

Figure 1: A flat rhombus P_ε, with $\lim_{\varepsilon \to 0} \mu_{P_\varepsilon}^*/\Delta(P_\varepsilon) = 1/6$.

Theorem 1 *For any convex body Q in the plane, we have $\mu_Q^* > \Delta(Q)/6$.*

Related work. Carmi, Har-Peled and Katz also show [3] that given a convex polygon Q with n vertices, and a parameter $\varepsilon > 0$, one can compute an ε-approximate Fermat-Weber center $q \in Q$ in $O(n + 1/\varepsilon^4)$ time such that $\mu_Q(q) \leq (1 + \varepsilon)\mu_Q^*$. Abu-Affash and Katz [1] obtained an $O(n)$ time algorithm for computing an approximate Fermat-Weber center q such that $\mu_Q(q) \leq \frac{25}{6\sqrt{3}}\mu_Q^*$. The same algorithm, combined with our Theorem 1 improves the approximation ratio to $\mu_Q(q) \leq \frac{4}{\sqrt{3}}\mu_Q^*$. It has been pointed out in [1] that the value of the constant c_1 plays a key role in a load balancing problem introduced by Aronov, Carmi and Katz [2].

2 Proof of Theorem 1

In a nutshell the proof goes as follows. Given a convex body Q, we take its Steiner symmetrization with respect to a supporting line of a diameter segment cd, followed by another Steiner symmetrization with respect to the perpendicular bisector of cd. The two Steiner symmetrizations preserve the area and the diameter, and do not increase the average distance from the corresponding Fermat-Weber centers. In the final step, we prove that the inequality holds for a convex body with two orthogonal symmetry axes.

Steiner symmetrization of a convex figure Q with respect to an axis (line) ℓ consists in replacing Q by a new figure $S(Q, \ell)$ with symmetry axis ℓ by means of

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the following construction: Each chord of Q orthogonal to ℓ is displaced along its line to a new position where its symmetric with respect to ℓ, see [4, pp. 64]. The resulting figure $S(Q,\ell)$ is also convex, and has the same area as Q. Let ℓ_x denote the x-axis, and ℓ_y denote the y-axis. A body Q is x-monotone if the intersection of Q with every vertical line is either empty or is connected.

Lemma 2 *Let Q be an x-monotone body in the plane with a diameter parallel or orthogonal to the x-axis, then* $\Delta(Q) = \Delta(S(Q, \ell_x))$.

Lemma 3 *If Q be an x-monotone body in the plane, then* $\mu^*_Q \geq \mu^*_{S(Q,\ell_x)}$.

Lemma 4 *Let T a right triangle in the first quadrant based on the x-axis, with vertices $(a, 0)$, (a, b), and $(1, 0)$, where $0 \leq a < 1$, and $b > 0$. Then $\mu_T(o) > 1/3$.*

Corollary 5 *For any rhombus P, $\mu^*_P > \Delta(P)/6$.*

Lemma 6 *Let T be a triangle in the first quadrant with a vertical side on the line $x = a$, where $0 \leq a < 1$, and a third vertex at $(1, 0)$. Then $\mu_T(o) > 1/3$.*

Proof of Theorem 1. Let Q be a convex body in the plane, and let $c, d \in Q$ be two points at $\Delta(Q)$ distance apart. We may assume that $c = (-1, 0)$ and $d = (1, 0)$. Apply a Steiner symmetrization with respect to the x-axis, and then a second Steiner symmetrization with respect to the y-axis. The resulting body $Q' = S(S(Q, \ell_x), \ell_y)$ is convex, and it is symmetric with respect to both coordinate axes. We have $\Delta(Q') = \Delta(Q) = 2$ by Lemma 2, and in fact $c, d \in Q'$. We also have $\mu^*_{Q'} \leq \mu^*_Q$ by Lemma 3.

Let Q_1 be the part of Q' lying in the first quadrant: $Q_1 = \{(x, y) \in Q' : x, y \geq 0\}$. By symmetry, $FW_{Q'} = o$ and we have $\mu^*_{Q'} = \mu_{Q'}(o) = \mu_{Q_1}(o)$. Let γ be the portion of the boundary of Q' lying in the first quadrant, between points $b = (0, h)$, with $0 < h \leq 1$, and $d = (1, 0)$. For any two points $p, q \in \gamma$ along γ, denote by $\gamma(p, q)$ the portion of γ between p and q. Let r be the intersection point of γ and the vertical line $x = 1/3$.

For a positive integer n, subdivide Q_1 into at most $2n + 2$ pieces as follows. Choose $n + 1$ points $b = q_1, q_2 \ldots, q_{n+1} = r$ along $\gamma(b, r)$ such that q_i is the intersection of γ and the vertical line $x = (i - 1)/3n$. Connect each of the $n + 1$ points to d by a straight line segment. These segments subdivide Q_1 into $n + 2$ pieces: the right triangle $T_0 = \Delta bod$; a convex body Q_0 bounded by rd and $\gamma(r, d)$; and n curvilinear triangles $\Delta q_i d q_{i+1}$ for $i = 1, 2, \ldots, n$. For simplicity, we assume that neither Q_0, nor any of the curvilinear triangles are degenerate; otherwise they can be safely ignored (they don't contribute to the value of $\mu^*_{Q'}$). Subdivide each curvilinear triangle $\Delta q_i d q_{i+1}$ along the vertical line through q_{i+1} into a small curvilinear triangle S_i on the left and a triangle T_i incident to point d on the right. The resulting subdivision has $2n + 2$ pieces, under the nondegeneracy assumption.

By Lemma 4, we have $\mu_{T_0}(o) > \frac{1}{3}$. Observe that the difference $\mu_{T_0}(o) - \frac{1}{3}$ does not depend on n, and let $\delta = \mu_{T_0}(o) - \frac{1}{3}$. By Lemma 6, we also have $\mu_{T_i}(o) > \frac{1}{3}$, for each $i = 1, 2, \ldots, n$. Since every point in Q_0 is at distance at least $\frac{1}{3}$ from the origin, we also have $\mu_{Q_0}(o) \geq \frac{1}{3}$.

For the n curvilinear triangles S_i, $i = 1, 2, \ldots, n$, we use the trivial lower bound $\mu_{S_i}(o) \geq 0$. We can also show that $s_n = \sum_{i=1}^{n} \text{area}(S_i) \leq 1/12n$.

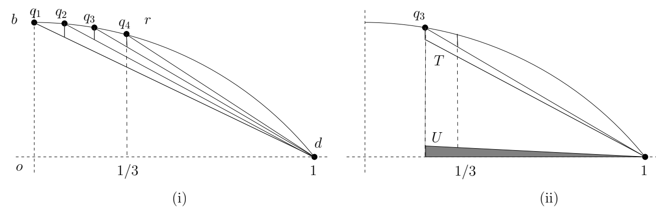

Figure 2: (i) The subdivision of Q_1 for $n = 3$. Here $o = (0, 0)$, $q_1 = b = (0, h)$, $q_4 = r$, $d = (1, 0)$. (ii) Transformation used in the proof of Lemma 6.

In particular, $s_n \leq \delta \cdot \text{area}(T_0)$ for a sufficiently large n. Then we can write

$$\begin{aligned}
\mu_{Q_1}(o) &= \frac{\int_{p \in Q_1} \text{dist}(op)\, \mathrm{d}p}{\text{area}(Q_1)} \\
&\geq \frac{\mu_{Q_0}(o) \cdot \text{area}(Q_0) + \sum_{i=0}^{n} \mu_{T_i}(o) \cdot \text{area}(T_i)}{\text{area}(Q_1)} \\
&\geq \frac{\frac{1}{3}(\text{area}(Q_1) - s_n) + \delta \cdot \text{area}(T_0)}{\text{area}(Q_1)} \\
&\geq \frac{1}{3} + \frac{2\delta \cdot \text{area}(T_0)}{3 \cdot \text{area}(Q_1)} > \frac{1}{3}.
\end{aligned}$$

This concludes the proof of Theorem 1.

References

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