A tight lower bound on the average distance from the Fermat-Weber center of a planar convex body

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Abstract

The Fermat-Weber center of a planar body $Q$ is a point in the plane from which the average distance to the points in $Q$ is minimal. We show that for any convex body $Q$ in the plane, the average distance from the Fermat-Weber center of $Q$ to the points of $Q$ is larger than $\Delta(Q)/6$, where $\Delta(Q)$ is the diameter of $Q$. This proves a conjecture of Paz Carmi, Sariel Har-Peled and Matthew Katz.

1 Introduction

The Fermat-Weber center of a measurable planar set $Q$ with positive area is a point in the plane that minimizes the average distance to the points in $Q$. Such a point is the ideal location for a base station (e.g., fire station or a supply station) serving the region $Q$, assuming the region has uniform density. Given a measurable set $Q$ with positive area and a point $p$ in the plane, let $\mu_Q(p)$ be the average distance between $p$ and the points in $Q$, namely,

$$\mu_Q(p) = \int_{q \in Q} \frac{\text{dist}(p,q)}{\text{area}(Q)} dq,$$

where $\text{dist}(p,q)$ is the Euclidean distance between $p$ and $q$. Let $FW_Q$ be the Fermat-Weber center of $Q$, and write

$$\mu_Q = \min\{\mu_Q(p) : p \in \mathbb{R}^2\} = \mu_Q(FW_Q).$$

Carmi, Har-Peled and Katz [3] showed that there exists a constant $c > 0$ such that $\mu_Q^2/\Delta(Q) \geq c$ for any convex body $Q$, where $\Delta(Q)$ denotes the diameter of $Q$. It is easy to construct nonconvex regions where the average distance from the Fermat-Weber center is arbitrarily small compared to the diameter. Let $c_1$ denote the infimum of $\mu_Q^2/\Delta(Q)$ over all convex bodies $Q$ in the plane. They also showed that $1/7 \leq c_1 \leq 1/6$ and conjectured that $c_1 = 1/6$. The inequality $c_1 \leq 1/6$ is given by an infinite sequence of rhombi, $P_\varepsilon$, where one diagonal has some fixed length, say 2, and the other diagonal tends to zero; see Fig. 1. By symmetry, the Fermat-Weber center of a rhombus is its center of symmetry, and one can verify that $\mu_{P_\varepsilon}/\Delta(P_\varepsilon)$ tends to $1/6$.

The lower bound for $c_1$ has been recently further improved by Abu-Affash and Katz from $1/7$ to $4/25$ [1]. Here we establish that $c_1 = 1/6$ and thereby confirm the above conjecture of Carmi, Har-Peled and Katz.

![Figure 1: A flat rhombus $P_\varepsilon$, with lim$\varepsilon\to0$ $\mu_{P_\varepsilon}/\Delta(P_\varepsilon) = 1/6.$](image)

Theorem 1 For any convex body $Q$ in the plane, we have $\mu_Q^2 > \Delta(Q)/6$.

Related work. Carmi, Har-Peled and Katz also showed [3] that given a convex polygon $Q$ with $n$ vertices, and a parameter $\varepsilon > 0$, one can compute an $\varepsilon$-approximate Fermat-Weber center $q \in Q$ in $O(n + 1/\varepsilon^4)$ time such that $\mu_Q(q) \leq (1 + \varepsilon)\mu_Q^\ast$. Abu-Affash and Katz [1] obtained an $O(n)$ time algorithm for computing an approximate Fermat-Weber center $q$ such that $\mu_Q(q) \leq \frac{25}{24}\mu_Q^\ast$. The same algorithm, combined with our Theorem 1 improves the approximation ratio to $\mu_Q(q) \leq \frac{4}{3\sqrt{3}}\mu_Q^\ast$. It has been pointed out in [1] that the value of the constant $c_1$ plays a key role in a load balancing problem introduced by Aronov, Carmi and Katz [2].

2 Proof of Theorem 1

In a nutshell the proof goes as follows. Given a convex body $Q$, we take its Steiner symmetrization with respect to a supporting line of a diameter segment $cd$, followed by another Steiner symmetrization with respect to the perpendicular bisector of $cd$. The two Steiner symmetrizations preserve the area and the diameter, and do not increase the average distance from the corresponding Fermat-Weber centers. In the final step, we prove that the inequality holds for a convex body with two orthogonal symmetry axes.

Steiner symmetrization of a convex figure $Q$ with respect to an axis (line) $\ell$ consists in replacing $Q$ by a new figure $S(Q, \ell)$ with symmetry axis $\ell$ by means of
the following construction: Each chord of $Q$ orthogonal to $\ell$ is displaced along its line to a new position where its symmetric with respect to $\ell$, see [4, pp. 64]. The resulting figure $S(Q, \ell)$ is also convex, and has the same area as $Q$. Let $\ell_x$ denote the $x$-axis, and $\ell_y$ denote the $y$-axis. A body $Q$ is $x$-monotone if the intersection of $Q$ with every vertical line is either empty or is connected.

**Lemma 2** Let $Q$ be an $x$-monotone body in the plane with a diameter parallel or orthogonal to the $x$-axis, then $\Delta(Q) = \Delta(S(Q, \ell_x))$.

**Lemma 3** If $Q$ be an $x$-monotone body in the plane, then $\mu^*_Q \geq \mu^*_S(Q, \ell_x)$.

**Lemma 4** Let $T$ be right triangle in the first quadrant based on the $x$-axis, with vertices $(a, 0)$, $(a, b)$, and $(1, 0)$, where $0 \leq a < 1$, and $b > 0$. Then $\mu_T(o) > 1/3$.

**Corollary 5** For any rhombus $P$, $\mu^*_P > \Delta(P)/6$.

**Lemma 6** Let $T$ be triangle in the first quadrant with a vertical side on the line $x = a$, where $0 \leq a < 1$, and a third vertex at $(1, 0)$. Then $\mu_T(o) > 1/3$.

**Proof of Theorem 1.** Let $Q$ be a convex body in the plane, and let $c, d \in Q$ be two points at $\Delta(Q)$ distance apart. We may assume that $c = (-1, 0)$ and $d = (1, 0)$. Apply a Steiner symmetrization with respect to the $x$-axis, and then a second Steiner symmetrization with respect to the $y$-axis. The resulting body $Q' = S(S(Q, \ell_x), \ell_y)$ is convex, and it is symmetric with respect to both coordinate axes. We have $\Delta(Q') = \Delta(Q) = 2$ by Lemma 2, and in fact $c, d \in Q'$. We also have $\mu^*_Q \leq \mu^*_Q$ by Lemma 3.

Let $Q_1$ be the part of $Q'$ lying in the first quadrant: $Q_1 = \{(x, y) \in Q' : x, y \geq 0\}$. By symmetry, $FW_{Q'} = o$ and we have $\mu^*_{Q'} = \mu^*_{Q'}(o) = \mu^*_Q(o)$. Let $\gamma$ be the portion of the boundary of $Q'$ lying in the first quadrant, between points $b = (0, h)$, with $0 < h \leq 1$, and $d = (1, 0)$. For any two points $p, q \in \gamma$ along $\gamma$, denote by $\gamma(p, q)$ the portion of $\gamma$ between $p$ and $q$. Let $r$ be the intersection point of $\gamma$ and the vertical line $x = 1/3$.

For a positive integer $n$, subdivide $Q_1$ into at most $2n + 2$ pieces as follows. Choose $n + 1$ points $b = q_1, q_2, \ldots, q_{n+1}$ along $\gamma(b, r)$ such that $q_i$ is the intersection of $\gamma$ and the vertical line $x = (i - 1)/3n$. Connect each of the $n + 1$ points to $d$ by a straight line segment. These segments subdivide $Q_1$ into $n + 2$ pieces: the right triangle $T_0 = \Delta bd$; a convex body $Q_0$ bounded by $rd$ and $\gamma(r, d)$; and $n$ curvilinear triangles $\Delta q_i q_{i+1}$ for $i = 1, 2, \ldots, n$. For simplicity, we assume that neither $Q_0$, nor any of the curvilinear triangles are degenerate; otherwise they can be safely ignored (they don’t contribute to the value of $\mu^*_Q$). Subdivide each curvilinear triangle $\Delta q_i q_{i+1}$ along the vertical line through $q_{i+1}$ into a small curvilinear triangle $S_i$ on the left and a triangle $T_i$ incident to point $d$ on the right. The resulting subdivision has $2n + 2$ pieces, under the nondegeneracy assumption.

By Lemma 4, we have $\mu_{T_i}(o) > 1/3$. Observe that the difference $\mu_{T_i}(o) - 1/3$ does not depend on $n$, and let $\delta = \mu_{T_i}(o) - 1/3$. By Lemma 6, we also have $\mu_{T_i}(o) > 1/3$, for each $i = 1, 2, \ldots, n$. Since every point in $Q_0$ is at distance at least $1/3$ from the origin, we also have $\mu_{Q_0}(o) \geq 1/3$.

For the $n$ curvilinear triangles $S_i$, $i = 1, 2, \ldots, n$, we use the trivial lower bound $\mu_{S_i}(o) \geq 0$. We can also show that $s_n = \sum_{i=1}^n \mu_{S_i}(o) \geq 1/12n$.

![Figure 2](image.png)

**Figure 2:** (i) The subdivision of $Q_1$ for $n = 3$. Here $o = (0, 0), q_1 = b = (0, h), q_2 = r, d = (1, 0)$. (ii) Transformation used in the proof of Lemma 6.

In particular, $s_n \leq \delta \cdot \text{area}(T_0)$ for a sufficiently large $n$. Then we can write

$$\mu_{Q_1}(o) = \frac{\int_{p \in Q_1} \text{dist}(op) \, dp}{\text{area}(Q_1)} \geq \frac{\mu_{Q_0}(o) \cdot \text{area}(Q_0) + \sum_{i=0}^n \mu_{T_i}(o) \cdot \text{area}(T_i)}{\text{area}(Q_1)} \geq \frac{1}{3} + \frac{2\delta \cdot \text{area}(T_0)}{3 \cdot \text{area}(Q_1)} > \frac{1}{3}.$$

This concludes the proof of Theorem 1.

**References**


