Computing Wrench Bounds Along a Curved Surface in 2D

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***Abstract*— Often it is useful to account for small error or variation in a physical simulation. We develop conservative bounds for the unit wrenches applied by pushing on a curved surface patch in two dimensions. We discuss subdividing the surface patch to obtain tighter bounds, and incorporating varying force directions caused by frictional contacts.**

I. Introduction

A. Motivation

The simulation of object contact and response is critical for a wide range of applications. Often these simulations compute one result for a specific set of object parameters and contact locations. However, in the real world contact locations and other parameters will never be exact, leading to variation in the final result. For this reason it can be useful to know what set of wrenches may be produced when pushing at any point along a small portion of an object's surface. This paper presents a method for determining a set containing all possible wrenches produced through contact along a surface patch in 2D.

If we are trying to identify locations on the object that produce wrenches within a set of solutions [1], then using our method it is now possible to prove that all points along a surface patch produce wrenches inside of the solution set. It is also possible to incorporate a frictional contact model where the force direction, as well as the contact location, is not known precisely. Previously, for curves of nominal complexity, it was only possible to prove that discrete points with a fixed force direction were contained in the solution set.

It is often fairly easy to bound the x and y components of the wrench, but bounding the moment of the contact force is more difficult, because it depends on both the force direction and the position of the line of action. Defining the moment analytically in terms of variable t is produces extremely complicated results even for very simple curves. When friction is present and force direction is uncertain, the moment of the contact force becomes a set-valued function of the variable t. In order to avoid the difficulties of determining the moment analytically, we instead generate conservative bounds.

B. Previous Work

Other researchers have done work in similar areas. When the side of one object is resting flat against another another object multiple contact points must be considered. One common question that arises from this scenario is deciding the stability of an object, or what forces are necessary to achieve stability [2], [3]. Constructing an assembly line, when the exact orientation of the part is unknown is another research area [4]. However, these approaches tend to make one or more of the following assumptions: we are only testing the stability of the object, all contacts are frictionless, or all objects are composed of polygonal sides. Our approach makes none of the above assumptions.

C. Structure of Paper

Since trying to find exact bounds for the moment at every point along a curve is difficult, we instead construct conservative bounds. We assume that it is possible to construct a bounding polygon that encloses the curve segment, and to construct a bound for all force directions. Using these bounds for position and force direction we can compute conservative bounds for the moment at all points on the curve segment.

We first derive the minimum and maximum moment for a fixed point with varying but bounded force direction. We then examine bounds for the moment at all points on a line segment with fixed force direction. This then leads to bounds for the moment for all points on a line segment with bounded force direction.

We show that the moment bounds for all points along the line segments of our bounding polygon also bounds all interior points, including the curve segment. We then briefly discuss how frictional contact interactions can be incorporated. After this we discuss subdividing the curve segment to achieve tighter bounds, as well as implementation details.

II. Background

A. Assumptions

Assume the following:

1) The surface can be split up into curve segments, where each segment s is defined parametrically by:

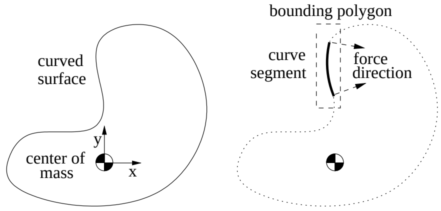
$$s(t) = (x(t), y(t)), \quad t \in [0, 1]. \tag{1}$$

2) We can bound all points of s within a simple polygon composed of vertices $(p_1, p_2, \ldots, p_n)$.

3) It is possible to calculate bounds for the direction of the force vector θ:

$$\theta_{start} \le \theta \le \theta_{end}, \tag{2}$$
$$0 \le \theta_{end} - \theta_{start} < 2\pi. \tag{3}$$

Methods for calculating the bounding polygon and force direction interval are discussed in sections VII and VIII.



(a) Surface (b) Curve Segment

Fig. 1. The part is shown on the left. On the right a curve segment from the shape, the bounding polygon surrounding the segment, and the unit force vectors at the endpoints are shown.

B. Variables

For a single point we construct a position vector $\vec{r}$ as well as an inward facing unit force vector $\hat{d}$. The wrench exerted by pushing at a point in the direction of $\hat{d}$ with magnitude f can be expressed as:

$$w = f \begin{bmatrix} \hat{d}_x \\ \hat{d}_y \\ \vec{r} \otimes \hat{d} \end{bmatrix}. \tag{4}$$

The moment, ψ, is the third element of the wrench w. For the time being we assume that the force magnitude $f = 1$:

$$\psi = \vec{r}_x \hat{d}_y - \vec{r}_y \hat{d}_x. \tag{5}$$

We designate ψ_{min} and ψ_{max} as the minimum and maximum moment values that can be produced. As the paper progresses we examine larger and larger domains where either position, force direction, or both may vary.

III. Fixed Point with Bounded Force Direction

We start by examining how ψ changes with respect to the angle of the force direction. This is equivalent to finding the moment bounds produced when pushing at a fixed point with varying force direction (such as in a frictional contact model). We look at two cases: when the force direction can be any angle, and when the force direction is bounded within an interval.

Since the unit force direction vector is length 1 it can be entirely expressed as an angle. We use the function atan2(y, x) to robustly find the polar angle for a vector $[x, y]^T$ in Cartesian coordinates. This function is well defined for all coordinates (unlike arctan) and always returns angles within the range $[-\pi, \pi]$. Using this function we define θ as the angle for the unit force vector $\hat{d}$:

$$\theta = \text{atan2}(\hat{d}_y, \hat{d}_x). \tag{6}$$

We can also define an angle for the position vector $\vec{r}$:

$$\phi = \text{atan2}(\vec{r}_y, \vec{r}_x). \tag{7}$$

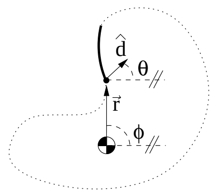


Fig. 2. An example showing $\vec{r}$, $\hat{d}$, θ and ϕ

Converting the cross-product term in Eqn. (5) to an equivalent sine expression:

$$\psi(\theta) = \|\vec{r}\| \|\hat{d}\| \sin(\theta - \phi). \tag{8}$$

In the above equation $\|\vec{r}\|$ and ϕ are fixed for a single point, and $\|\hat{d}\|$ is 1 for all points. This lets us express the moment at a fixed point as a scaled and offset sine function involving only θ, as can be seen in Eqn. (8).

A. Unbounded Force Direction

We designate θ^* and θ^{**} as the angles of force direction that respectively minimize and maximize ψ at a single point $\vec{r}$. We first examine the case where the force direction is completely unrestricted, allowing θ to be any value within $\mathbb{R}$:

$$\theta^* = \text{argmin}\{\psi(\theta) : \theta \in \mathbb{R}\}, \tag{9}$$
$$\theta^{**} = \text{argmax}\{\psi(\theta) : \theta \in \mathbb{R}\}. \tag{10}$$

Examining Eqn. (8) and using well known properties of the sine function we can state $\psi(\theta)$ is a periodic function with period 2π, it is monotonic between local minima and maxima and all minima and maxima are global minima and maxima. Furthermore, the global minima and maxima have values

$$\psi_{min} = -\|\vec{r}\|, \tag{11}$$
$$\psi_{max} = \|\vec{r}\|. \tag{12}$$

Finally, expressing the set of all integers as $\mathbb{Z}$, the global minima and maxima occur at the following angles:

$$\theta^* = \phi + \frac{3\pi}{2} + k(2\pi), k \in \mathbb{Z}, \tag{13}$$
$$\theta^{**} = \phi + \frac{\pi}{2} + k(2\pi), k \in \mathbb{Z}. \tag{14}$$

This confirms the intuitive notion that the moment is either maximized or minimized when the force direction is perpendicular to the position vector. It is also important to note that all possible angles θ^* refer to one force vector direction since all angles are separated by 2π.

B. Bounded Force Direction

If the force direction is restricted to lie within the angles $[\theta_{start}, \theta_{end}]$ for a fixed point there are two possibilities. Either a global minimum for ψ can be achieved by a force direction within $[\theta_{start}, \theta_{end}]$, or it cannot. We assume for the moment that we want to find a minimal moment ψ_{min}.

A global minimum of ψ for a fixed point $\vec{r}$ is achieved when $\theta = \phi + \frac{3\pi}{2}$, as shown in Eqn. (13). If $\theta_{start} \leq \phi + \frac{3\pi}{2} \leq \theta_{end}$ then $\psi_{min} = \psi(\phi + \frac{3\pi}{2})$. If $\phi + \frac{3\pi}{2}$ is not in the interval, then since there are no other local minima the minimum moment must be either $\psi(\theta_{start})$ or $\psi(\theta_{end})$. From this we can express θ^* and θ^{**} for bounded force direction as the following:

$$\theta^* = \left\{ \begin{array}{ll} \theta_{start} \leq \phi + \frac{3\pi}{2} \leq \theta_{end}, & \phi + \frac{3\pi}{2} \\ \text{else}, & \text{argmin}\{\psi(\theta) : \theta_{start}, \theta_{end}\} \end{array} \right\}, \quad (15)$$

$$\theta^{**} = \left\{ \begin{array}{ll} \theta_{start} \leq \phi + \frac{\pi}{2} \leq \theta_{end}, & \phi + \frac{\pi}{2} \\ \text{else}, & \text{argmax}\{\psi(\theta) : \theta_{start}, \theta_{end}\} \end{array} \right\}. \quad (16)$$

IV. Line Segment with Fixed Force Direction

In this section we examine the moment bounds for all points along a line segment using a fixed force direction θ. This is equivalent to finding the moment bounds for applying force at any point along a line segment with a frictionless contact model.

Given two non-equivalent points (p_{1x}, p_{1y}) and (p_{2x}, p_{2y}) we can define a line segment as:

$$u(t) = ((t)p_{1x} + (1-t)p_{2x}, (t)p_{1y} + (1-t)p_{2y}), \quad t \in [0, 1]. \quad (17)$$

The moment can then be defined as:

$$a = \frac{-(p_{1x} - p_{2x})^2 - (p_{1y} - p_{2y})^2}{\sqrt{(p_{1x} - p_{2x})^2 + (p_{1y} - p_{2y})^2}}, \quad (18)$$

$$b = \frac{p_{2x}(-p_{1x} + p_{2x}) - (p_{1y} - p_{2y})p_{2y}}{\sqrt{(p_{1x} - p_{2x})^2 + (p_{1y} - p_{2y})^2}}, \quad (19)$$

$$\psi(t) = at + b. \quad (20)$$

This gives the interesting result that the moment for each point along a line segment with a fixed force direction varies linearly with t. Since linear functions do not have local minima or maxima it is only necessary to check the values of ψ at the two endpoints of the line segment to find the minimum and maximum values of ψ.

V. Line Segment with Bounded Force Direction

In this section we examine the moment bounds for all points along a line segment using a bounded force direction. This is relevant to finding the moment bounds obtained by pushing anywhere along a line segment with a frictional contact model. This will be useful later when we examine the bounding polygon of a curve segment.

Looking at Eqns. (15) and (16) it would seem important to know if angles $\phi + \frac{3\pi}{2}$, $\phi + \frac{\pi}{2}$, or both lie within the interval $[\theta_{start}, \theta_{end}]$. Originally we split the line segment up into sub-segments based on the existence of θ extrema. It turns out that this is not necessary.

As was discussed earlier, if we choose one force direction and graph the moment produced versus the parametric variable t along a line segment we get a straight line. In this case we have chosen one value of θ, but when graphing maximal moment we can select any value of θ within $[\theta_{start}, \theta_{end}]$ to find the maximum value of ψ at each position $\vec{r}$. This means that the graph of maximal moment will have values that are greater than or equal to the graph of moment for a fixed force direction. Another way of saying this is that we know that all values for the graph of maximal moment must lie above the half-space produced by the graph of moment for a fixed force direction. If we graph two lines we know that all values for the graph of maximal moment must lie within the intersection of the upper half-spaces produced by the lines. When graphing minimal moment we would use the intersection of the lower half-spaces of the lines.

The graph of maximal moment along a line segment can be seen as the boundary of the intersection of many half-spaces produced by individual force directions. Stating it this way we can make a few observations about the final graph of maximal moment. Since a half-space is a convex set, and we are taking the intersection of convex sets, the set of all points that lie above the maximal moment graph is convex. Since the graph of maximal moment is the boundary below a convex set, there may be a local minimum, but there can never be a local maximum. Similarly, the graph of minimal moment may contain a local maximum, but no local minimum. In both cases the local extrema are of no interest since we are searching for the largest value of maximal moment along a line segment, and the smallest value of minimal moment along a line segment. Since we do not have to worry about local extrema it is only necessary to find minimal and maximal moment values at the two endpoints of the line segment, $t = 0$ and $t = 1$. This methodology would not work if the set of valid forces changed for different points along the line segment.

When θ extrema are present along a portion of the line segment the graph of moment ψ versus t can be computed by plugging in the line segment definition defined by Eqn. (17) into Eqns. (11) and (12). In the end we get $\pm\sqrt{at^2 + bt + c}$, where a, b and c are constants that depend on the location of the line segment, and the sign depends on whether we are finding a minimum or maximum value for ψ.

In Fig. 3 we see an example of these ideas. At the spot pictured in the left image the force direction A maximizes ψ. However, for other points along the line segment, A is non-optimal. This relationship is shown in the right image. The value of ψ produced by using fixed force direction A intersects the maximal curve at exactly one point. Every force direction that is not perpendicular to the line segment corresponds to a line that is tangent to either the maximal or minimal moment curves.

While we do not need worry about local minima or maxima for points along the interior of the line segment, we do need to worry about θ extrema on the endpoints of the line segment, as documented in Eqns. (15) and (16). To do this we must check if $\phi + \frac{3\pi}{2}$ or $\phi + \frac{\pi}{2}$ are within $[\theta_{start}, \theta_{end}]$. This can be seen

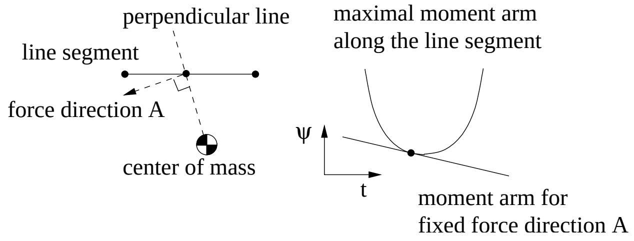


Fig. 3. The left shows a line segment with force direction A, the right shows a graph of moment for all maximal force directions and for the fixed force direction A

visually by drawing two lines through the origin at angles perpendicular θ_{start} and θ_{end}. Any points that fall between these lines will have either $\phi + \frac{3\pi}{2}$ or $\phi + \frac{\pi}{2}$ within $[\theta_{start}, \theta_{end}]$. In Fig. 4 left endpoint of the line segment tests three different values of θ. The $\phi + \frac{\pi}{2}$ is tested to find the maximum value of ψ, and both θ_{start} and θ_{end} are tested to find minimum values of ψ.

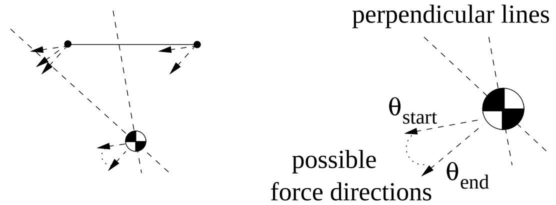


Fig. 4. A third value of θ must be tested for the left endpoint because the extra θ value is perpendicular to ϕ and within $[\theta_{start}, \theta_{end}]$.

VI. Wrench Bounds for a Curve

We assume that the curve segment $s(t)$ is contained within some known closed polygon. We solve for the minimum and maximum moments on each line segment of the bounding polygon. Comparing all values of ψ_{min} and ψ_{max} for each line segment we find values for ψ_{min} and ψ_{max} that bound the moment for all points on the boundary of the polygon. From here we can prove that we have also bounded all points contained in the interior of the polygon.

For any point within the polygon, draw a line in any direction and find the intersection with the line and the polygon boundary. We have found that the possible values of ψ for a point on a line segment are bounded by the values possible at the endpoints of the line segment. For this reason we can say that ψ_{min} and ψ_{max} for the internal point will be bounded by the points on the boundary of the polygon. Since the two endpoints lie on the boundary of the polygon, and since we have bounds for all points on the boundary of the polygon we can say that our final global bounds $[\psi_{min}, \psi_{max}]$ bound the moment for all points interior points using any force direction $[\theta_{start}, \theta_{end}]$.

At this point we have three bounds, the intersection of which make up our final conservative bounds in the $xy\psi$ wrench space. First we use the interval $[\theta_{start}, \theta_{end}]$ to bound the x and y direction of all wrenches. Secondly, we know $\|\hat{d}\| = 1$ so we bound all wrenches to be within the $x^2 + y^2 = 1$ cylinder. Thirdly we use the interval $[\psi_{min}, \psi_{max}]$ to bound the ψ component of the wrench. This creates a quadrilateral on the $x^2 + y^2 = 1$ cylinder. To normalize all wrenches to unit length we project onto the unit sphere as shown in Fig. 5. On the unit sphere the longitude is restricted to be within $[\theta_{start}, \theta_{end}]$ and the latitude is restricted to be within $[\arctan(\psi_{min}), \arctan(\psi_{max})]$. If the force magnitude f can be any non-negative value, then the set of of all resulting wrenches is a four sided convex polyhedral cone.

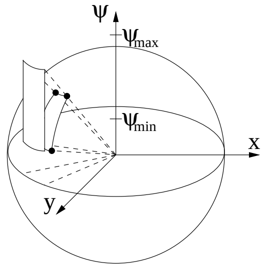


Fig. 5. Bounds are projected onto the sphere of unit wrenches

VII. Varying Force Direction Due to Friction

For a frictionless contact model the inward facing normal will always equal the inward facing force direction vector. For a single point the force direction θ will always equal the normal direction which we will call γ:

$$\theta_{start} = \gamma_{start}, \tag{21}$$

$$\theta_{end} = \gamma_{end}. \tag{22}$$

If we use a frictional contact model, things are slightly more complicated. Using a Coulomb friction contact model with positive coefficient of friction μ, the force direction can be any vector that lies lies inside of the friction cone. The angle between the edge and center of the cone is $\arctan(\mu)$. Using this we can can bound the possible force directions with the normal direction and coefficient of friction:

$$\theta_{start} = \gamma - \arctan(\mu), \tag{23}$$

$$\theta_{end} = \gamma + \arctan(\mu). \tag{24}$$

If we want to bound force direction for more than one point, and points can have different normal directions γ, then our force direction interval $[\theta_{start}, \theta_{end}]$ needs to contain any value in $[-\arctan(\mu), \arctan(\mu)]$ summed with any value in $[\gamma_{start}, \gamma_{end}]$:

$$\theta_{start} = \gamma_{start} - \arctan(\mu), \tag{25}$$

$$\theta_{end} = \gamma_{end} + \arctan(\mu). \tag{26}$$

VIII. Implementation

Finding force direction bounds robustly for a curve can be surprisingly difficult. It is easy to make an implementation which produces incorrect results such as $\theta_{start} = \frac{7}{8}\pi$, $\theta_{end} = -\frac{7}{8}\pi$. In this case $\theta_{start} > \theta_{end}$, most likely due to the discontinuity of atan2 at $\theta = \pi$. Offsets of 2π can be used to properly order the intervals. Rotating the curve such that $\theta_{start} = -\pi$ is another option.

The easiest way to calculate the normal direction interval $[\gamma_{start}, \gamma_{end}]$ for a curve segment is to split the curve into entirely concave or convex sub-segments. In this case the direction of the normal vectors at the endpoints bound all normal directions. However, depending on the curve, finding inflection points may not be trivial. For cubic curves it is possible to find whether an inflection point exists by solving a quadratic equation [5]. Using this, one can classify most of the surface patch as either convex or concave using a divide-and-conquer method. It is also possible to solve for the exact location of an inflection point along a cubic curve segment by solving a cubic equation [6]. The presence of straight surfaces and cusps require special cases. For straight surfaces only one surface normal is possible along the line segment, and when dealing with cusps many possible surface normals may exist for a single point. Force applied along a line or at a single point is discussed in earlier sections of the paper.

Once curve segments are split at inflection points it is important to know whether the curve segment is concave or convex, since this will determine whether the normal direction will sweep clockwise or counter-clockwise. One way to check is by examining whether the second derivative points into or away from the the surface. Another approach to finding which way the normal direction sweeps is to test the normal direction at internal points along the curve segment.

Our method uses uniform B-splines to define curved surfaces. Using B-splines the convex hull of the four control points for any curve segment creates a bounding polygon for the curve segment. It is also fairly easy to determine areas of concavity and convexity using B-splines. Finally, a single B-spline curve segment can be subdivided into two smaller curve segments. This is useful if the bounds for a curve segment overlap with a given solution set of wrenches. If conservative bounds are too large to lie entirely within or outside of our solution set of unit wrenches we recursively split the curve in half until the bounds do not overlap the solution set or we reach some predefined termination level.

Any curve segment where the possible force directions sweep beyond a full revolution should be avoided, since this will break our assumption in Eqn. (3), (see Fig. 6). A self-intersecting B-spline curve can be made with four control points, but in our experience it seems impossible to create a curve that sweeps beyond a full circle using only four control points.

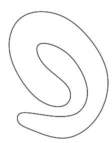
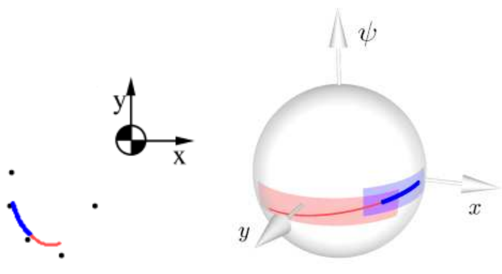


Fig. 6. A large enough curve segment on this surface may have values of θ that sweep past 2π

Since the conservative boundary is a convex cone, checking whether the conservative bounds are contained within a desired solution set can be most easily accomplished when the solution set is convex. In this case it is only necessary to check whether the four edge wrenches of the polygonal cone lie within the solution set. If the solution set is not convex, then convex decomposition can be used to try to find convex regions to test against. Otherwise other geometric methods need to be employed.

In Fig. 7(a) and 7(b) we show results from our implementation. In Fig. 7(a) we show a Java program that allows the specification of B-spline control points, and then calculates the wrench bounds for every curve segment. In Fig. 7(b) we show another application that visualizes both the conservative bounds (the shaded rectangles), as well as the results obtained from high resolution point sampling (thick lines).



(a) Surface (b) Wrenches

Fig. 7. Two curve segments along a surface are shown in the left figure. In the right figure the frictionless wrenches and bounds are mapped onto the unit sphere.

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