Sensor Selection in Arbitrary Dimensions

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***Abstract*— We address the sensor selection problem which arises in tracking and localization applications. In sensor selection, the goal is to select a small number of sensors whose measurements provide a good estimate of a target's state (such as location). We focus on the bounded uncertainty sensing model where the target is a point in the d dimensional Euclidean space. Each sensor measurement corresponds to a convex, polyhedral subset of the space. The measurements are merged by intersecting corresponding sets. We show that, on the plane, four sensors are sufficient (and sometimes necessary) to obtain an estimate whose area is at most twice the area of the best possible estimate (obtained by intersecting all measurements). We also extend this result to arbitrary dimensions and show that a constant number of sensors suffice for a constant factor approximation in arbitrary dimensions. Both constants depend on the dimensionality of the space but are independent of the total number of sensors in the network.**

Note to Practitioners

In many applications, sensing and communication constraints may render using all available sensors infeasible. In such scenarios, selecting a small number of sensors – whose collaborative performance in estimating the state of a target is comparable to the best possible achievable error – becomes important. This paper focuses on sensors whose measurements can be specified as an intersection of halfspaces (e.g. cameras, whose measurements correspond to cones). It is proven that a "small" set of good sensors can be selected from an arbitrary set of measurements in any dimension d. Of practical importance are the two cases: $d = 2$ (where four sensors suffice for a good estimate) and $d = 3$ (eight sensors are enough).

***Index Terms*— Sensor networks: camera networks and sensor selection; Computational Geometry and Object Modeling: Geometric algorithms, languages, and systems: minimum enclosing simplex, polytope approximation.**

I. Introduction

A sensor-network consists of sensing devices with communication, computation and sensing capabilities. One of the primary applications of sensor-networks is tracking. In most systems, multiple nodes participate in the tracking task and collaboratively estimate the location of the target. On the other hand, power and bandwidth limitations may prevent the utilization of a large number of sensor nodes at a given time. Consequently, many researchers focused on *sensor selection* so as to choose a small number of sensors while guaranteeing high quality estimates.

The sensor selection problem is typically formulated as follows. We are given the location of the sensors as well as prior information about the location of the target. In addition, we are given a sensing model, which gives us the quality of an estimate of the target's state (e.g. position) for a given set of chosen sensors and the target's true state. The goal is to select a small number of sensors so that the quality of the estimate is high.

We address the sensor selection problem in the bounded uncertainty sensing model. In the planar version of this model, each sensor measurement corresponds to a convex subset of the plane. We merge measurements by intersecting corresponding subsets and the quality of the estimation is inversely proportional to the area of the intersection. This formulation generalizes naturally to higher dimensions: The state of the target is represented by a point in $\mathbb{R}^d$. The measurement from a sensor s identifies a subset of the space $U(s) \subset \mathbb{R}^d$ which contains the true state of the target. For example, in camera-network applications, $U(s)$ is a proper cone in 3D. In general, the target's state can be higher dimensional. For example, it can contain its location and additional attributes such as its temperature. If a single sensor node contains both position and temperature sensors, it is natural to minimize the number of active sensor nodes so as to minimize the total communication in the network. Therefore, sensor selection in arbitrary dimensions may be of interest in certain applications.

Recently, sensor selection in the bounded uncertainty model has been addressed in [14]. The authors showed that when the measurements correspond to convex, polygonal subsets of the plane, one can choose six sensors such that the resulting uncertainty from these measurements is at most twice the uncertainty that would have been obtained by querying *all* the sensors [14] – no matter how large the number of sensors is, six sensors suffice for a 2-approximation. In the present work, we improve on this result in the following directions.

1) We show that, in the planar case four sensors suffice for a 2-approximation. We also show that this result is tight: there are instances where at least four sensors are needed to obtain any bounded approximation.

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2) In the 3-d case (e.g. cameras in 3d space), we show that 8 sensors suffice for a 9-approximation, and at least 6 are needed to guarantee bounded approximation.
3) In higher dimensions, we obtain an analogous result. Let n be the number of sensors in the network and d be the dimensionality of the space. We show that with a *constant* number of measurements, independent of n, one can obtain a *constant* factor approximation, also independent of n. Both constants depend on d. The main tool we use is a construction of an enclosing simplex of a convex polygon, which may be of independent interest.

II. Related Work

Sensor selection has received significant recent attention. In [10], a selection algorithm is presented where the minimum mean squared error of the best linear estimate of the object position in 2-D is the metric for selection. The work in [5] addresses a generic utility-based sensor selection scheme and presents $\log$ factor approximation algorithms for a class of set-weighted utility functions. Sensor selection in the bounded uncertainty model on the plane was studied in [14]. The present work improves on this result and generalizes it to arbitrary dimensions.

In [21], an *information driven sensor query* approach was proposed. In this approach, at any given time, only a single sensor (leader) is active. After obtaining a measurement, the leader selects the most informative node in the network and passes its measurement to this node which becomes the new leader. In subsequent work, researchers addressed leader election, state representation, and aggregation issues [20], [28]. A sensor selection method based on the mutual information principle is presented in [11]. Recently, an *entropy based heuristic approach* was proposed [27] which greedily selects the next sensor to reduce overall uncertainty. The bounded uncertainty model, which we focus on in this paper, is frequently used for localization in the robotics and sensor-networks literature. Examples can be found in [8], [24].

A. Related Geometric Results

Here we consider enclosing a convex polytope given by its redundant H-representation (linear inequalities). Enclosing convex objects is a well researched topic. Typically the convex object is given by a redundant V-representation (convex hull of vertices). The type V and H canonical representations of convex polytopes, and moving between the two are discussed in [1].

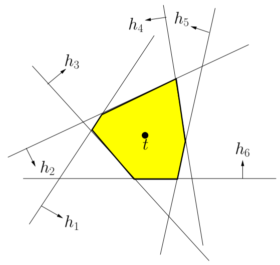
Optimal, or near optimal, linear algorithms exist for constructing enclosing simplices in 2 and 3 dimensions, [22], [30]. The centroidal property of minimum enclosing simplices in d-dimensions was given in [16] which was exploited in analyzing the degrees of freedom of minimal simplices in [25]. There are no results on finding minimum enclosing simplices efficiently for general d By the result of Klee [16], any minimum simplex S intersects the convex body C at every one of its facet centroids. The centroidal simplex S_c with vertices at these centroids has volume equal to $volume(S)/d^d$ (this folklore result may be deduced from the result in [7], alternatively see the proof of Lemma 6.3). By convexity, $S_c \subset C$, and so it immediately follows that the minimum enclosing simplex is a d^d volume approximation to C. We give an explicit construction for an enclosing simplex with a better volume bound by an extra factor of d. Our construction goes through a *locally* maximal inscribed simplex. Dudley [9] gives an efficient construction of enclosing polytopes for a convex polytope in arbitrary dimension, where the approximation ratio is a decreasing function of the number of vertex points in the approximating body. In particular, in 2 and 3 dimensions, a polytope with $O(1/\epsilon)$ vertices suffices (constructive) for an $O(\epsilon)$ approximation. We study what can be done with a small (constant) number of vertices.

Other useful, simple enclosing bodies are parallelepipeds, ellipsoids and balls, which have been the focus of significant research. Minimal enclosing parallelepipeds in 2 and 3 dimensions are studied in [2], [23], [26]. Approximations to minimal enclosing balls have been studied in arbitrary dimension [18], [29], and it is shown in [13] that the ellipsoid method can be used to construct an affine transformation such that the unit ball is contained in the convex body which in turn is contained in the $d\sqrt{d}$ ball. This immediately gives a construction for an enclosing ellipsoid with volume approximation $d^{3d/2}$. Efficient $(1+\epsilon)$-approximations to minimum volume ellipsoids are given in [19] and it is shown in [15] that the minimum volume ellipsoid gives a d^d volume approximation to the convex polytope. There is no bound on the number of intersection points of the convex body and the enclosing ball or ellipsoid, thus simplices and parallelepipeds are more suited to obtaining good volume approximations with a small subset of the halfspaces. Other types of constraints, such as axial symmetry [3], have also been studied. Applications of constructions which tightly enclose a set of points or balls have become prevalent, e.g. proximity based algorithms and kernel methods for clustering [4], [12].

III. Problem Formulation

In this section, we formulate the sensor selection problem. We are given a set of sensors as well as an estimate of the state of the target. A query to a sensor s, localizes the object to a subset of the space $U(s) \subset \mathbb{R}^d$ which contains the state of the target. We call $U(s)$ the *measurement* corresponding to sensor s. We assume that $U(s)$ is an intersection of halfspaces, i.e., the region to which a sensor localizes an object is given by a convex polytope (possibly unbounded). This certainly applies to many sensor models which identify the sensed region with a proper cone.

After querying a subset Q of the sensors, the target can be "localized" to the set $\cap_{s\in Q} U(s)$. It is natural to define the uncertainty of the measurement as $volume(\cap_{s\in Q} U(s))$. Since intersection is monotonic, it is optimal to query every sensor. Unfortunately, in most sensor-network applications, this is not feasible due to communication and power constraints. We study what can be achieved with querying only a small, specifically constant, number of sensors.



We restrict the *sensor selection problem* to halfspace measurements. This definition immediately generalizes to measurements which are arbitrary convex polytopes, since any convex polytope is an intersection of a finite number of halfspaces. Let $\mathcal{H}$ be a set of n halfspaces in $\mathbb{R}^d$, whose intersection is bounded and non-empty. Each halfspace $h_i \in \mathcal{H}$ as a measurement. The setup is illustrated in the figure above, where t is the target object. For any subset of the measurements, $\mathcal{H}' \subseteq \mathcal{H}$, we define the uncertainty $\mathcal{E}(\mathcal{H}')$ as the d-dimensional volume of the intersection of all halfspaces in $\mathcal{H}'$ (if it is finite, and ∞ otherwise). $\mathcal{H}'$ is a ρ-approximation to $\mathcal{H}$ if $\mathcal{E}(\mathcal{H}') \leq \rho \cdot \mathcal{E}(\mathcal{H})$.

IV. Sensor Selection on the Plane

We first consider the 2-d problem and show that 4 measurements are enough for a 2-approximation to $\mathcal{H}$. These 4 measurements can be determined in $O(n^4)$ by selecting the subset of size 4 with minimum uncertainty. Practically, this means that

> *No matter how many sensors are available, a carefully chosen set of four sensors suffices to localize to within twice the uncertainty attainable using all the sensors*

In 2 dimensions, the volume of a convex polytope is its area, so $\mathcal{E}(\mathcal{H}') = area(\mathcal{H}')$. We will explicitly use $area$ as the uncertainty measure in the results of this section. We also assume from now on that the uncertainty when using all the hyperplanes in $\mathcal{H}$ is bounded, i.e., the hyperplanex in $\mathcal{H}$ define a convex polygon. The main tool which will establish our result is Lemma 1 which bounds the area of the minimum enclosing triangle (MET) for any convex polygon.

Lemma 1 (Minimum Enclosing Triangle (MET)): Let P be any convex polygon. Then, there is a triangle T which contains P satisfying the following two properties:

(i) $area(T) \leq 2 \cdot area(P)$;
(ii) at least two edges of T are parallel to two sides of P.

Lemma 1 part (i) holds even if the convex polygon P is replaced by an arbitrary bounded convex shape C. Lemma 1 part (ii) is proven in the next section. The remainder of the argument to establish the advertised result using Lemma 1 is analogous to the analysis in [14]. We paraphrase some of the results in [14] below.

Theorem 2 (Isler, Bajcsy [14]): Suppose that for any convex polygon P, one can find a minimum enclosing convex polygon Q with r edges satisfying the following two properties:

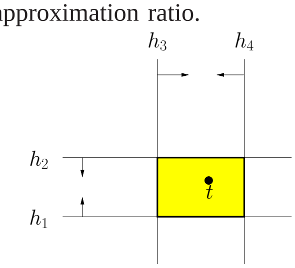
(i) $area(Q) \leq \lambda \cdot area(P)$;
(ii) at least $k \leq r$ edges of Q intersect P at edges and the remaining (at most) $r - k$ edges intersect P at a vertex.

Then, for any set of measurements $\mathcal{H}$, there exists a subset $\mathcal{H}'$ with $|\mathcal{H}'| \leq 2r - k$ for which $area(\mathcal{H}') \leq \lambda \cdot area(\mathcal{H})$.
The basic idea in the proof is that for the edges of Q which intersect edges of P, one selects the measurements corresponding to those edges in P. The remaining edges of Q intersect vertices of P and each vertex of P corresponds to two measurements. Let $m \geq k$ be the number of edges of Q which intersect edges of P. Then the total number of measurements is $m + 2(r - m) = 2r - m \leq 2r - k$. To conclude, note that these measurements form a convex polygon which is enclosed in Q and therefore has area at most that of Q.

Corollary 4.1: Any set of measurements in 2-dimensions can be 2-approximated with a subset of at most 4 measurements.

Proof: Apply of Lemma 1 with $r = 3$, $k = 2$ and $\lambda = 2$ in Theorem 2. ■

Isler and Bajcsy [14] used a result similar to Lemma 1 for minimum enclosing parallelograms with $r = 4$, $k = 2$ and $\lambda = 2$ which gave that six measurements was enough. One of our contributions is to reduce the number of required sensors by 2, without sacrificing on the approximation ratio.

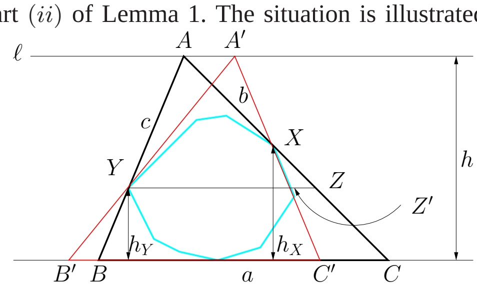


Finally, we note that the 4-measurement result is optimal, i.e., there exist settings where any collection of three measurements cannot provide a constant factor approximation to the error. To see this, consider the arrangement of four measurements shown on the right, with the target object localized in the shaded box. Intuitively, two measurements serve to localize the position in one of the dimensions and the other two localize the position in the other dimension. It is easy to verify that any subset of 3 measurements has an unbounded uncertainty, and hence an infinite approximation error.

It is natural to extend this result to higher dimensions, most practically 3 dimensions. The comments above suggest that two sensors are needed to localize in each dimension, and in fact a similar example shows that at least $2d$ sensors are necessary for a bounded approximation in d dimensions. We conjecture that this is also an upper bound on the required number of sensors to obtain a constant factor approximation, however, our analysis will only yield an upper bound of $d(d+1)$ for general d.

A. *Proof of the MET Lemma*

Let T be an MET for a convex polygon P with $area(P) = 1$. We can assume that every edge of T must intersect with an edge or vertex of P (if not we can accomplish this by shrinking T). First, we show that one can always select a new triangle T' with $area(T') \leq area(T)$ satisfying property (ii) of Lemma 1. Then, all that will remain is to show that there exists at least one triangle satisfying property (i) of Lemma 1. The basic proof idea is to take any enclosing triangle T and alter it to an enclosing triangle T' without increasing the area and such that T' had one additional side flush with P. Repeating this argument one more time then gives part (ii) of Lemma 1. The situation is illustrated below.



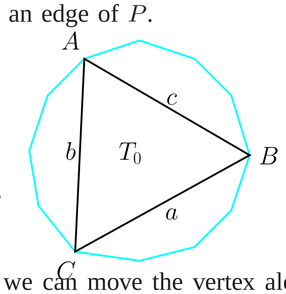
Let the vertices of T be A, B, C with respective opposite edges a, b, c and suppose that fewer than two edges of T intersect with edges of P. We now show how to increase the number of edges of T which intersect with edges of P by at least one. So suppose that two edges b, c of T intersect P at the vertices X, Y of P. Orient the triangle with base a and consider the heights h_X, h_Y of X, Y with respect to the base a. The setup is illustrated in the figure above. Without loss of generality, we can assume that $h_X \geq h_Y$ and let h be the height of A above a. Draw the line ℓ through A parallel to a and consider the point A' which is A shifted toward X on ℓ. As we shift A', we consider the triangle $A'B'C'$ in which $A'B'$ passes through Y, $A'C'$ passes through X and $B'C'$ is on the line passing through a. As we shift A', either $A'B'$ will intersect the upper edge of P at Y or $A'C'$ will intersect the lower edge of P at X. We stop shifting A' when one of these situations occurs (both conditions could also occur simultaneously). Suppose that $A'C'$ intersects the lower edge of P at X ($A'B'$ may or may not intersect the edge at Y). An identical argument applies in the other case in which $A'B'$ intersects the upper edge of P at Y. Construct the line parallel to a through Y which intersects AC at Z and $A'C'$ at Z' with $YZ' \leq YZ$ (equality occurs if $h_Y = h_X$).

AYZ and ABC are similar, therefore $h/h_Y = BC/YZ$; $A'YZ'$ and $A'B'C'$ are similar therefore $h/h_Y = B'C'/YZ'$; thus, we conclude that

$$B'C' = \frac{YZ'}{YZ} BC \leq BC.$$

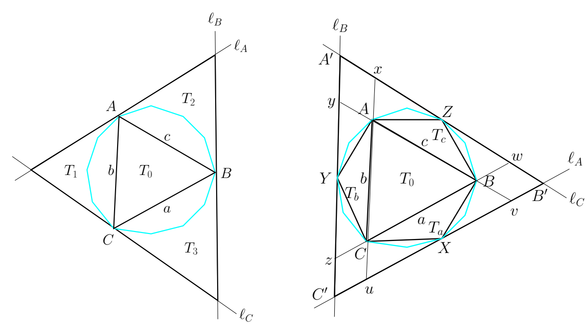
Therefore, $area(A'B'C') \leq area(ABC)$ and $A'B'C'$ is an enclosing triangle with at least one more edge intersecting an edge of P. Iterating this argument, property (ii) follows.

We have not found any published proof of Property (i). Therefore we present a proof here which will be easy to generalize to arbitrary dimension. We now show that there exists a triangle that encloses P with area at most 2. Let T_0 be a *maximum* area triangle that is enclosed by P. Without loss of generality, we can assume that the vertices A, B, C of T_0 are also vertices of P. (If not, then some vertex of T_0 is on an edge of P.



This edge must be parallel to the base of T_0 opposite the vertex, for if not then we can move the vertex in the direction of increasing height, increasing the area of T_0, which is a contradiction. If the edge is parallel to the base, then we can move the vertex along the edge to a vertex of P, without changing the area of T_0.) The final arrangement is illustrated in the figure to the right.

Construct the lines ℓ_A, ℓ_B, ℓ_C passing through A, B, C respectively and parallel to the edges a, b, c respectively. Let T be the triangle formed by ℓ_A, ℓ_B, ℓ_C as illustrated in figure (a) below. If any point of P lies outside T, then T_0 is not a maximum area inscribed triangle, so every point of P must be inside T, hence T encloses P.



(a) (b)

The triangles T_1, T_2, T_3 illustrated in figure (a) above are all congruent to T_0, hence $area(T) = 4 \cdot area(T_0)$. Thus, if $area(T_0) \leq \frac{1}{2}$, then $area(T) \leq 2$ and we are done. So suppose that $area(T_0) > \frac{1}{2}$. We use a different construction to obtain T. We define three triangles T_a, T_b, T_c as shown in figure (b) above. Let ℓ_A be the line parallel to a and tangent to P at vertex X. Thus, T_a is the triangle BCX. Note that P is divided into two sub-polygons by a (one which contains A and one which does not). The sub-polygon which does not contain A could be empty, and so T_a could be empty. This does not affect the argument. BCX is a maximum area triangle with base a that can be embeded into the sub-polygon of P that does not contain A. T_b and T_c are constructed similarly. Note that $area(P) = 1 \geq area(T_0) + area(T_a) + area(T_b) + area(T_c)$. Let h_A be the altitude in T_0 from A to a, and similarly define h_B, h_C. Let h_X be the altitude from X to a in T_a, and similarly define h_Y, h_Z. Then $area(T_a) = area(T_0) \cdot h_X/h_A$, and similarly for $area(T_a), area(T_b)$. Thus, we have

$$1 \geq area(T_0) \cdot \left(1 + \frac{h_X}{h_A} + \frac{h_Y}{h_B} + \frac{h_Z}{h_C}\right). \qquad (1)$$

Since $area(T_0) > \frac{1}{2}$ by assumption, we have that $\frac{h_X}{h_A} + \frac{h_Y}{h_B} + \frac{h_Z}{h_C} < 1$. Triangles ABC and $A'B'C'$ are similar. We now bound $area(A'B'C')$. Consider enlarging ABC into $A'B'C'$ in three steps through a sequence of similar triangles: $ABC \rightarrow Auv \rightarrow yvC' \rightarrow A'B'C'$. Let the three length scale factors for these enlargements be $\lambda_1, \lambda_2, \lambda_3$. It is easy to verify that

$$\lambda_1 = 1 + \frac{h_X}{h_A}, \quad \lambda_2 = 1 + \frac{h_Y}{\lambda_1 h_B}, \quad \lambda_3 = 1 + \frac{h_Z}{\lambda_1 \lambda_2 h_C}.$$

The length scale factor for the entire enlargement $ABC \rightarrow A'B'C'$ is $\lambda_1\lambda_2\lambda_3$ which after some manipulation reduces to $\lambda_1\lambda_2\lambda_3 = \lambda_1\lambda_2 + \frac{h_Z}{h_C} = \lambda_1 + \frac{h_Y}{h_B} + \frac{h_Z}{h_C} = 1 + \frac{h_X}{h_A} + \frac{h_Y}{h_B} + \frac{h_Z}{h_C}$. Since area scales as length squared, $area(T) = (\lambda_1\lambda_2\lambda_3)^2 \cdot area(T_0)$, we have that

$$\begin{aligned} area(T) &= \left(1 + \frac{h_X}{h_A} + \frac{h_Y}{h_B} + \frac{h_Z}{h_C}\right)^2 \cdot area(T_0) \\ &\stackrel{(a)}{\leq} \left(1 + \frac{h_X}{h_A} + \frac{h_Y}{h_B} + \frac{h_Z}{h_C}\right) < 2, \end{aligned}$$

concluding the proof (inequality (a) above follows from (1)).

V. Arbitrary Dimension

In this section we show that $d(d+1)$ measurements suffice to obtain a d^{d-1}-approximation for sensor selection in $\mathbb{R}^d$. For 2 and 3 dimensions, tighter results can be shown. We have seen that in 2-dimensions, 4 measurements suffice for a 2-approximation. We will shortly show that in 3-dimensions, 8 sensors (as opposed to 12) suffice for a 9-approximation.

The main tool we will need is a bound on the volume of a minimum enclosing simplex (MES), which is given in the lemma below.

Lemma 3: Let P be a bounded convex polytope in $\mathbb{R}^d$ with minimum enclosing simplex S. Then $volume(S) \leq d^{d-1} \cdot volume(P)$.

We present here a sketch of the proof of Lemma 3. The proof idea is analogous to the 2d-case, and we defer the full technical details to Section VI. Our proof constructs an enclosing simplex with the required volume bound from a *locally* maximal inscribed simplex. In our context, P is a bounded convex polytope, however our proof applies to an arbitrary bounded convex body. We note that finding the maximum enclosed simplex for a convex polytope is NP-hard [17]. However, finding a locally maximal simplex (Definition 6.1) is a differentiable local optimization problem, and hence can be solved efficiently using convex optimization techniques [6]. Tightly enclosing convex bodies using simple geometric objects is an important problem, especially as a precursor to collision detection of point sets, with applications in computational geometry, machine learning, etc. By Lemma 3, the feasible set of any number of linear inequalities (assuming it is non-empty and bounded) is approximated by the feasible set of a constant number of carefully chosen inequalities. Thus, Lemma 3 may be of independent interest.

Proof Sketch: We begin with the locally largest simplex M which can be inscribed inside P. If $volume(M)$ is small (at most $\frac{1}{d}$), then analogous to the 2-dimensional case, we show how to cover P with a simplex whose volume is d^d times larger than $volume(M)$. Thus, any MES has volume at most $d^d volume(M)$. On the other hand, if $volume(M)$ is large (at least $\frac{1}{d}$), we show how to expand every height (perpendicular length from a vertex to a face) in M slightly so as to enclose P. This results in a new simplex M' which is a homothet of M. We show that the length scale factor for the enlargement is $1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i}$, where for each height h_i in M, the corresponding height in M' is $h_i + \delta_i$, increased by δ_i. Thus, in this enlargement, the volume increases by a factor $(1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i})^d$. Since $volume(M)$ is large, the δ_i's are not large, and infact it is the case that $\left(1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i}\right) volume(M) \leq 1$. It then follows that $volume(M') = \left(1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i}\right)^d volume(M) \leq \left(1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i}\right)^{d-1}$. The result follows because $volume(M) \geq \frac{1}{d}$, and so $1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i} \leq d$. Lemma 3 gives a d^{d-1}-approximation for the measurement selection:

Theorem 4: There exists a subset $\mathcal{H}' \subseteq \mathcal{H}$ with $|\mathcal{H}'| \leq d(d+1)$ and $\mathcal{E}(\mathcal{H}') \leq d^{d-1} \cdot \mathcal{E}(\mathcal{H})$.

Proof: The simplex S is the intersection of $d+1$ halfspaces $f_0, \ldots, f_d$, with boundaries $\partial f_0, \ldots, \partial f_d$. Each hyperplane ∂f_i can be chosen to intersect with P, i.e. ∂f_i contains a face g_i of P with $0 \leq deg(g_i) \leq d-1$ (in the worst case, ∂f_i contains only a vertex of P); g_i is defined by the intersection of $d - deg(g_i)$ halfspaces in $\mathcal{H}$, denoted by $h_1^i, \ldots, h_{d-deg(g_i)}^i$. Therefore $P \subset \cap_j h_j^i \subseteq f_i$, and hence $P \subseteq \cap_{i,j} h_j^i \subseteq \cap_i f_i = S$. Using Lemma 3, we have $volume(\cap_{i,j} h_j^i) \leq volume(S) \leq d^{d-1} \cdot volume(P)$. To conclude, let $\mathcal{H}' = \{h_j^i\}_{i,j}$ and note that $|\mathcal{H}'| = \sum_{i=0}^{d} d - deg(g_i) \leq d(d+1)$. ∎

The sum $\deg(S) = \sum_{i=0}^{d} d - deg(g_i)$ which determines $|\mathcal{H}'|$ in the proof above is often referred to as *the number of degrees of freedom* of the enclosing simplex S. If S is minimal, tighter upper bounds (than the trivial $d(d+1)$) for $\deg(S)$ can be used to strengthen the result. In particular, for $d = 2$, $\deg(S) \leq 4$ [22], and for $d = 3$, $\deg(S) \leq 8$ [25]. Therefore, in 2 dimensions, we have a 2-approximation with 4 measurements; in 3 dimensions, a 9-approximation with 8 measurements; and, for $d > 3$, a d^{d-1}-approximation with $d(d+1)$ measurements. By considering hyperplanes supporting the faces of a d-dimensional parallelepiped as $\mathcal{H}$ ($|\mathcal{H}| = 2d$), we immediately get the lower bound of $2d$ measurements to obtain a bounded approximation. Thus, the results for $d = 1, 2$ are tight. Further, by letting P be a ball, it is clear that one cannot expect more than an exponential approximation ratio with a constant number of halfspaces.

VI. Proof of Lemma 3

A simplex $S(\mathbf{v}_0, \ldots, \mathbf{v}_d) = \{\mathbf{x} = \sum_{i=0}^{d} \lambda_i \mathbf{v}_i | \lambda_i \in \mathbb{R}^+, \sum_{i=0}^{d} \lambda_i = 1\}$ is the convex closure of $d + 1$ points $\mathbf{v}_0, \ldots, \mathbf{v}_d$. (We will usually suppress the vectors defining S when the context is clear, and will use $\mathbf{v}_i$ to refer to the vector of coordinates of the vertices of the simplex as well as the vertices themselves.) The hypervolume of S is given by

$$volume_d(S(\mathbf{v}_0, \ldots, \mathbf{v}_d)) = \frac{1}{d!} |\det(\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_d - \mathbf{v}_0)|,$$

where the subscript d (which will usually be omitted) indicates that the volume is d-dimensional. For each vertex $\mathbf{v}_i$, we define the opposite face f_i as the convex closure of the remaining d vertices, and let $\mathbf{e}_i$ be a unit normal to f_i in the direction of $\mathbf{v}_i$. Let h_i be the height of $\mathbf{v}_i$ above f_i, and let $\mathbf{u}_0^i, \ldots, \mathbf{u}_{d-1}^i$ be the vertices defining f_i. f_i defines a $d-1$ dimensional space, and by projecting $\mathbf{u}_i$ onto an orthogonal basis for this space, we obtain a $(d-1)$-dimensional simplex whose $(d-1)$-dimensional volume we define as the $d-1$ dimensional hyperarea of f_i, denoted by $\mathcal{A}_i$, $\mathcal{A}_i = volume_{d-1}(\mathbf{u}_0^i, \ldots, \mathbf{u}_{d-1}^i)$. In terms of $\mathcal{A}_i$, we have $volume(S) = \frac{1}{d} \cdot h_i \cdot \mathcal{A}_i$.

Let S be a minimum enclosing simplex (MES) for the convex polygon P with $volume(P) = 1$. We can assume that every edge of S must intersect P (if not we can shrink S). Our proof on the volume bound of S will be to construct an enclosing simplex S' with small volume. Our construction will use a *maximal inscribed simplex*.

Definition 6.1 (Maximal Inscribed Simplex (MIS)): A simplex $S_0(\mathbf{v}_0, \ldots, \mathbf{v}_d)$ inscribed in P is *maximal* if for every $\mathbf{v}_i$, and some sufficiently small ball $B_\epsilon(\mathbf{v}_i)$ centered at $\mathbf{v}_i$, $S_0(\mathbf{v}_0, \ldots, v_d)$ has maximum volume among all other simplices whose vertex $\mathbf{v}_i$ is replaced by any other $\mathbf{v}_i \in B_\epsilon(\mathbf{v}_i) \cap P$.

From now on, $S_0(\mathbf{v}_0, \ldots, \mathbf{v}_d)$ will denote an MIS for P. We now present a useful property of an MIS, which allows us to construct enclosing simplices from it.

Lemma 6.2: Let f_i' be the hyperplane parallel to f_i and passing through $\mathbf{v}_i$ for the MIS S_0. Let q_i^+ denote the closed halfspace bounded by f_i' which contains $\mathbf{v}_i$. Then q_i^+ contains P.

Proof: Suppose that q_i^+ does not contain P, so some point $\mathbf{z} \in P$ resides in the complementary open halfspace to q_i^+. So

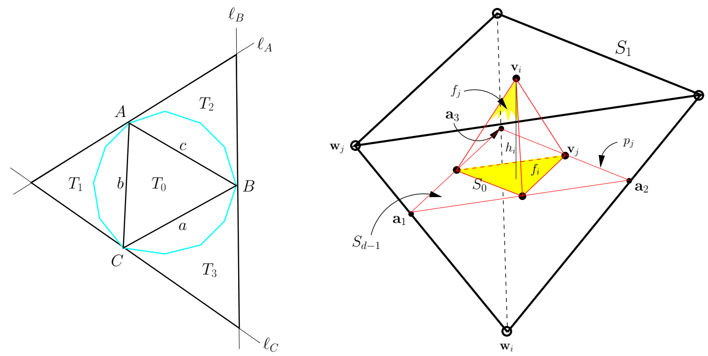


Fig. 1. The reflected homothetic simplex corresponding to S_0 for $d = 2, 3$.

$\mathbf{z} \cdot \mathbf{e}_i > \mathbf{v}_i \cdot \mathbf{e}_i$. For any $\lambda \in (0, 1]$, let $\mathbf{z}(\lambda) = \mathbf{v}_i + \lambda(\mathbf{z} - \mathbf{v}_i)$. Then $\mathbf{z}(\lambda) \cdot \mathbf{e}_i > \mathbf{v}_i \cdot \mathbf{e}_i$, i.e. the simplex S_0' in which $\mathbf{v}_i$ is replaced by $\mathbf{z}(\lambda)$ has larger height above f_i, and hence larger volume (because $\mathcal{A}_i$, the hyperarea of f_i, is not changed). Since P is convex, and $\mathbf{v}_i, \mathbf{z} \in P$, $\mathbf{z}(\lambda) \in P$, and hence the simplex $S_0' \subseteq P$ for all $\lambda \in (0, 1]$. Every ball of radius ϵ about $\mathbf{v}_i$ contains $\mathbf{z}(\lambda)$ for $\lambda \leq \epsilon$ and hence S_0 cannot have maximum volume among all choices of $\mathbf{v}_i$ in this ball, contradicting the maximality of S_0. ■

By Lemma 6.2, the simplex $S_1 = \cap_i q_i^+$ contains P, and hence we can construct an enclosing simplex from *any* MIS. We refer to S_1 as the *reflected homothetic simplex* corresponding to S_0 – since all the faces of S_1 are parallel to faces of S_0, S_1 is a homothet of S_0. We illustrate the reflected homothetic simplex for the 2 and 3 dimensional cases in Figure 1. The next lemma bounds $volume(S_1)$ in terms of $volume(S_0)$.

Lemma 6.3: $volume(S_1) = d^d \cdot volume(S_0)$.

Proof: We refer to the notation in Figure 1. Since S_1 and S_0 are homothets, the lemma amounts to the length scale factor being d. For $d = 2$, it is clear that f_i partition S_1 into 4 congruent triangles, and so the length scale factor is 2.

We proceed by induction on d, so suppose that the claim holds in $d - 1$ dimensions for $d \geq 3$ (i.e., the length scale factor is $d - 1$), and consider d dimensions. Consider any vertex $\mathbf{v}_i$ of S_0 and its opposite face f_i; the face f_i' is parallel to f_i and passes through $\mathbf{v}_i$. Now consider any other vertex $\mathbf{v}_j$, and its corresponding hyperplane f_j' parallel to its opposite face f_j and passing through $\mathbf{v}_j$. This hyperplane f_j' intersects the hyperplane containing f_i at the $d-2$ dimensional hyperplane denoted by p_j in Figure 1. In 3 dimensions, p_j is a line as illustrated in Figure 1. We will consider the $d-2$-dimensional surfaces $\{p_j\}$ for all $j \neq i$.

Vertex $\mathbf{v}_j$ is a vertex of the $d - 1$ dimensional simplex f_i. Since f_j and f_j' are parallel, so are their intersections with the hyperplane f_i. Thus, for the $(d-1)$-simplex f_i, p_j is the $(d-2)$-dimensional hyperplane parallel to the $(d-2)$-dimensional opposite face of the vertex $\mathbf{v}_j$ *in the simplex* f_i. Let h_j^+ be the

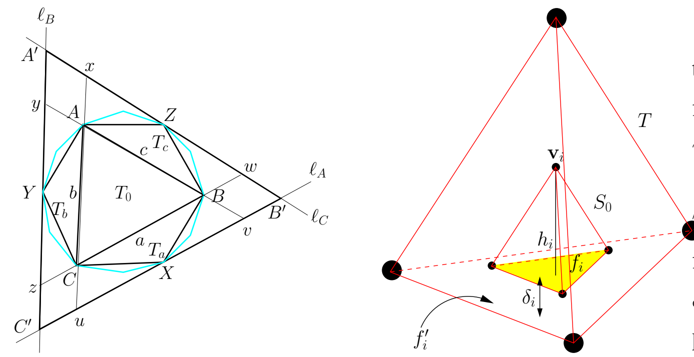


Fig. 2. Expansion of S_0 to the enclosing simplex T in $d = 2, 3$.

$(d-1)$-dimensional halfspace bounded by p_j which contains f_i. Then, the $(d-1)$-simplex $S_{d-1} = \cap_j h_i^+$ contains f_i in exactly the same way that S_1 contains S_0, i.e. S_{d-1} is the enclosing reflected homothetic $d-1$-simplex for the $(d-1)$-simplex f_i, to which we can apply the induction hypothesis. Thus, the length scale factor from f_i to S_{d-1} is $(d-1)$.

Now consider the simplex S' defined by the vertices of S_{d-1} and $\mathbf{w}_i$, the vertex of S_1 opposite f_i'. S' is clearly a homothet of S_1, and hence is also a homothet of S_0. The base of S' is S_{d-1} and the base of S_0 is f_i, and these two bases are related by the length scale factor $(d-1)$, which must therefore also be the length scale factor for the heights. Thus, $height(S') = (d-1) \cdot h_i$. Since $height(S_1) = h_i + height(S')$, we conclude that $height(S_1) = d \cdot h_i$, i.e. the length scale factor relating S_0 to S_1 is d. ■

Continuing with the proof of Lemma 3, if $volume(S_0) \le \frac{1}{d}$, then S_1 which encloses P has a volume at most d^{d-1}. We now consider the case $volume(S_0) > \frac{1}{d}$. In this case we use a different construction to obtain an enclosing simplex. This second construction does not require that the simplex S_0 be an MIS.

Let S_0 be any simplex enclosed in P (eg. an MIS), with the faces $f_0, \ldots, f_d$ and normals $\mathbf{e}_0, \ldots, \mathbf{e}_d$, where, for each face f_i, $\mathbf{e}_i$ is directed from f_i towards its corresponding vertex $\mathbf{v}_i$. Let $\mathbf{p}_i \in P$ be a maximizer of $-\mathbf{p}_i \cdot \mathbf{e}_i$, i.e. a point of maximum height in P which is below f_i. Let $\delta_i = -\mathbf{p}_i \cdot \mathbf{e}_i$ be the height of $\mathbf{p}_i$ below f_i, and consider the hyperplane q_i parallel to f_i containing $\mathbf{p}_i$. Let q_i^+ be the halfspace bounded by q_i which contains $\mathbf{v}_i$. Analogous to the proof of Lemma 6.2, since $\mathbf{p}_i$ has maximum height below f_i, it follows that q_i^+ must contain P. Therefore, we have

Lemma 6.4: Let $T = \cap_i q_i^+$. Then $P \subseteq T$.

Lemma 6.4 gives another construction of an enclosing simplex. Further, T is a homothet of S_0 (all its faces q_i are parallel to f_i, pushed out by a distance δ_i). We refer to T as the *expanded homothetic simplex* corresponding to S_0 and P. The next lemma bounds the volume of T. The situation is illustrated in Figure 2 for $d = 2, 3$. setting.

Lemma 6.5: Let S_0 be an arbitrary simplex, and let T be the homothetic simplex obtained from S_0 by translating each face out by a height δ_i. Then, $volume(T) \le \left(1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i}\right)^d \cdot volume(S_0)$.

Proof: It suffices to prove that the length scale factor relating T to S_0 is $1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i}$. To see this we view the transformation from S_0 to T as a sequence of enlargements, the first is centered at $\mathbf{v}_0$ with scale factor $\lambda_0 = (h_0 + \delta_0)/h_0$, which corresponds to pushing out the face f_0 to the plane containing q_0 by a distance δ_0. In this enlargement, all other faces get enlarged, but remain on the same plane. The next enlargement is centered at the new position of $\mathbf{v}_1$ and has scale factor λ_1 such that the new enlarged face f_1 is pushed out to the plane containing q_1 by an amount δ_1. Since h_1 increased to $\lambda_0 \cdot h_1$, we conclude that $\lambda_1 = (\delta_1 + \lambda_0 \cdot h_1)/\lambda_0 \cdot h_1$. We continue with an enlargement centered at the new position of $\mathbf{v}_2$ with scale factor $\lambda_2 = (\delta_2 + \lambda_0\lambda_1 \cdot h_2)/\lambda_0\lambda_1 \cdot h_1$; and so on, we have enlargements succesively at the the new positions of $\mathbf{v}_3, \ldots, \mathbf{v}_d$ until we finally obtain T. Suppose that the scale factor for the first k enlargements is $\lambda_0, \ldots, \lambda_{k-1}$. Then the scale factor for the $(k+1)$th enlargement is $\lambda_k = \frac{\delta_k + h_k \cdot \prod_{i=0}^{k-1} \lambda_i}{h_k \cdot \prod_{i=0}^{k-1} \lambda_i} = 1 + \frac{\delta_k}{h_k \cdot \prod_{i=0}^{k-1} \lambda_i}$. The scale factor for the transformation from S_0 to T is given by $\prod_{k=0}^{d} \lambda_k$. We evaluate this product as follows: $\prod_{k=0}^{d} \lambda_k = \lambda_d \cdot \prod_{k=0}^{d-1} \lambda_k, = \left(1 + \frac{\delta_d}{h_d \cdot \prod_{i=0}^{d-1} \lambda_i}\right) \cdot \prod_{k=0}^{d-1} \lambda_k, = \prod_{k=0}^{d-1} \lambda_k + \frac{\delta_d}{h_d}$. It follows by induction that $\prod_{k=0}^{d} \lambda_k = 1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i}$, concluding the proof. ■

The next lemma bounds the sum $1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i}$ which appears in the lemma above.

Lemma 6.6: $volume(S_0) \cdot \left(1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i}\right) \le 1$.

Proof: Define the simplices $T_0, \ldots, T_d$ as follows. T_i is the convex closure of $\mathbf{p}_i$ and f_i – T_i is a simplex with base f_i and height δ_i. The body $Q = S_0 \cup T_0 \cup \cdots \cup T_d$ is enclosed in P, hence $volume(Q) \le volume(P) = 1$. The simplices T_i and T_j are disjoint except on a set of measure zero. This follows from the fact that the height of $\mathbf{p}_i$ above f_i is at least as large as the height of $\mathbf{p}_j$ above f_i (and vice-versa) and Lemma 6.9 which is a technical result which we will prove later. Hence, $volume(T_i \cap T_j) = 0$. Similarly T_i and S_0 intersect at f_i which has measure zero, hence $volume(Q) = volume(S_0) + \sum_{i=0}^{d} volume(T_i) \le 1$. To conclude, note that by (VI), $volume(T_i) = \frac{1}{d} \cdot \mathcal{A}_i \cdot \delta_i = \frac{\delta_i}{h_i} \cdot volume(S_0)$ ■

An immediate corollary of Lemmas 6.5 and 6.6 is

Corollary 6.7: $volume(T) \le \left(1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i}\right)^{d-1}$.

To complete the proof of Lemma 3, suppose that $volume(S_0) > \frac{1}{d}$; then, by Lemma 6.6 $1 + \sum_{i=0}^{d} \frac{\delta_i}{h_i} < d$, and by Corollary 6.7, we have that $volume(T) < d^{d-1}$. We recap all these results in the following theorem.

Theorem 6.8: Let P be a bounded convex polytope. Then the following algorithm constructs an enclosing simplex S satisfying

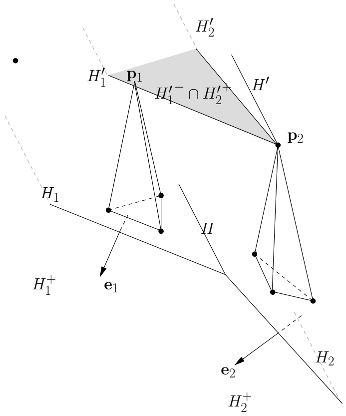


Fig. 3. Disjointness of simplices subtended by non-parallel faces.

$volume(S) \le d^{d-1} \cdot volume(P)$.

Construct S_0, a locally maximal inscribed simplex for P.
if $volume(S_0) \le \frac{1}{d}$ **then**
  Let S be the reflected homothetic simplex corresponding to S_0.
else
  Let S be the expanded homothetic simplex corresponding to S_0 and P.

A Technical Lemma on the Disjointness of Maximum Height Simplices.: We now present the technical lemma which is used in the proof of Lemma 6.6. This lemma shows that the T_i are disjoint. To be specific, let H_1 and H_2 be two non-parallel $(d-1)$-dimensional hyperplanes with unit normals $\mathbf{e}_1, \mathbf{e}_2$. Let H_1 and H_2 intersect at the $d-2$ dimensional hyperplane H. Let H_1^+ and H_2^+ be two halfspaces defined by H_1 and H_2. Define two regions $\mathcal{R} = H_1^+ \cap H_2^+$, and its complement $\overline{\mathcal{R}} = \overline{\mathcal{H}_1^+} \cup \overline{H_2^+}$. Assume that $\mathbf{e}_1, \mathbf{e}_2$ are in the direction of H_1^+, H_2^+ respectively. Let F_1 and F_2 be sets of points in $\mathcal{R}$ which reside on H_1 and H_2 respectively. For a point $\mathbf{p} \in \overline{\mathcal{R}}$, we define its heights above H_1 and H_2 respectively as $h_1(\mathbf{p}) = -\mathbf{p} \cdot \mathbf{e}_1$ and $h_2(\mathbf{p}) = -\mathbf{p} \cdot \mathbf{e}_2$. Let $\mathbf{p}_1, \mathbf{p}_2 \in \overline{\mathcal{R}}$ be two points such that $\mathbf{p}_1$ is higher than $\mathbf{p}_2$ with respect to H_1 and the reverse is true with respect to H_2, i.e. $h_1(\mathbf{p}_1) \ge h_1(\mathbf{p}_2)$ and $h_2(\mathbf{p}_1) \le h_2(\mathbf{p}_2)$. Let T_1 be the convex closure of $F_1 \cup \mathbf{p}_1$ and T_2 the convex closure of $F_2 \cup p_2$. Then T_1 and T_2 are disjoint (up to a set of measure zero). The situation is illustrated in Figure 3.

Lemma 6.9: $vol(T_1 \cap T_2) = 0$.

Proof: Define H_1^- as $H_1 \cup \overline{H_1^+}$ and similarily H_2^-. Consider point $\mathbf{p}_2$ and let H_1' be the hyperplane containing $\mathbf{p}_2$ which is parallel to H_1, and similarily define H_2'. Let $H' = H_1' \cap H_2'$, which is parallel to H. Also define ${H_1'}^+, {H_1'}^-, {H_2'}^+, {H_2'}^-$ in a similar way. Since $h_1(\mathbf{p}_1) \ge h_1(\mathbf{p}_2)$ and $h_2(\mathbf{p}_1) \le h_2(\mathbf{p}_2)$, $\mathbf{p}_1$ must be lie in ${H_1'}^- \cap {H_2'}^+$ as illustrated by the shaded region in Figure 3. Now consider the hyperplane G which contains H and H', i.e. G intersects H_1 and H_2 at H and G intersects H_1' and H_2' at H'. F_1 and F_2 lie on opposite sides of G, as do ${H_1'}^- \cap {H_2'}^+$ and $\mathbf{p}_2$. Note that G separates $H_1^- \cap H_2^+$ contains F_1 and since ${H_1'}^- \cap {H_2'}^+$ is a translate of $H_1^- \cap H_2^+$ along a line joining H to H', it follows that F_1 and ${H_1'}^- \cap {H_2'}^+$ are on the same side of G. Since $\mathbf{p}_1 \in {H_1'}^- \cap {H_2'}^+$, it follows that F_1 and $\mathbf{p}_1$ are on the same side of G, and so G separates $F_1 \cup \mathbf{p}_1$ from $F_2 \cup \mathbf{p}_2$. Since G separates $F_1 \cup \mathbf{p}_1$ from $F_2 \cup \mathbf{p}_2$, it also separates their convex closures. Thus, the intersection of their convex closures is a subset of G, and since $vol(G) = 0$, this intersection must also have zero volume. ■

A. Algorithm Analysis

We briefly discuss the running time of the algorithm summarized in Theorem 6.8. The first step to compute a single maximal inscribed simplex is a local optimization problem of a differentiable objective over a convex set. Since it is a local search problem, it can be solved efficiently, and we discuss some approaches to this in Section VI-B. Computing $volume(S_0)$ involves computing a d-dimensional determinant which is $O(d^3)$. We will shortly show that all the other tasks that need to be solved can be reduced to solving $O(d)$ d-dimensional linear programs with $n = |\mathcal{H}|$ inequality constraints. Solving one such program takes $O(d^2 n)$ operations, hence the entire running time is given by $M(n,d) + O(d^3 n)$, where M is the complexity of finding the maximal inscribed simplex.

We now walk through the tasks in the algorithm.

If $volume(S_0) \le \frac{1}{d}$, we construct the reflected homothetic simplex for S_0. This can be accomplished because: we can compute $\mathbf{e}_i$ by projecting $\mathbf{v}_i - \mathbf{v}_j$ to the space orthogonal to f_i in $O(d^3)$; $(\mathbf{v}_i, \mathbf{e}_i)$ then defines f_i', which in turn gives the reflected homothet. However, it is not the reflected homothet which we desire, but its intersection point with P. This task can be solved by simply augmenting $\mathcal{H}$ with an additional equality constraint $(\mathbf{x} - \mathbf{v}_i) \cdot \mathbf{e}_i = 0$ and finding a feasible point which is a linear program. Thus, we have $(d+1)$ linear programs, each with n constraints.

If $volume(S_0) > \frac{1}{d}$, we do not actually need the expanded homothet. We only need its points of intersection with P, which are exactly the points $\mathbf{p}_i$ described in the previous section. The $\mathbf{p}_i$ are exactly the solutions to the $(d+1)$ linear programs $\min_{\mathbf{x}} \mathbf{x} \cdot \mathbf{e}_i$ such that $\mathbf{x} \in \mathcal{H}$, again $(d+1)$ linear programs with n constraints.

Once the points of intersection $\mathbf{p}_i$ have been constructed, it only remains to recover the constraints in $\mathcal{H}$ which are active. For each $\mathbf{p}_i$, this is an $O(dn)$ task, for a total time $O(d^2 n)$. One final note is that more than d active constraints may be recovered for each point of intersection. In this case, any subset of the active constraints of size d whose interesection is contained in the corresponding halfspace q_i^+ suffices. At least one such subset exists.

B. Constructing a Maximal Inscribed Simplex

The first step in our construction is to obtain a locally maximum inscribed simplex. This is a standard, differentiable local optimization problem

$$\max_V \det V, \qquad \text{such that } V \in \mathcal{H},$$

where $V = [\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_d - \mathbf{v}_0]$ and $V \in \mathcal{H}$ *iff* $\mathbf{v}_i \in \mathcal{H}$ for all $i = 0, \ldots, d$. The domain of V is convex, as is easily verified, and the determinant is differentiable, hence ellipsoid algorithms can be used to obtain a local minimum. From a practical perspective, it is better to maximize $\log\det V^T V$. An added benefit of choosing $\log\det V^T V$ is that $\log\det$ is concave on S^d_{++} (positive definite symmetric matrices), hence maximizing it on any convex subset of S^d_{++} is a convex optimization problem.

VII. Conclusion

In the bounded uncertainty model, using measurements from all sensors gives the optimal uncertainty for localizing a target. In this paper, we showed that, one can always select a *constant* number of sensors and guarantee a localization uncertainty close to optimal (bounded by a constant times optimal). In particular, we showed that 4 sensors suffice for a 2-approximation in 2-dimensions and 8 sensors suffice for a 9-approximation in 3-dimensions. Both of these sensors sets can be computed efficiently. We also showed how these results can be generalized to arbitrary dimensions and that a constant factor approximation can be obtained by a constant number of sensors. Both constants depend on the dimensionality but are independent from the total number of available sensors.

An important issue which remains unaddressed is robustness. In this paper, we assumed that the locations of all sensors are known. Sensor selection in the presence of uncertainties regarding sensor locations is an important future research direction.

VIII. Acknowledgments

The proof of Lemma 1(ii) arose out of discussions with Ling Li of the Caltech Learning Systems Group. We also like to thank him for other some pointers to useful references.

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