

Navier-Stokes & Flow Simulation

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Last Time?

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Today

- Flow Simulations in Computer Graphics
 - water, smoke, viscous fluids
- Navier-Stokes Equations
- Fluid Representations
- Basic Algorithm
- Data Representation

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Flow Simulations in Graphics

- Random velocity fields
 - with averaging to get simple background motion
- Shallow water equations
 - height field only, can't represent crashing waves, etc.
- Full Navier-Stokes
- *note: typically we ignore surface tension and focus on macroscopic behavior*

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Today

- Flow Simulations in Computer Graphics
- Navier-Stokes Equations
 - incompressibility, conservation of mass
 - conservation of momentum & energy
- Fluid Representations
- Basic Algorithm
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Flow in a Voxel Grid

- conservation of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

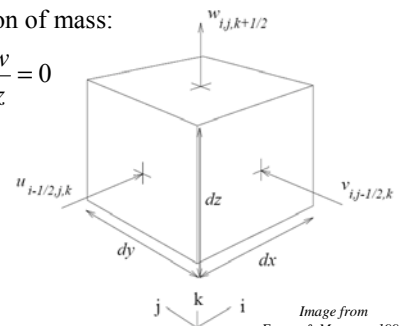


Image from
Foster & Maxax, 1996

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Navier-Stokes Equations

- conservation of momentum:

$$\begin{aligned}
 \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} &= -\frac{\partial p}{\partial x} + g_x + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
 \frac{\partial v}{\partial t} + \frac{\partial vu}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} &= -\frac{\partial p}{\partial y} + g_y + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
 \frac{\partial w}{\partial t} + \frac{\partial wu}{\partial x} + \frac{\partial wv}{\partial y} + \frac{\partial w^2}{\partial z} &= -\frac{\partial p}{\partial z} + g_z + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
 \end{aligned}$$

acceleration convection drag

pressure gravity (& other external forces) viscosity

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Modeling the Water Surface

- Volume-of-fluid tracking
 - a scalar saying how “full” each cell is
- Particle In Cell (PIC)
 - the particles have mass
- Marker and Cell (MAC)
 - the particles don’t effect computation, just identify which cells the surface passes through
 - Harlow & Welch, “Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface”, *The Physics of Fluids*, 1965.

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PIC method (2-dimensional): At time t_n the state of the fluid is represented by N particles with the property vectors $\mathbf{u}_k^n \equiv \{r_k^n, v_k^n, e_k^n\}$ ($k = 1, \dots, N$). In each Eulerian cell of side length Δl there should be at least ≈ 100 particles.

1. Compute, for each Euler cell (i, j) , the cell properties

$$\begin{aligned}
 \rho_{i,j}^n &= \frac{m}{(\Delta l)^2} \sum_{k=1}^N \delta[r_k^n(i, j)] \\
 (\rho v)_{i,j}^n &= \frac{m}{(\Delta l)^2} \sum_{k=1}^N v_k^n \delta[r_k^n(i, j)] \\
 (\rho e)_{i,j}^n &= \frac{m}{(\Delta l)^2} \sum_{k=1}^N e_k^n \delta[r_k^n(i, j)]
 \end{aligned}$$

2. Using the equation of state to evaluate cell pressures $p_{i,j}$, compute new (preliminary) flow velocities according to

$$\begin{aligned}
 v_{i,j}^{n+1} &= v_{i,j}^n - a (p_{i+1,j}^n - p_{i-1,j}^n) \\
 v_{i,j}^{n+1} &= v_{i,j}^n - a (p_{i,j+1}^n - p_{i,j-1}^n)
 \end{aligned}$$

with $a \equiv \Delta t / 2(\Delta l)\rho_{i,j}^n$. For each fluid particle k we now have $\mathbf{v}_k^{n+1} = \mathbf{v}_k^n + \mathbf{e}_k^{n+1} = \mathbf{v}_k^n / (\rho_k^n)$.

3. From the time-centered cell velocities $\mathbf{v}_{i,j}^{n+1/2} \equiv [\mathbf{v}_{i,j}^{n+1} + \mathbf{v}_{i,j}^n] / 2$, compute for each particle k an intermediate velocity

$$\mathbf{v}_k^{n+1/2} = \frac{1}{(\Delta l)^2} \sum_{(i,j)} a_{(i,j)} \mathbf{v}_{(i,j)}^{n+1/2}$$

using suitable weights $a_{(i,j)}$ (see text); calculate new particle positions

$$\mathbf{r}_k^{n+1} = \mathbf{r}_k^n + \Delta t \mathbf{v}_k^{n+1/2}$$

4. Each particle is now given the new state vector

$$\mathbf{u}_k^{n+1} \equiv \{r_k^{n+1}, v_k^{n+1}, e_k^{n+1}\}$$

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Comparing Representations

- How do we render the resulting surface?
- Are we guaranteed not to lose mass/volume? (is the simulation incompressible?)
- How is each affected by the grid resolution and timestep?
- Can we guarantee stability?

Demos

- Nice Marker and Cell (MAC) videos at:
<http://panoramix.ift.uni.wroc.pl/~maq/eng/cfdthesis.php>

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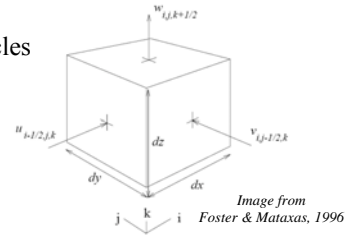
Today

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Each Grid Cell Stores:

- Velocity at the cell faces (offset grid)
- Pressure
- List of particles



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Initialization

- Choose a voxel resolution
- Choose a particle density
- Place the particles
- Initialize pressure & velocity of each cell
- Set the viscosity & gravity
- Choose a timestep & go!

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At each Timestep:

- Identify which cells are Empty, Full, or on the Surface
- Compute new velocities
- Adjust the velocities to maintain an incompressible flow
- Move the particles
 - Interpolate the velocities at the faces
- Render the geometry and repeat!

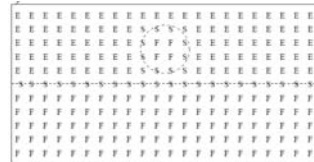
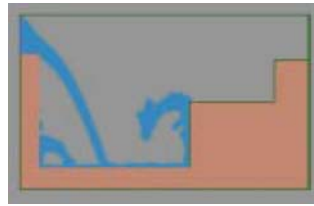
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Empty, Surface & Full Cells



Images from Foster & Matas, 1996

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Compute New Velocities

$$\begin{aligned} \tilde{u}_{i+1/2,j,k} = & u_{i+1/2,j,k} + \delta t \{ (1/\delta x) [(u_{i,j,k})^2 - (u_{i+1,j,k})^2] \\ & + (1/\delta y) [(uv)_{i+1/2,j-1/2,k} - (uv)_{i+1/2,j+1/2,k}] \\ & + (1/\delta z) [(uw)_{i+1/2,j,k-1/2} - (uw)_{i+1/2,j,k+1/2}] + g_x \\ & + (1/\delta x) (p_{i,j,k} - p_{i+1,j,k}) + (\nu/\delta x^2) (u_{i+3/2,j,k} \\ & - 2u_{i+1/2,j,k} + u_{i-1/2,j,k}) + (\nu/\delta y^2) (u_{i+1/2,j+1,k} \\ & - 2u_{i+1/2,j,k} + u_{i+1/2,j-1,k}) + (\nu/\delta z^2) (u_{i+1/2,j,k+1} \\ & - 2u_{i+1/2,j,k} + u_{i+1/2,j,k-1}) \}, \end{aligned}$$

Note: some of these values are the average velocity within the cell rather than the velocity at a cell face

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Adjusting the Velocities

- Calculate the divergence of the cell (the extra in/out flow)
- The divergence is used to update the pressure within the cell
- Adjust each face velocity uniformly to bring the divergence to zero
- Iterate across the entire grid until divergence is $< \epsilon$

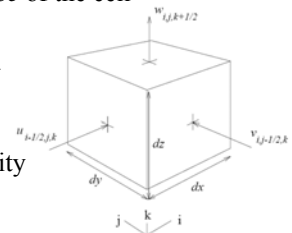
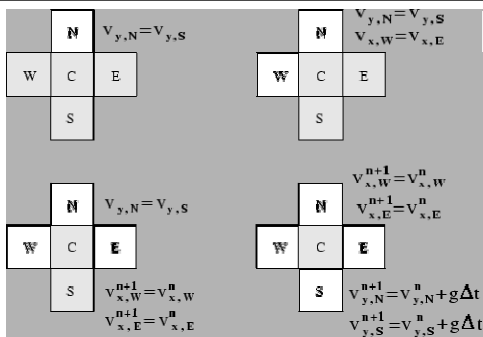


Image from Foster & Matas, 1996

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Handling Boundaries with MAC



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At each Timestep:

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- **Move the particles**
 - Interpolate the velocities at the faces
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Velocity Interpolation

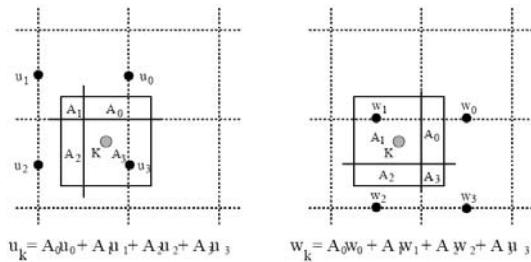


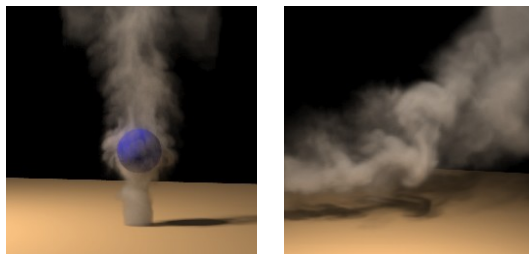
Image from Foster & Metaxas, 1996

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Efficient Smoke Simulation



"Visual Simulation of Smoke"
Fedkiw, Stam & Jensen
SIGGRAPH 2001

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Solid/Liquid: Time-varying viscosity



Click to Play Movie

"Melting and Flowing"
Carlson, Mucha, Van Horn III & Turk
Symposium on Computer Animation 2002



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