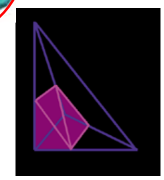
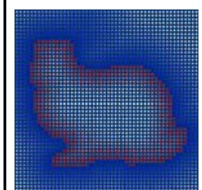
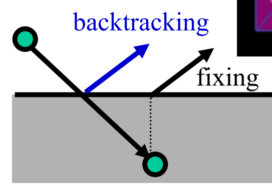
Navier-Stokes
&
Flow Simulation

Last Time?

- Implicit Surfaces
- Marching Cubes/Tetras
- Collision Detection & Response
- Conservative Bounding Regions







Today

- Flow Simulations in Computer Graphics
  - water, smoke, viscous fluids
- Navier-Stokes Equations
  - incompressibility, conservation of mass
  - conservation of momentum & energy
- Fluid Representations
- Basic Algorithm
- Data Representation

Flow Simulations in Graphics

- Random velocity fields
  - with averaging to get simple background motion
- Shallow water equations
  - height field only, can't represent crashing waves, etc.
- Full Navier-Stokes

- *note: typically we ignore surface tension and focus on macroscopic behavior*

Flow in a Voxel Grid

- conservation of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

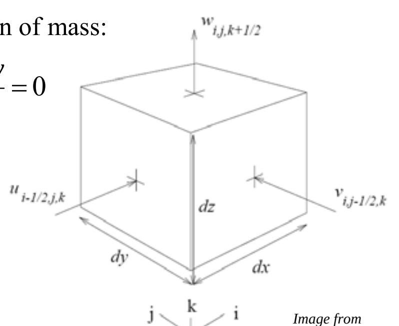


Image from Foster & Mataxas, 1996

Navier-Stokes Equations

- conservation of momentum:

gravity (& other external forces)
pressure
viscosity

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = -\frac{\partial p}{\partial x} + g_x + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial vu}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} = -\frac{\partial p}{\partial y} + g_y + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + \frac{\partial wu}{\partial x} + \frac{\partial wv}{\partial y} + \frac{\partial w^2}{\partial z} = -\frac{\partial p}{\partial z} + g_z + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

acceleration
Convection: internal movement in a fluid (e.g., caused by variation in density due to a transfer of heat)
drag

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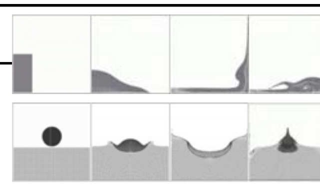
Modeling the Water Surface

- Volume-of-fluid tracking
  - a scalar saying how "full" each cell is
- Particle In Cell (PIC)
  - the particles have mass
- Marker and Cell (MAC)
  - the particles don't effect computation, just identify which cells the surface passes through
  - Harlow & Welch, "Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface", *The Physics of Fluids*, 1965.

Comparing Representations

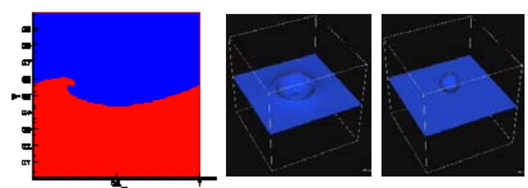
- How do we render the resulting surface?
- Are we guaranteed not to lose mass/volume? (is the simulation incompressible?)
- How is each affected by the grid resolution and timestep?
- Can we guarantee stability?

Demos



- Nice Marker and Cell (MAC) videos at:

http://panoramix.ift.uni.wroc.pl/~maq/eng/cfdthesis.php



http://mme.uwaterloo.ca/~fslien/free_surface/free_surface.htm

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Each Grid Cell Stores:

- Velocity at the cell faces (offset grid)
- Pressure
- List of particles

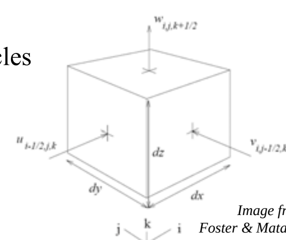


Image from Foster & Mataxas, 1996

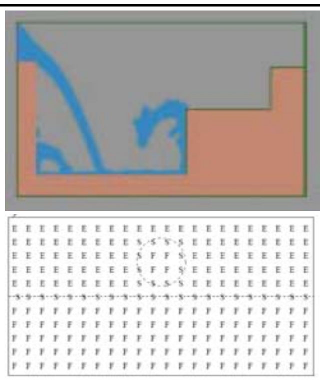
Initialization

- Choose a voxel resolution
- Choose a particle density
- Create grid & place the particles
- Initialize pressure & velocity of each cell
- Set the viscosity & gravity
- Choose a timestep & go!

At each Timestep:

- Identify which cells are Empty, Full, or on the Surface
- Compute new velocities
- Adjust the velocities to maintain an incompressible flow
- Move the particles
  - Interpolate the velocities at the faces
- Render the geometry and repeat!

Empty, Surface & Full Cells



Images from Foster & Mataxas, 1996

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Compute New Velocities

$$
\begin{aligned}
\tilde{u}_{i+1/2,j,k} = u_{i+1/2,j,k} + \delta t\{&(1/\delta x)[(u_{i,j,k})^2 - (u_{i+1,j,k})^2] \\
&+(1/\delta y)[(uv)_{i+1/2,j-1/2,k} - (uv)_{i+1/2,j+1/2,k}] \\
&+(1/\delta z)[(uw)_{i+1/2,j,k-1/2} - (uw)_{i+1/2,j,k+1/2}] + g_x \\
&+(1/\delta x)(p_{i,j,k} - p_{i+1,j,k}) + (\nu/\delta x^2)(u_{i+3/2,j,k} \\
&-2u_{i+1/2,j,k} + u_{i-1/2,j,k}) + (\nu/\delta y^2)(u_{i+1/2,j+1,k} \\
&-2u_{i+1/2,j,k} + u_{i+1/2,j-1,k}) + (\nu/\delta z^2)(u_{i+1/2,j,k+1} \\
&-2u_{i+1/2,j,k} + u_{i+1/2,j,k-1})\},
\end{aligned}
$$

Note: some of these values are the *average velocity* within the cell rather than the velocity at a cell face

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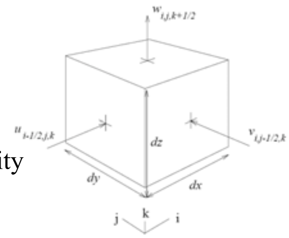
Adjusting the Velocities

- Calculate the *divergence* of the cell (the extra in/out flow)
- The divergence is used to update the *pressure* within the cell
- Adjust each face velocity uniformly to bring the divergence to zero
- Iterate across the entire grid until divergence is < ε



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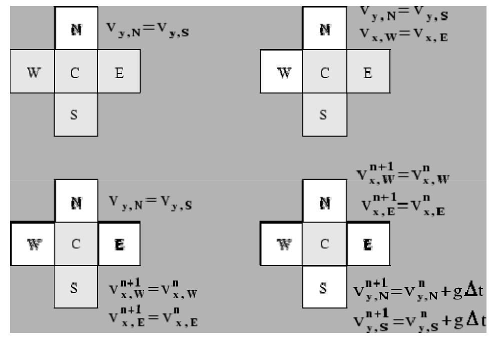
Handing Boundaries with MAC



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Velocity Interpolation

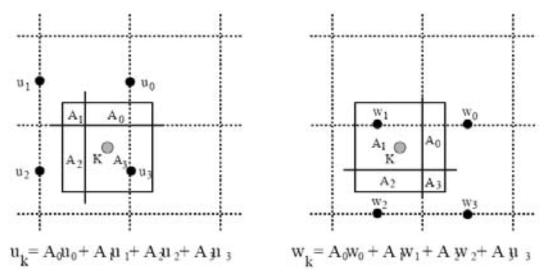
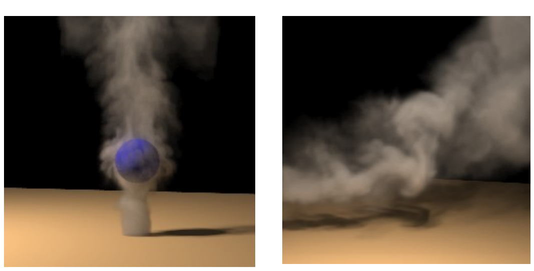


Image from Foster & Mataxas, 1996

Efficient Smoke Simulation



"Visual Simulation of Smoke"
Fedkiw, Stam & Jensen
SIGGRAPH 2001

Solid/Liquid: Time-varying viscosity

"Melting and Flowing"
Carlson, Mucha, Van Horn III & Turk
Symposium on Computer Animation 2002