Spline Curves

Last Time?
• Adjacency Data Structures
  – Geometric & topologic information
  – Dynamic allocation
  – Efficiency of access
• Mesh Simplification
  – edge collapse/vertex split
  – geomorphs
  – progressive transmission
  – view-dependent refinement

Today
• Interpolating Color & Normals in OpenGL
• Limitations of Polygonal Models
• Some Modeling Tools & Definitions
• What’s a Spline?
• Linear Interpolation
• Interpolation Curves vs. Approximation Curves
• Bézier Spline
• BSpline (NURBS)

Color Interpolation
• Interpolate colors of the 3 vertices
• Linear interpolation, barycentric coordinates

glShadeModel (GL_SMOOTH);
• From OpenGL Reference Manual:
  – Smooth shading, the default, causes the computed
colors of vertices to be interpolated as the primitive is
rasterized, typically assigning different colors to each
resulting pixel fragment.
  – Flat shading selects the computed color of just one
vertex and assigns it to all the pixel fragments
generated by rasterizing a single primitive.
  – In either case, the computed color of a vertex is the
result of lighting if lighting is enabled, or it is the
current color at the time the vertex was specified if
lighting is disabled.

Normal Interpolation

Normal colors

```glBegin(GL_TRIANGLES);
glColor3f(1.0,0.0,0.0);
glVertex3f(…);
glColor3f(0.0,1.0,0.0);
glVertex3f(…);
glColor3f(0.0,0.0,1.0);
glVertex3f(…);
glEnd();```
Gouraud Shading

- Instead of shading with the normal of the triangle, we'll shade the vertices with the average normal and interpolate the shaded color across each face.

- How do we compute Average Normals? Is it expensive??

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Limitations of Polygonal Meshes

- Planar facets (& silhouettes)
- Fixed resolution
- Deformation is difficult
- No natural parameterization (for texture mapping)

Gouraud not always good enough

- Still low, fixed resolution (missing fine details)
- Still have polygonal silhouettes
- Intersection depth is planar (e.g. ray tracing visualization)
- Collisions problems for simulation
- Solid Texturing problems
- ...

Some Non-Polygonal Modeling Tools

- Extrusion
- Surface of Revolution
- Spline Surfaces/Patches
- Quadrics and other implicit polynomials

Continuity definitions:

- \( C^0 \) continuous
  - curve/surface has no breaks/gaps/holes
- \( G^1 \) continuous
  - tangent at joint has same direction
- \( C^1 \) continuous
  - curve/surface derivative is continuous
  - tangent at joint has same direction and magnitude
- \( C^n \) continuous
  - curve/surface derivative through \( n \)th derivative is continuous
  - important for shading

"Shape Optimization Using Reflection Lines", Tosun et al., 2007
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Questions?

Definition: What's a Spline?

- Smooth curve defined by some control points
- Moving the control points changes the curve

Interpolation Curves / Splines

- Curve is constrained to pass through all control points
- Given points $P_0, P_1, \ldots, P_n$, find lowest degree polynomial which passes through the points

$$x(t) = a_{n-1}t^{n-1} + \ldots + a_2t^2 + a_1t + a_0$$
$$y(t) = b_{n-1}t^{n-1} + \ldots + b_2t^2 + b_1t + b_0$$

Linear Interpolation

- Simplest "curve" between two points

$$Q(t) = p + t(p_2 - p_1)$$

$$Q(t) = \begin{pmatrix} Q_x(t) \\ Q_y(t) \end{pmatrix} = (p_3 - p_1) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + (p_2 - p_1) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$Q(t) = GBT(t) = \text{Geometry G} \cdot \text{Spline Basis B} \cdot \text{Power Basis T}(t)$$
Interpolation vs. Approximation Curves

- **Interpolation**
  - Curve must pass through control points

- **Approximation**
  - Curve is influenced by control points

Interpolation vs. Approximation Curves

- Interpolation Curve – over constrained → lots of (undesirable?) oscillations
- Approximation Curve – more reasonable?

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Cubic Bézier Curve

- 4 control points
- Curve passes through first & last control point
- Curve is tangent at $P_1$ to $(P_2-P_1)$ and at $P_4$ to $(P_4-P_3)

A Bézier curve is bounded by the convex hull of its control points.

Cubic Bézier Curve

- de Casteljau's algorithm for constructing Bézier curves
Cubic Bézier Curve

\[ Q(t) = (1-t)^3 P_1 + 3(1-t)^2 t P_2 + 3(1-t) t^2 P_3 + t^3 P_4 \]

Bernstein Polynomials

\[ B_0(t) = (1-t)^3; B_1(t) = 3t(1-t)^2; B_2(t) = 3(1-t)t^2; B_3(t) = t^3 \]

Connecting Cubic Bézier Curves

- Asymmetric: Curve goes through some control points but misses others
- How can we guarantee \( C^0 \) continuity?
- How can we guarantee \( G^1 \) continuity?
- How can we guarantee \( C^1 \) continuity?
- Can’t guarantee higher \( C^2 \) or higher continuity

Connecting Cubic Bézier Curves

- Where is this curve
  - \( C^0 \) continuous?
  - \( G^1 \) continuous?
  - \( C^1 \) continuous?
- What’s the relationship between:
  - the \# of control points, and
  - the \# of cubic Bézier subcurves?

Higher-Order Bézier Curves

- > 4 control points
- Bernstein Polynomials as the basis functions

\[ B_n^i(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \quad 0 \leq i \leq n \]
- Every control point affects the entire curve
  - Not simply a local effect
  - More difficult to control for modeling

Questions?

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Cubic BSplines

• ≥ 4 control points
• Locally cubic
• Curve is not constrained to pass through any control points

Cubic BSplines

• Iterative method for constructing BSplines

A BSpline curve is also bounded by the convex hull of its control points.

Connecting Cubic BSpline Curves

• Can be chained together
• Better control locally (windowing)

Connecting Cubic BSpline Curves

• What’s the relationship between
  – the # of control points, and
  – the # of cubic BSpline subcurves?

BSpline Curve Control Points

Default BSpline

BSpline with Discontinuity

BSpline which passes through end points

Repeat interior control point

Repeat end points
Bézier is not the same as BSpline

- Relationship to the control points is different

Bézier

BSpline

Converting between Bézier & BSpline

- Using the basis functions:

\[
B_{\text{Bézier}} = \begin{pmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 3 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
B_{\text{BSpline}} = \frac{1}{6} \begin{pmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 1 \\
-3 & 3 & 0 & 4 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

\[Q(t) = \text{GBT}(t) = \text{Geometry} \cdot \text{Spline Basis} \cdot \text{Power Basis} \cdot T(t)\]

NURBS (generalized BSplines)

- BSpline: uniform cubic BSpline

- NURBS: Non-Uniform Rational BSpline
  - non-uniform = different spacing between the blending functions, a.k.a. knots
  - rational = ratio of polynomials (instead of cubic)

Neat Bezier Spline Trick

- A Bezier curve with 4 control points:
  - \( P_0 \), \( P_1 \), \( P_2 \), \( P_3 \)

- Can be split into 2 new Bezier curves:
  - \( P_0 \), \( P'_1 \), \( P'_2 \), \( P'_3 \)
  - \( P'_3 \), \( P'_4 \), \( P'_5 \), \( P_3 \)

A Bézier curve is bounded by the convex hull of its control points.
Questions?

Reading for Today

• "Teddy: A Sketching Interface for 3D Freefrom Design", Igarashi et al., SIGGRAPH 1999

Reading for Friday (1/25)

• Hoppe et al., “Piecewise Smooth Surface Reconstruction” SIGGRAPH 1994

• Post a comment or question on the LMS discussion by 10am on Friday 1/25