Spline Curves

Last Time?
• Adjacency Data Structures
  – Geometric & topologic information
  – Dynamic allocation
  – Efficiency of access
• Mesh Simplification
  – edge collapse/vertex split
  – geomorphs
  – progressive transmission
  – view-dependent refinement

Today
• Interpolating Color & Normals in OpenGL
• Limitations of Polygonal Models
• Some Modeling Tools & Definitions
• What's a Spline?
• Linear Interpolation
• Interpolation Curves vs. Approximation Curves
• Bézier Spline
• BSpline (NURBS)

Color Interpolation
• Interpolate colors of the 3 vertices
• Linear interpolation, barycentric coordinates

Normal Interpolation

glShadeModel (GL_SMOOTH);
• From OpenGL Reference Manual:
  – Smooth shading, the default, causes the computed colors of vertices to be interpolated as the primitive is rasterized, typically assigning different colors to each resulting pixel fragment.
  – Flat shading selects the computed color of just one vertex and assigns it to all the pixel fragments generated by rasterizing a single primitive.
  – In either case, the computed color of a vertex is the result of lighting if lighting is enabled, or it is the current color at the time the vertex was specified if lighting is disabled.
Gouraud Shading

• Instead of shading with the normal of the triangle, we’ll shade the vertices with the average normal and interpolate the shaded color across each face

• How do we compute Average Normals? Is it expensive??

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Limitations of Polygonal Meshes

• Planar facets (& silhouettes)
• Fixed resolution
• Deformation is difficult
• No natural parameterization (for texture mapping)

Gouraud not always good enough

• Still low, fixed resolution (missing fine details)
• Still have polygonal silhouettes
• Intersection depth is planar (e.g. ray tracing visualization)
• Collisions problems for simulation
• Solid Texturing problems
• ...

Some Non-Polygonal Modeling Tools

Extrusion
Surface of Revolution
Spline Surfaces/Patches
Quadrics and other implicit polynomials

Continuity definitions:

• $C^0$ continuous
  – curve/surface has no breaks/gaps/holes
• $G^1$ continuous
  – tangent at joint has same direction
• $C^1$ continuous
  – curve/surface derivative is continuous
  – tangent at joint has same direction and magnitude
• $C^n$ continuous
  – curve/surface through $n$th derivative is continuous
  – important for shading

“Shape Optimization Using Reflection Lines”, Tosun et al., 2007
Questions?

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Definition: What's a Spline?

• Smooth curve defined by some control points
• Moving the control points changes the curve

Interpolation

Bézier (approximation)

BSpline (approximation)

Interpolation Curves / Splines

Interpolation Curves

• Curve is constrained to pass through all control points
• Given points $P_0, P_1, \ldots, P_n$, find lowest degree polynomial which passes through the points

$$x(t) = a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$$

$$y(t) = b_{n-1} t^{n-1} + \ldots + b_1 t + b_0$$

Linear Interpolation

• Simplest "curve" between two points

$$Q(t) = (1 - s) P_1 + s P_2$$

Spline Basis Functions

a.k.a. Blending Functions

$$Q(t) = \begin{bmatrix} Q_x(t) \\ Q_y(t) \\ Q_z(t) \end{bmatrix} = \begin{pmatrix} (P_0) & (P_1) \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$Q(t) = \mathbf{G} \mathbf{B} \mathbf{T}(t) = \text{Geometry} \ \mathbf{G} \ \cdot \ \text{Spline Basis} \ \mathbf{B} \ \cdot \ \text{Power Basis} \ \mathbf{T}(t)$$

www.abm.org
Interpolation vs. Approximation Curves

Interpolation curve must pass through control points

Approximation curve is influenced by control points

• Interpolation Curve – over constrained → lots of (undesirable?) oscillations

• Approximation Curve – more reasonable?

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Cubic Bézier Curve

• 4 control points
• Curve passes through first & last control point
• Curve is tangent at \( P_1 \) to \((P_2-P_1)\) and at \( P_4 \) to \((P_4-P_3)\)

A Bézier curve is bounded by the convex hull of its control points.
Cubic Bézier Curve

\[ Q(t) = (1-t)^3P_1 + 3(1-t)^2tP_2 + 3(1-t)t^2P_3 + t^3P_4 \]
\[ \beta_{\text{Bezier}} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]
\[ B_0(t) = (1-t)^3; B_1(t) = 3(1-t)^2t; B_2(t) = 3(1-t)t^2; B_3(t) = t^3 \]

Connecting Cubic Bézier Curves

Asymmetric: Curve goes through some control points but misses others

- How can we guarantee \( C^0 \) continuity?
- How can we guarantee \( G^1 \) continuity?
- How can we guarantee \( C^1 \) continuity?
- Can’t guarantee higher \( C^2 \) or higher continuity

Connecting Cubic Bézier Curves

- Where is this curve \( C^0 \) continuous?
- \( G^1 \) continuous?
- \( C^1 \) continuous?
- What’s the relationship between:
  - the \( \# \) of control points, and
  - the \( \# \) of cubic Bézier subcurves?

Higher-Order Bézier Curves

- \( > 4 \) control points
- Bernstein Polynomials as the basis functions

\[ B_n^i(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \quad 0 \leq i \leq n \]

- Every control point affects the entire curve
  - Not simply a local effect
  - More difficult to control for modeling

Questions?

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Cubic BSplines

- ≥ 4 control points
- Locally cubic
- Curve is not constrained to pass through any control points

A BSpline curve is also bounded by the convex hull of its control points.

Cubic BSplines

- Iterative method for constructing BSplines

Shirley, Fundamentals of Computer Graphics

Connecting Cubic BSpline Curves

- Can be chained together
- Better control locally (windowing)

BSpline Curve Control Points

- What’s the relationship between
  - the # of control points, and
  - the # of cubic BSpline subcurves?

Default BSpline  
BSpline with Discontinuity  
BSpline which passes through end points

Repeat interior control point  
Repeat end points
Bézier is not the same as BSpline

- Relationship to the control points is different

NURBS (generalized BSplines)

- BSpline: uniform cubic BSpline
- NURBS: Non-Uniform Rational BSpline
  - non-uniform = different spacing between the blending functions, a.k.a. knots
  - rational = ratio of polynomials (instead of cubic)

Converting between Bézier & BSpline

- Using the basis functions:

\[ B_{\text{Bézier}} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]

\[ B_{\text{BSpline}} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]

Q(t) = G(t)B(t) = Geometry G • Spline Basis B • Power Basis T(t)

Neat Bezier Spline Trick

- A Bezier curve with 4 control points:
  - \( P_0, P_1, P_2, P_3 \)
- Can be split into 2 new Bezier curves:
  - \( P_0, P'_1, P'_2, P'_3 \)
  - \( P'_3, P'_4, P'_5, P_3 \)

A Bézier curve is bounded by the convex hull of its control points.
### Misc. Mesh/Surface Vocabulary

- **Genus**: The maximum number of disjoint simple closed curves which can be cut from an orientable surface of genus $g$ without disconnecting it is $g$.

- Homeomorphic/Topological equivalence: a continuous stretching and bending of the object into a new shape


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### Readings for Today (pick one)

- "Geometry Images", Gu, Gortler, & Hoppe, SIGGRAPH 2002
- "Teddy: A Sketching Interface for 3D Freeform Design", Igarashi et al., SIGGRAPH 1999

- Post a comment or question on the LMS discussion by 10am on Tuesday

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### Reading for Friday (2/5)

- DeRose, Kass, & Truong, "Subdivision Surfaces in Character Animation", SIGGRAPH 1998

- Post a comment or question on the LMS discussion by 10am on Friday