Spline Curves

Last Time?
• Adjacency Data Structures
  – Geometric & topologic information
  – Dynamic allocation
  – Efficiency of access
• Mesh Simplification
  – edge collapse/vertex split
  – geomorphs
  – progressive transmission
  – view-dependent refinement

Today
• Interpolating Color & Normals in OpenGL
• Limitations of Polygonal Models
• Some Modeling Tools & Definitions
• What’s a Spline?
• Linear Interpolation
• Interpolation Curves vs. Approximation Curves
• Bézier Spline
• BSpline (NURBS)

Color Interpolation
• Interpolate colors of the 3 vertices
• Linear interpolation, barycentric coordinates

glShadeModel (GL_SMOOTH);
• From OpenGL Reference Manual:
  – Smooth shading, the default, causes the computed colors of vertices to be interpolated as the primitive is rasterized, typically assigning different colors to each resulting pixel fragment.
  – Flat shading selects the computed color of just one vertex and assigns it to all the pixel fragments generated by rasterizing a single primitive.
  – In either case, the computed color of a vertex is the result of lighting if lighting is enabled, or it is the current color at the time the vertex was specified if lighting is disabled.

Normal Interpolation
```c
 glBegin(GL_TRIANGLES);
 glNormal3f(1.0, 0.0, 0.0);
 glVertex3f(…);
 glNormal3f(-1.0, 0.0, 0.0);
 glVertex3f(…);
 glNormal3f(0.0, 1.0, 0.0);
 glVertex3f(…);
 glEnd();
```
Gouraud Shading

- Instead of shading with the normal of the triangle, we'll shade the vertices with the average normal and interpolate the shaded color across each face.
- How do we compute Average Normals? Is it expensive??

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Limitations of Polygonal Meshes

- Planar facets (& silhouettes)
- Fixed resolution
- Deformation is difficult
- No natural parameterization (for texture mapping)

Gouraud not always good enough

- Still low, fixed resolution (missing fine details)
- Still have polygonal silhouettes
- Intersection depth is planar (e.g. ray tracing visualization)
- Collisions problems for simulation
- Solid Texturing problems
- ...

Some Non-Polygonal Modeling Tools

Extrusion
Surfaces of Revolution
Spline Surfaces/Patches
Quadrics and other implicit polynomials

Continuity definitions:

- \( C^0 \) continuous
  - curve/surface has no breaks/gaps/holes
- \( G^1 \) continuous
  - tangent at joint has same direction
- \( C^2 \) continuous
  - curve/surface derivative is continuous
  - tangent at joint has same direction and magnitude
- \( C^n \) continuous
  - curve/surface through \( n \)th derivative is continuous
  - important for shading

“Shape Optimization Using Reflection Lines”, Tosun et al., 2007
Questions?

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Definition: What's a Spline?

- Smooth curve defined by some control points
- Moving the control points changes the curve

Interpolation Curves / Splines

Interpolation Curves

- Curve is constrained to pass through all control points
- Given points $P_0, P_1, \ldots, P_n$, find lowest degree polynomial which passes through the points

$$
\begin{align*}
    x(t) &= a_{n-1}t^{n-1} + \ldots + a_2t^2 + a_1t + a_0 \\
y(t) &= b_{n-1}t^{n-1} + \ldots + b_2t^2 + b_1t + b_0
\end{align*}
$$

Linear Interpolation

- Simplest "curve" between two points

$$Q(t) = (1 - t)P_0 + tP_1$$

$$Q(t) = \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} = \begin{bmatrix} (P_0) & (P_1) \\ \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$Q(t) = GBT(t) = \text{Geometry } G \cdot \text{ Spline Basis } B \cdot \text{ Power Basis } T(t)$$
Interpolation vs. Approximation Curves

- **Interpolation**
  - Curve must pass through control points
- **Approximation**
  - Curve is influenced by control points

**Questions?**

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**Cubic Bézier Curve**

- 4 control points
- Curve passes through first & last control point
- Curve is tangent at \( P_1 \) to \((P_2-P_1)\) and at \( P_4 \) to \((P_4-P_3)\)

A Bézier curve is bounded by the convex hull of its control points.

**Today**

- Interpolation Curve – over constrained → lots of (undesirable?) oscillations
- Approximation Curve – more reasonable?

Cubic Bézier Curve

- de Casteljau's algorithm for constructing Bézier curves
Cubic Bézier Curve

\[ Q(t) = (1-t)^3 P_1 + 3(1-t)^2 t P_2 + 3(1-t) t^2 P_3 + t^3 P_4 \]

\[ B(t) = \begin{bmatrix}
    -1 & 3 & -3 & 1 \\
    3 & -6 & 3 & 0 \\
    -3 & 3 & 0 & 0 \\
    1 & 0 & 0 & 0
\end{bmatrix} \]

Bernstein Polynomials

\[ B_i(t) = (1 - t)^3; B_i(t) = 3t(1-t)^2; B_3(t) = 3t^2(1-t); B_4(t) = t^3 \]

Connecting Cubic Bézier Curves

Asymmetric: Curve goes through some control points but misses others

- How can we guarantee \( C^0 \) continuity?
- How can we guarantee \( G^1 \) continuity?
- How can we guarantee \( C^1 \) continuity?
- Can’t guarantee higher \( C^2 \) or higher continuity

Connecting Cubic Bézier Curves

- Where is this curve
  - \( C^0 \) continuous?
  - \( G^1 \) continuous?
  - \( C^1 \) continuous?
- What’s the relationship between:
  - the \# of control points, and
  - the \# of cubic Bézier subcurves?

Higher-Order Bézier Curves

- \( > 4 \) control points
- Bernstein Polynomials as the basis functions

\[ B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \quad 0 \leq i \leq n \]

- Every control point affects the entire curve
  - Not simply a local effect
  - More difficult to control for modeling

Questions?

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Cubic BSplines
- ≥ 4 control points
- Locally cubic
- Curve is not constrained to pass through any control points

A BSpline curve is also bounded by the convex hull of its control points.

Connecting Cubic BSpline Curves
- Can be chained together
- Better control locally (windowing)

BSpline Curve Control Points
- Default BSpline
- BSpline with Discontinuity
- BSpline which passes through end points
Bézier is not the same as BSpline

- Relationship to the control points is different

Converting between Bézier & BSpline

- Using the basis functions:

\[
B_{\text{Bez}} = \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 3 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B_{\text{BSpline}} = \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
6 & -9 & 3 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

Q(t) = GBT(t) = Geometry G · Spline Basis B · Power Basis T(t)

NURBS (generalized BSplines)

- BSpline: uniform cubic BSpline
- NURBS: Non-Uniform Rational BSpline
  - non-uniform = different spacing between the blending functions, a.k.a. knots
  - rational = ratio of polynomials (instead of cubic)

Neat Bezier Spline Trick

- A Bezier curve with 4 control points:
  - \( P_0 \), \( P_1 \), \( P_2 \), \( P_3 \)
- Can be split into 2 new Bezier curves:
  - \( P_0^* \), \( P_1^* \), \( P_2^* \), \( P_3^* \)
  - \( P_3^* \), \( P_4^* \), \( P_5^* \), \( P_6^* \)

A Bezier curve is bounded by the convex hull of its control points.
Questions?

Readings for Today (pick one)

- "Geometry Images", Gu, Gortler, & Hoppe, SIGGRAPH 2002
- "Teddy: A Sketching Interface for 3D Freeform Design", Igarashi et al., SIGGRAPH 1999
- Post a comment or question on the LMS discussion by 10am on Tuesday

Reading for Friday (2/4)

- DeRose, Kass, & Truong, "Subdivision Surfaces in Character Animation", SIGGRAPH 1998
- Post a comment or question on the LMS discussion by 10am on Friday