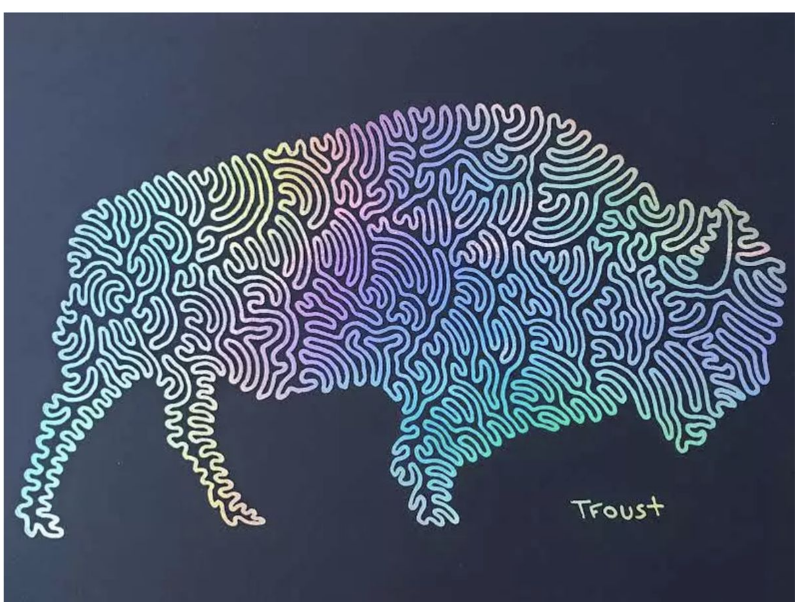
Curves & Surfaces

https://www.moillusions.com/glass-water-optical-illusion/

One Line Bison, Tyler Foust, 2020

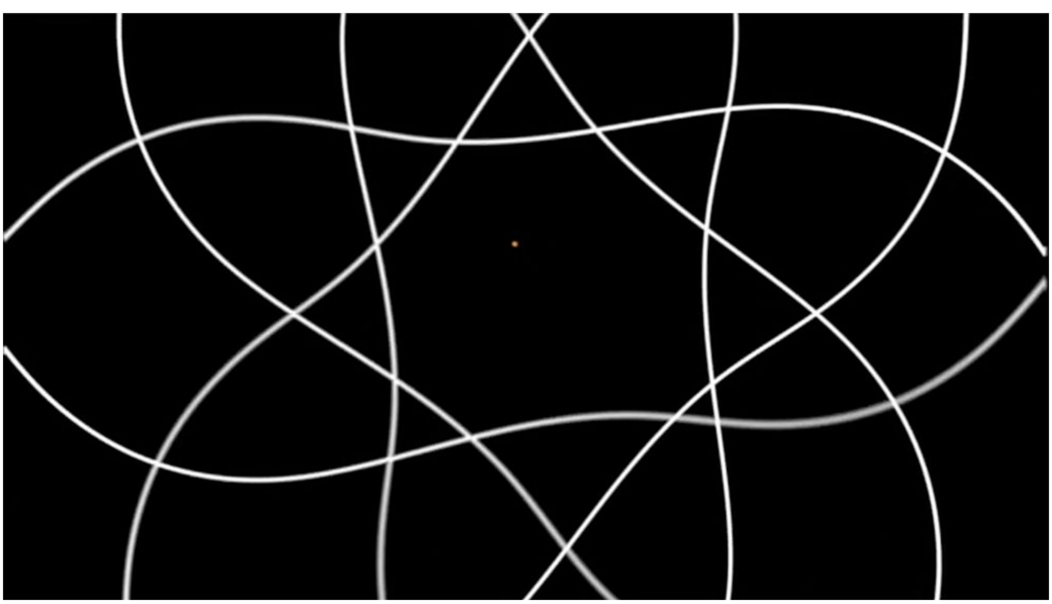


https://www.tylerfoust.com/

Squaring the Circle, Troika, 2013

http://troika.uk.com/work/troika-squaring-the-circle/

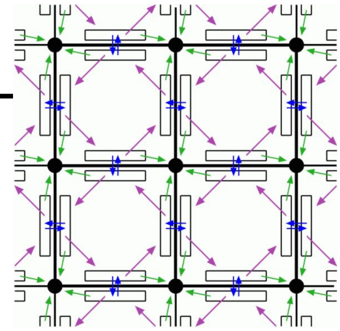
Herbstlaub

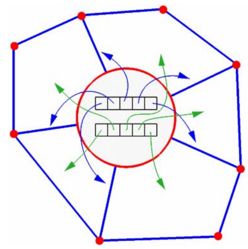


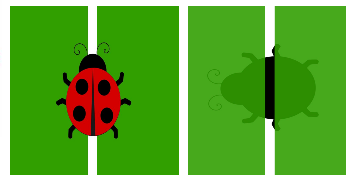
Oliver Vogel, Siggraph 2007

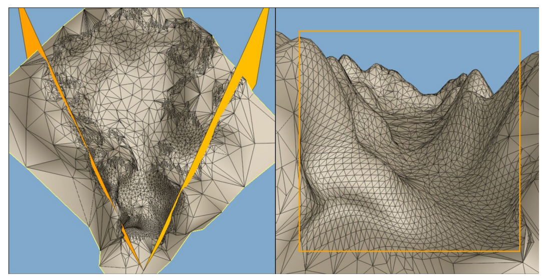
Last Time?

- Adjacency Data Structures
  - Geometric & topologic information
  - Dynamic allocation
  - Efficiency of access
- Mesh Simplification
  - edge collapse/vertex split
  - geomorphs
  - progressive transmission
  - view-dependent refinement









Today

- Reading: "Teddy: A Sketching Interface for 3D Freeform Design"
- Limitations of Polygonal Models
- What's a Spline?
- Bézier Spline
- BSpline (NURBS)
- Extending to Surfaces & Paper for Friday
- Worksheet: Shortest Edge Collapse

Reading for Today

- "Teddy: A Sketching Interface for 3D Freeform Design", Igarashi et al., SIGGRAPH 1999

- How do we represent objects that don’t have flat polygonal faces & sharp corners? What are the right tools to design/construct digital models of blobby, round, or soft things? What makes a user interface intuitive, quick, and easy-to-use for beginners?

Today

- Reading: “Teddy: A Sketching Interface for 3D Freeform Design”
- Limitations of Polygonal Models
  - Interpolating Color & Normals in OpenGL
  - Some Modeling Tools & Definitions
- What's a Spline?
- Bézier Spline
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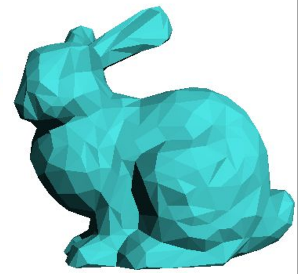
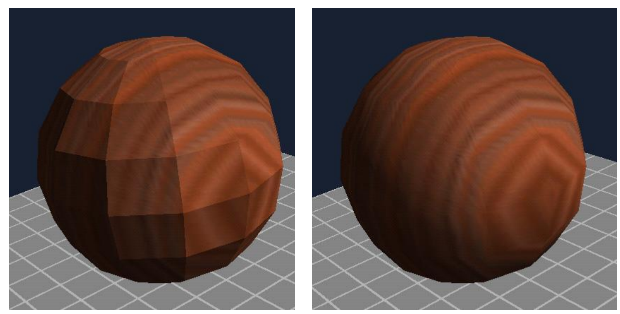
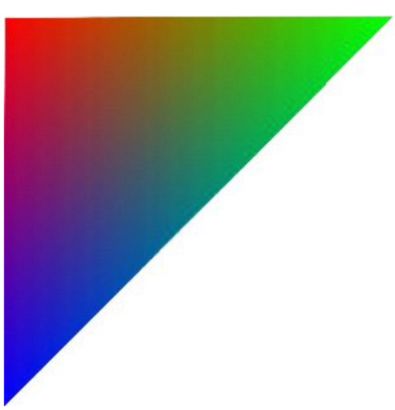
Limitations of Polygonal Meshes

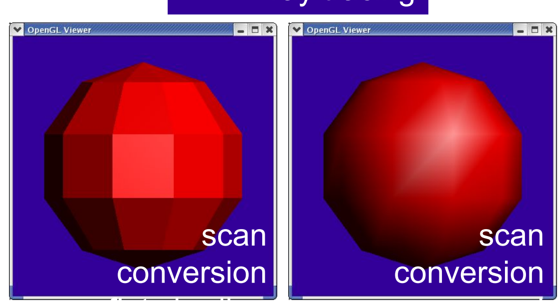
- Planar facets (& silhouettes)
- Fixed resolution
- Deformation is difficult
- No natural parameterization (for texture mapping)
- Incorrect collision detection
- Solid texturing problems

Color & Normal Interpolation

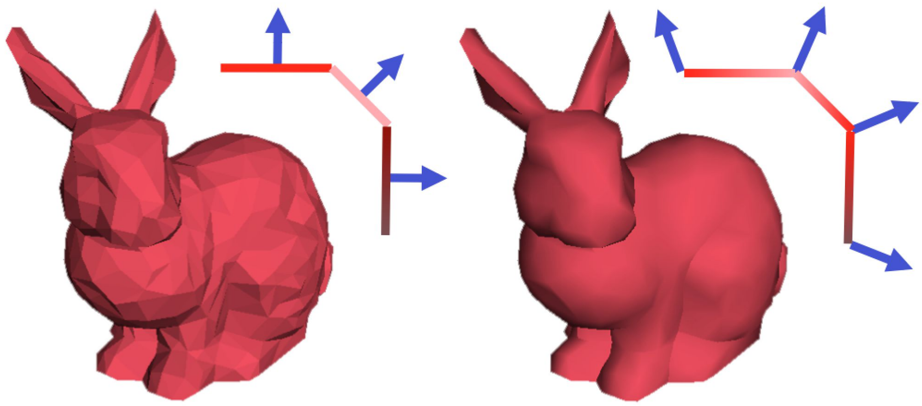
- It's easy in OpenGL to specify different colors and/or normals at the vertices of triangles:
- Why is this useful?



What is Gouraud Shading?

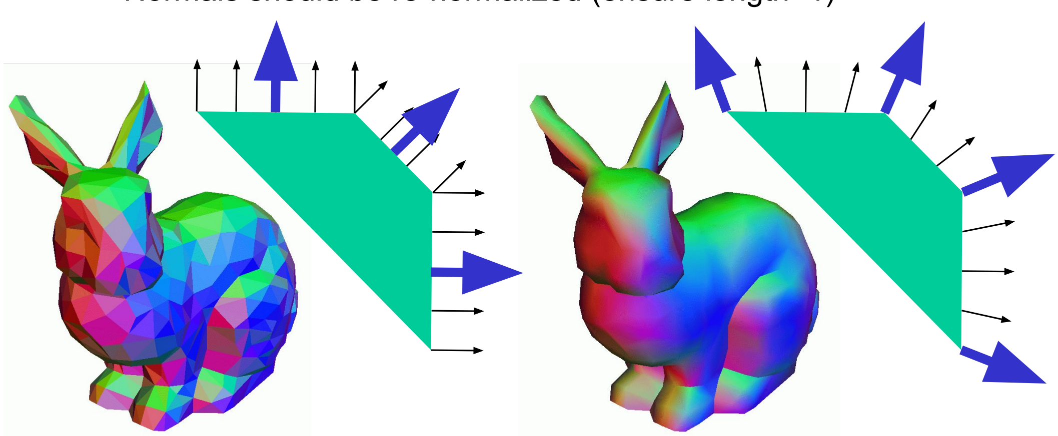
- Instead of shading with the normal of the triangle, we'll shade the vertices with the *average normal* and *interpolate the shaded color* across each face
  - This gives the *illusion of a smooth surface* with smoothly varying normals



- How do we compute Average Normals? Is it expensive??

Phong Normal Interpolation (Not Phong *Shading*)

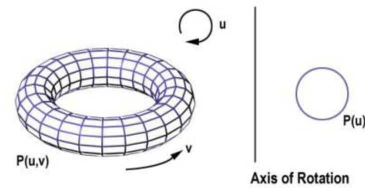
- *Interpolate the average vertex normals* across the face and compute *per-pixel shading*
  - Normals should be re-normalized (ensure length=1)



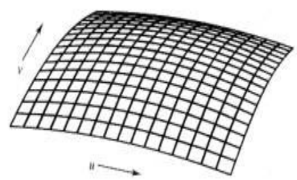
- Before shaders, per-pixel shading was not possible in hardware (Gouraud shading is actually a decent substitute!)

Some Non-Polygonal Modeling Tools

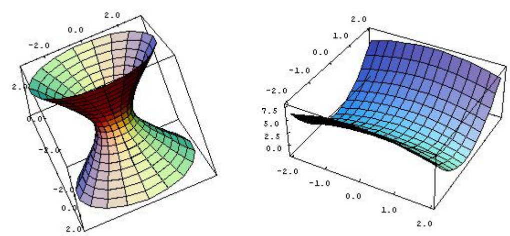
Extrusion



Surface of Revolution

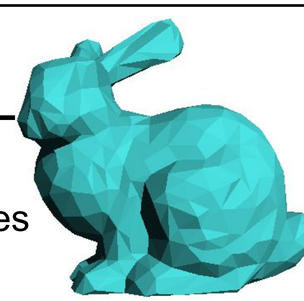


Spline Surfaces/Patches

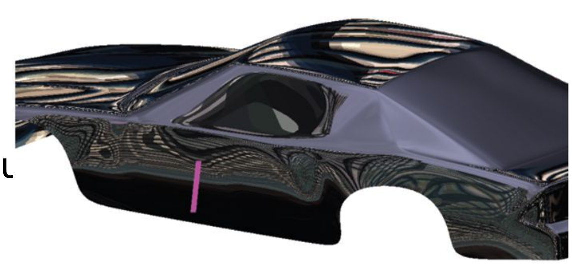


Quadrics and other implicit polynomials

Continuity definitions:



- C^0 continuous
  - curve/surface has no breaks/gaps/holes
- G^1 continuous
  - tangent at joint has same direction
- C^1 continuous
  - curve/surface derivative is continuous
  - tangent at joint has same direction *and* magnitude
- C^n continuous
  - curve/surface through n^{th} derivative is continuou
  - important for shading



"Shape Optimization Using Reflection Lines", Tosun et al., 2007

Questions?

Today

- Reading: “Teddy: A Sketching Interface for 3D Freeform Design”
- Limitations of Polygonal Models
- What's a Spline?
  - Interpolation Curves vs. Approximation Curves
  - Linear Interpolation
- Bézier Spline
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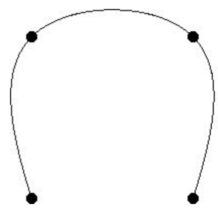
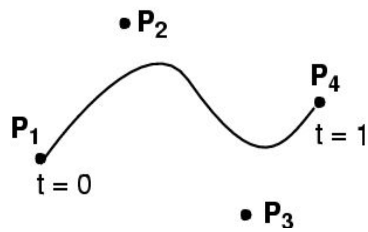
Definition: What's a Spline?

- Smooth curve defined by some control points
- Moving the control points changes the curve

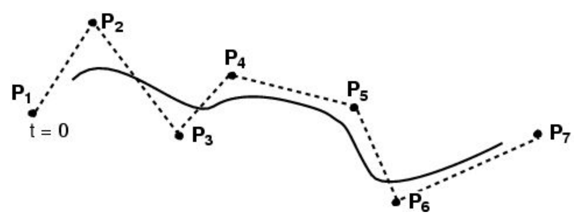


Interpolation

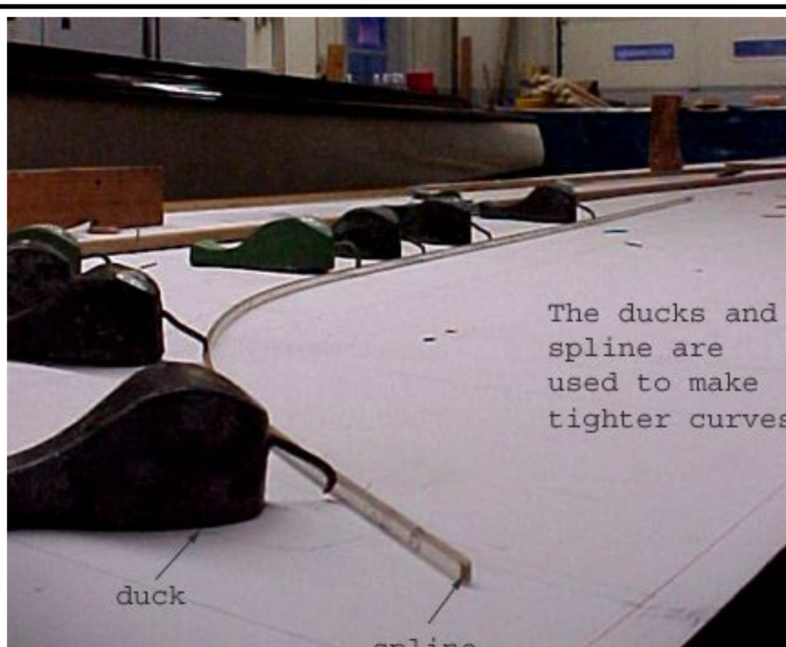


Bézier (approximation)

BSpline
(approximation)



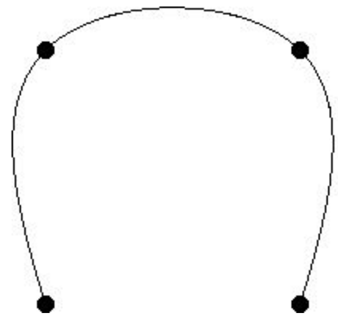
Interpolation Curves / Splines



www.abm.org

Interpolation Curves

- Curve is constrained to pass through all control points

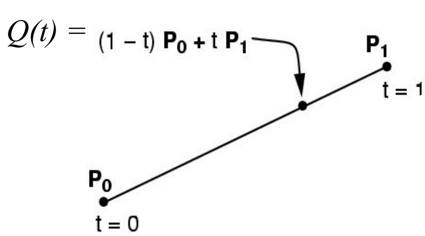


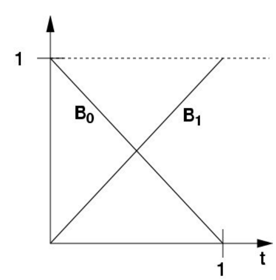
- Given points P_0, P_1, ... P_n, find lowest degree polynomial which passes through the points

$$x(t) = a_{n-1}t^{n-1} + ... + a_2t^2 + a_1t + a_0$$
$$y(t) = b_{n-1}t^{n-1} + ... + b_2t^2 + b_1t + b_0$$

Linear Interpolation

- Simplest "curve" between two points





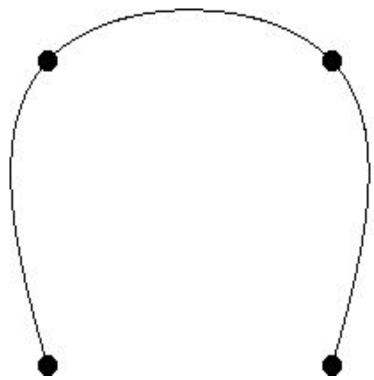
Spline Basis Functions

a.k.a. Blending Functions

$$Q(t) = \begin{pmatrix} Q_x(t) \\ Q_y(t) \\ Q_z(t) \end{pmatrix} = \begin{pmatrix} (P_0) & (P_1) \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

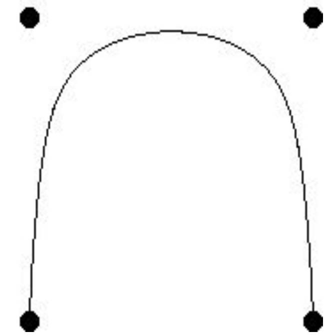
$$Q(t) = \mathbf{GBT(t)} = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

Interpolation vs. Approximation Curves



Interpolation

curve must pass
through control points

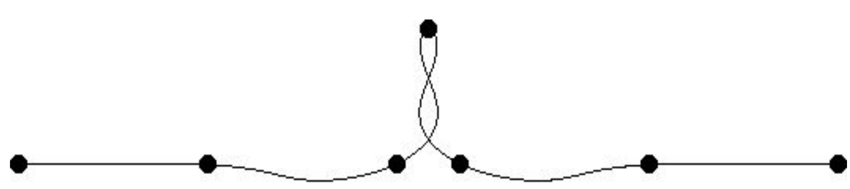


Approximation

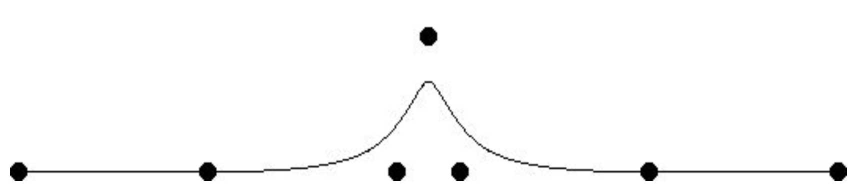
curve is influenced
by control points

Interpolation vs. Approximation Curves

- Interpolation Curve – over constrained → lots of (undesirable?) oscillations



- Approximation Curve – more reasonable?



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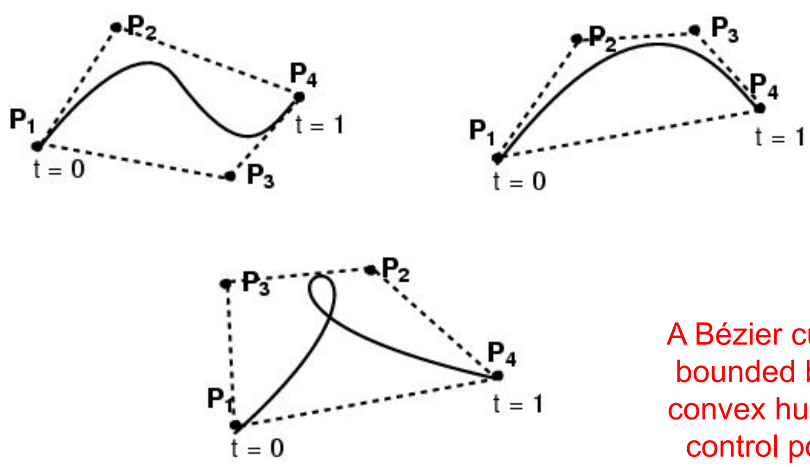
Cubic Bézier Curve

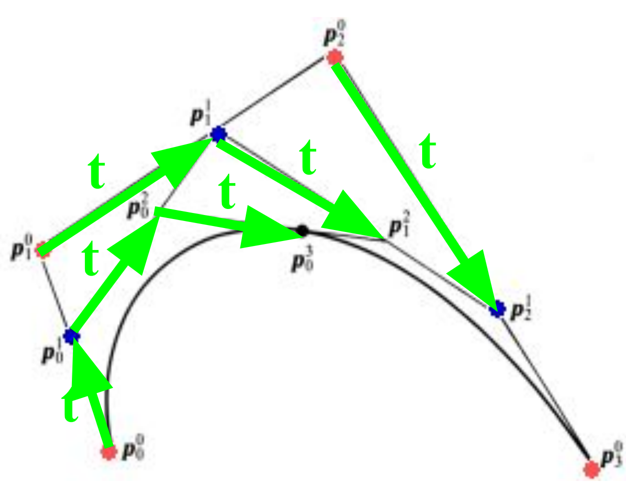
- 4 control points
- Curve passes through first & last control point
- Curve is tangent at $\mathbf{P}_1$ to ($\mathbf{P}_2$-$\mathbf{P}_1$) and at $\mathbf{P}_4$ to ($\mathbf{P}_4$-$\mathbf{P}_3$)



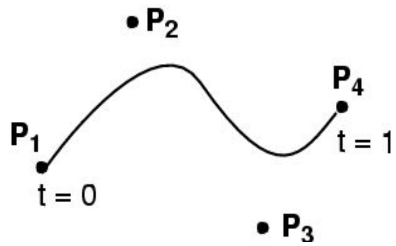
A Bézier curve is bounded by the convex hull of its control points.

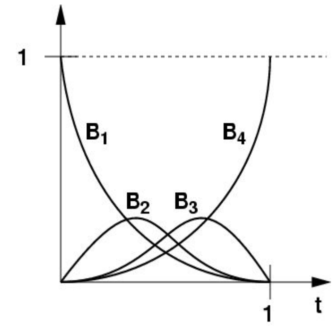
Cubic Bézier Curve

- de Casteljau's algorithm for constructing Bézier curves



Cubic Bézier Curve





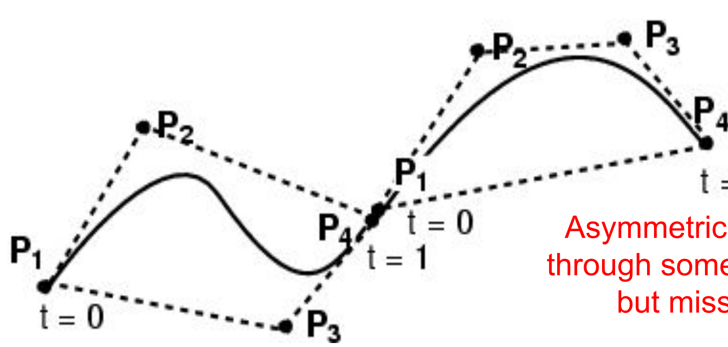
$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

$$Q(t) = \mathbf{GBT(t)} \qquad B_{Bezier} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Bernstein Polynomials

$$B_1(t) = (1-t)^3;\ B_2(t) = 3t(1-t)^2;\ B_3(t) = 3t^2(1-t);\ B_4(t) = t^3$$

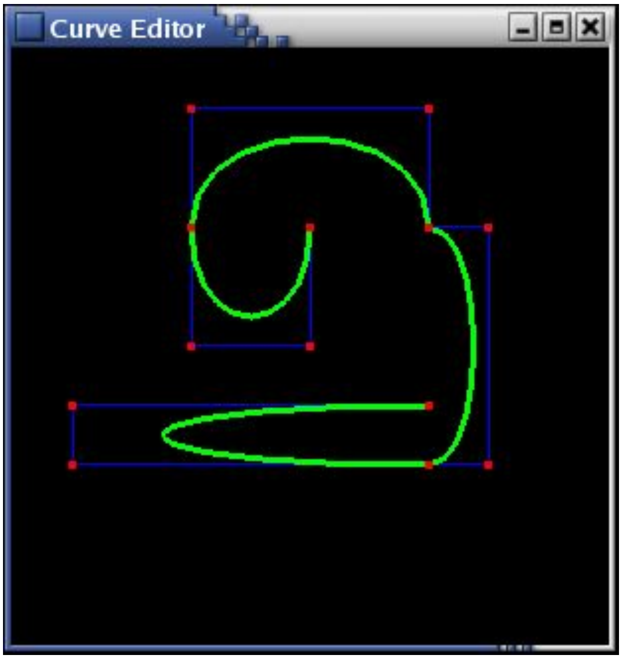
Connecting Cubic Bézier Curves



Asymmetric: Curve goes through some control points but misses others

- How can we guarantee C^0 continuity?
- How can we guarantee G^1 continuity?
- How can we guarantee C^1 continuity?
- Can't guarantee higher C^2 or higher continuity

Connecting Cubic Bézier Curves



- Where is this curve
  - C^0 continuous?
  - G^1 continuous?
  - C^1 continuous?
- What's the relationship between:
  - the # of control points, and
  - the # of cubic Bézier subcurves?

Higher-Order Bézier Curves

- > 4 control points
- Bernstein Polynomials as the basis functions

$$B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \qquad 0 \leq i \leq n$$

- Every control point affects the entire curve
  - Not simply a local effect
  - More difficult to control for modeling

Questions?

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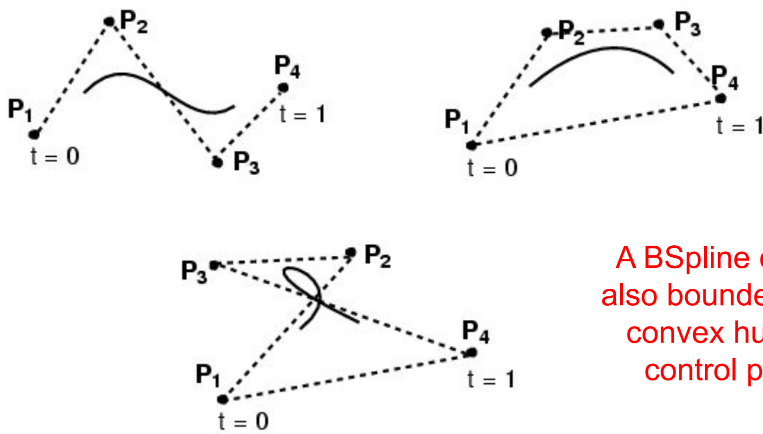
Cubic BSplines

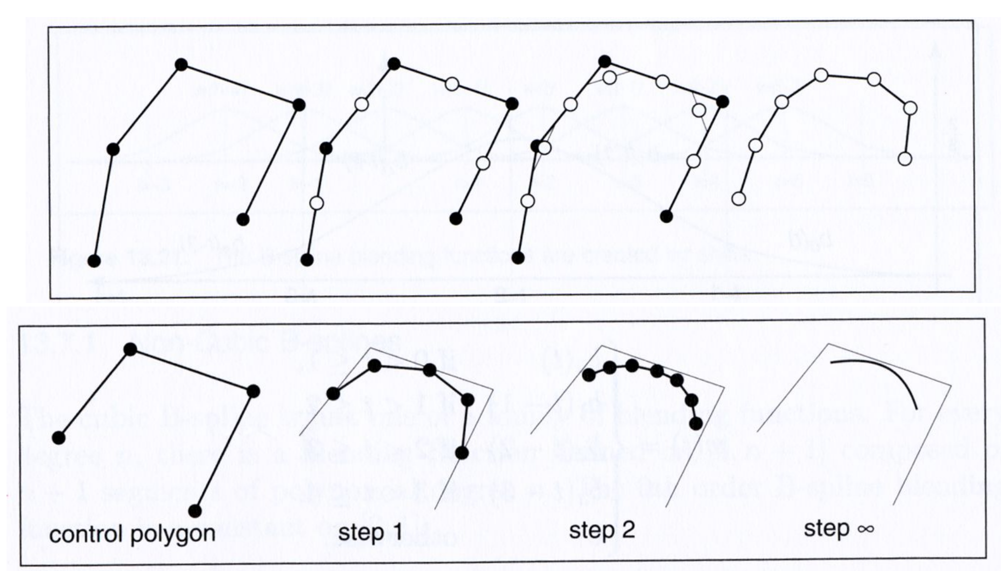
- ≥ 4 control points
- Locally cubic
- Curve is not constrained to pass through any control points



A BSpline curve is also bounded by the convex hull of its control points.

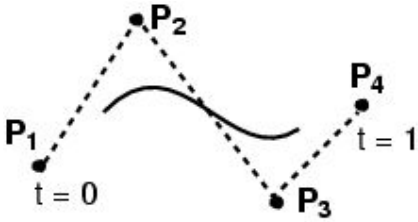
Cubic BSplines

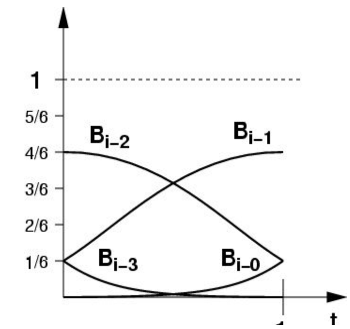
- Iterative method for constructing BSplines



Shirley, Fundamentals of Computer Graphics

Cubic BSplines



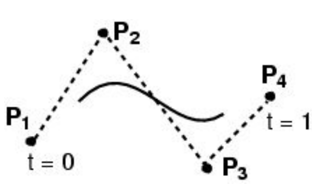


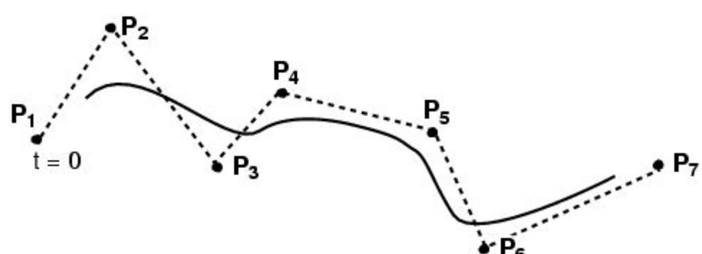
$$Q(t) = \frac{(1-t)^3}{6}P_{i-3} + \frac{3t^3 - 6t^2 + 4}{6}P_{i-2} + \frac{-3t^3 + 3t^2 + 3t + 1}{6}P_{i-1} + \frac{t^3}{6}P_i$$

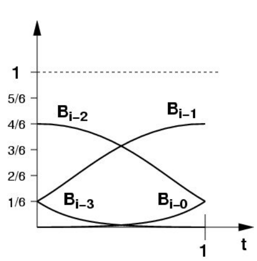
$$Q(t) = \mathbf{GBT(t)} \qquad B_{B-Spline} = \frac{1}{6}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

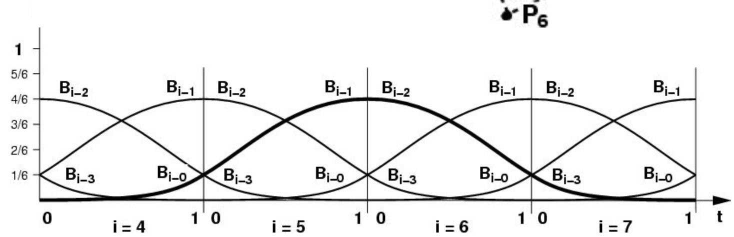
Connecting Cubic BSpline Curves

- Can be chained together
- Better control locally (windowing)

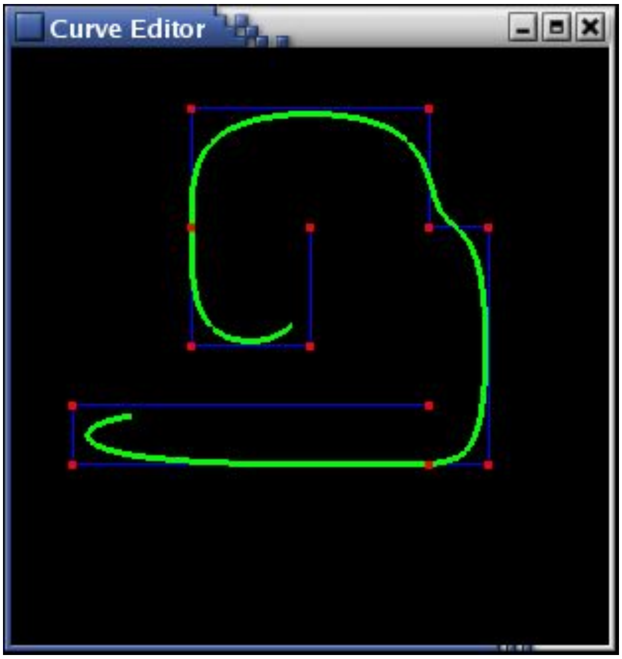






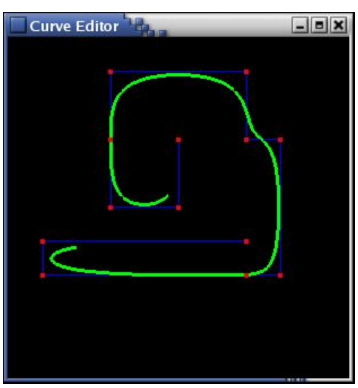


Connecting Cubic BSpline Curves

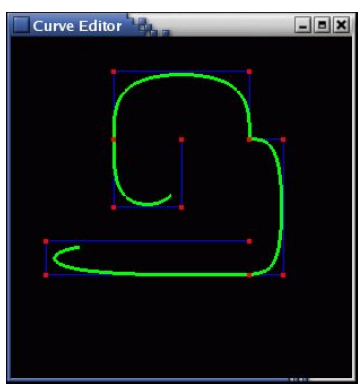


- What's the relationship between
  - the # of control points, and
  - the # of cubic BSpline subcurves?

BSpline Curve Control Points

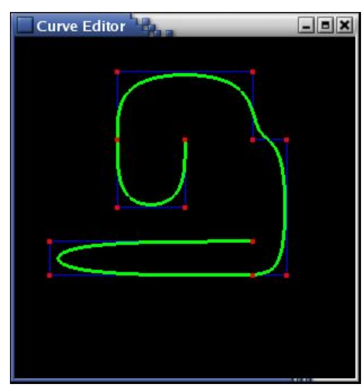


Default BSpline



BSpline with Discontinuity

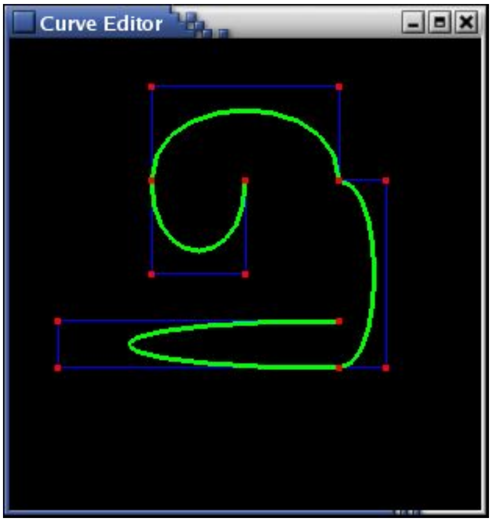
Repeat interior control point



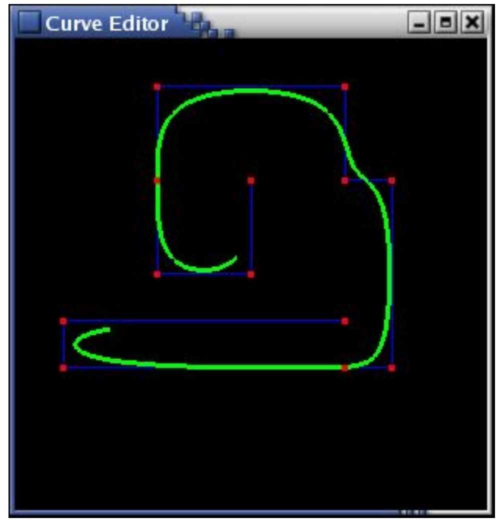
BSpline which passes through end points

Repeat end points

Bézier is not the same as BSpline



Bézier

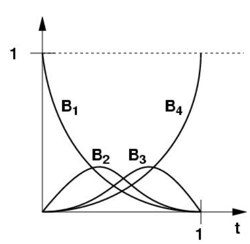


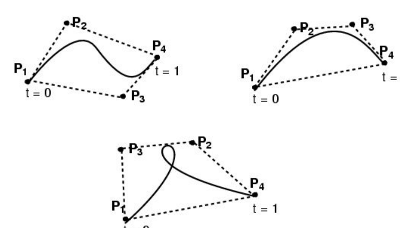
BSpline

Bézier is not the same as BSpline

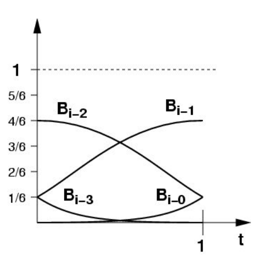
- Relationship to the control points is different

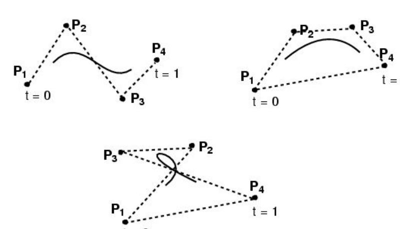
Bézier



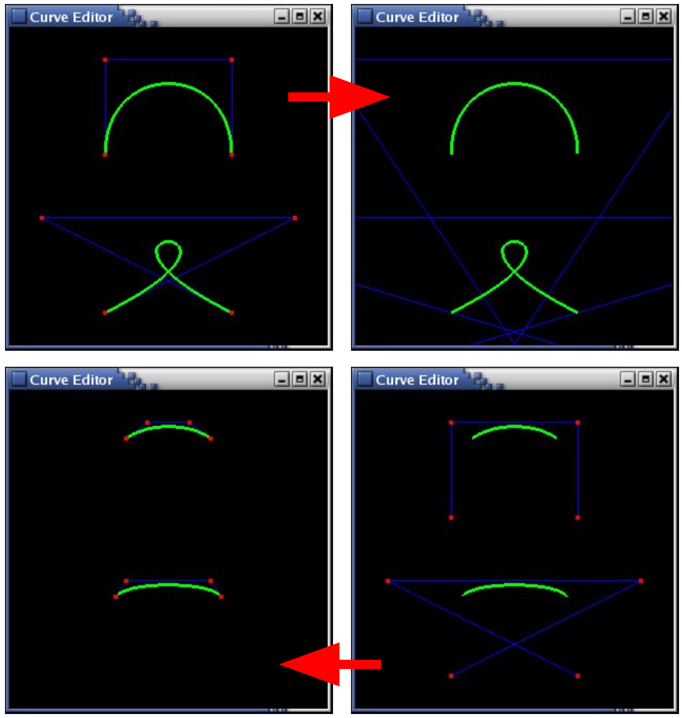


BSpline





Converting between Bézier & BSpline



original control points as Bézier

new BSpline control points to match Bézier

new Bézier control points to match BSpline

original control points as BSpline

Converting between Bézier & BSpline

Using the basis functions:

$$Q(t) = \mathbf{GBT(t)} = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

$$G_{Bezier} \cdot B_{Bezier} \cdot T = G_{BSpline} \cdot B_{BSpline} \cdot T$$

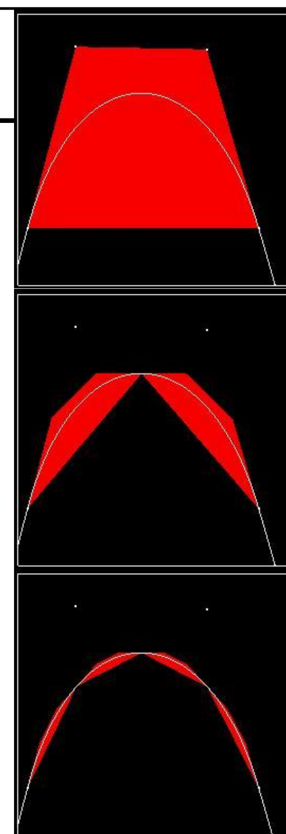
$$G_{Bezier} = \frac{G_{BSpline} \cdot B_{BSpline} \cdot T}{B_{Bezier} \cdot T}$$

$$B_{Bezier} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad B_{B-Spline} = \frac{1}{6}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

NURBS (generalized BSplines)

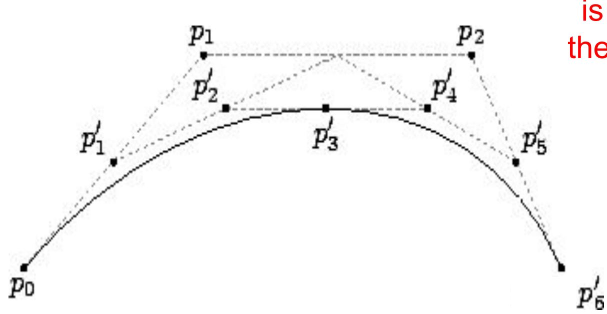
- BSpline: uniform cubic BSpline
- NURBS: Non-Uniform Rational BSpline
  - non-uniform = different spacing between the blending functions, a.k.a. knots
  - rational = ratio of polynomials (instead of cubic)

Neat Bezier Spline Trick



- A Bezier curve with 4 control points:
  - $P_0 \quad P_1 \quad P_2 \quad P_3$
- Can be split into 2 new Bezier curves:
  - $P_0 \quad P'_1 \quad P'_2 \quad P'_3$
  - $P'_3 \quad P'_4 \quad P'_5 \quad P_3$

A Bézier curve is bounded by the convex hull of its control points.



Today

- Reading: “Teddy: A Sketching Interface for 3D Freeform Design”
- Limitations of Polygonal Models
- What's a Spline?
- Bézier Spline
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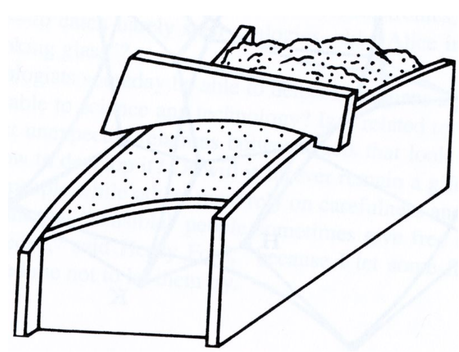
- Worksheet: Shortest Edge Collapse

Spline Surface via Tensor Product

- Of two vectors:

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \otimes \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_2 b_1 & a_3 b_1 \\ a_1 b_2 & a_2 b_2 & a_3 b_2 \\ a_1 b_3 & a_2 b_3 & a_3 b_3 \\ a_1 b_4 & a_2 b_4 & a_3 b_4 \end{bmatrix}$$

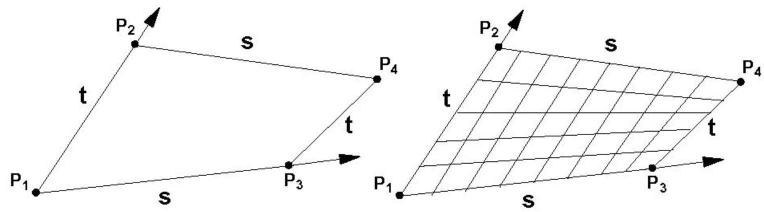
- Similarly, we can define a surface as the tensor product of two curves....



Farin, Curves and Surfaces for Computer Aided Geometric Design

Bilinear Patch

Bi-lerp a (typically non-planar) quadrilateral

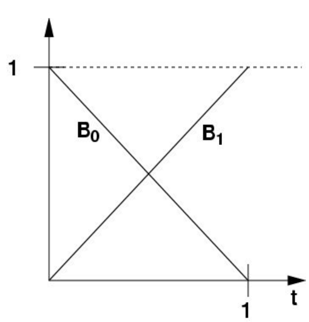


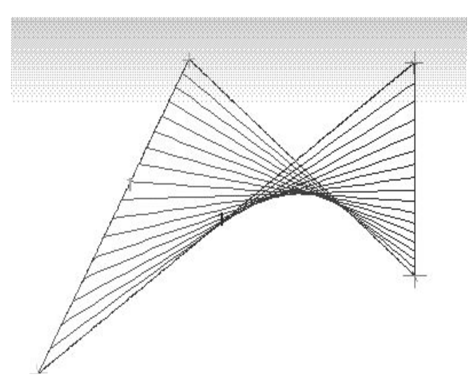
Notation: $\mathbf{L}(P_1, P_2, \alpha) \equiv (1-\alpha)P_1 + \alpha P_2$

$$Q(s,t) = \mathbf{L}(\mathbf{L}(P_1, P_2, t), L(P_3, P_4, t), s)$$

Bilinear Patch

- Smooth version of quadrilateral with non-planar vertices...





  - But will this help us model smooth surfaces?
  - Do we have control of the derivative at the edges?

Ruled Surfaces in Art & Architecture

http://www.bergenwood.no/wp-content/media/images/frozenmusic.jpg

Chiras Iulia
Astri Isabella
Matiss Shteinerts

Antoni Gaudi
Children's School
Barcelona

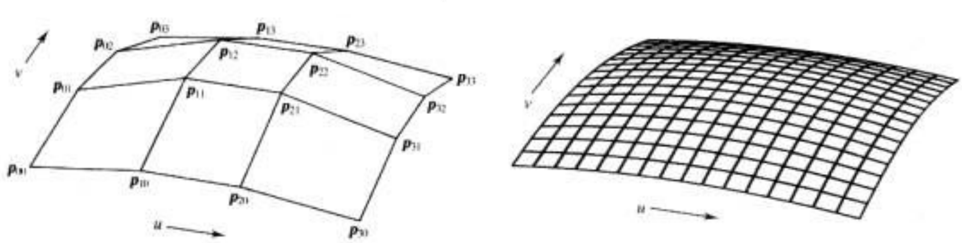
http://www.lonelyplanetimages.com/images/399954

Bicubic Bezier Patch

Notation: $\mathbf{CB}(P_1, P_2, P_3, P_4, \alpha)$ is Bézier curve with control points P_i evaluated at α

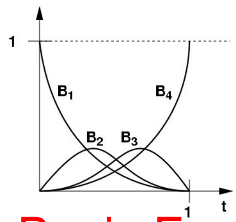
Define "Tensor-product" Bézier surface

$$
\begin{aligned}
Q(s,t) = \mathbf{CB}(\quad & \mathbf{CB}(P_{00}, P_{01}, P_{02}, P_{03}, t), \\
& \mathbf{CB}(P_{10}, P_{11}, P_{12}, P_{13}, t), \\
& \mathbf{CB}(P_{20}, P_{21}, P_{22}, P_{23}, t), \\
& \mathbf{CB}(P_{30}, P_{31}, P_{32}, P_{33}, t), \\
& s)
\end{aligned}
$$

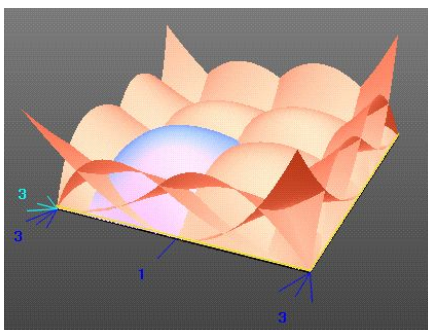


(a) (b)

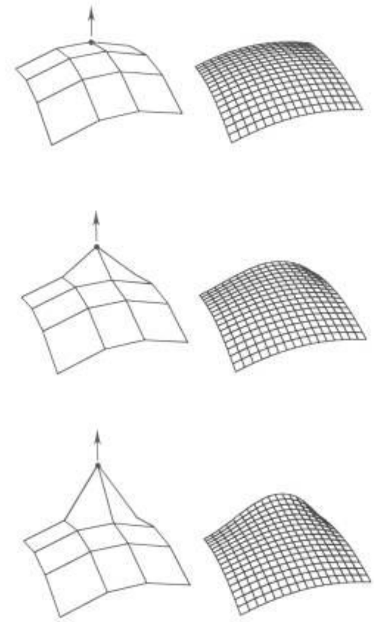
Editing Bicubic Bezier Patches



Curve Basis Functions

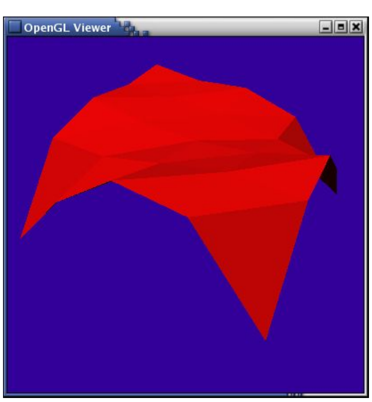


Surface Basis Functions

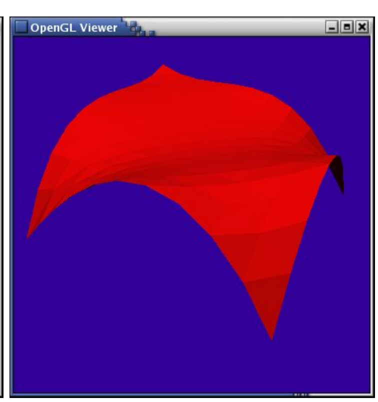


Bicubic Bezier Patch Tessellation

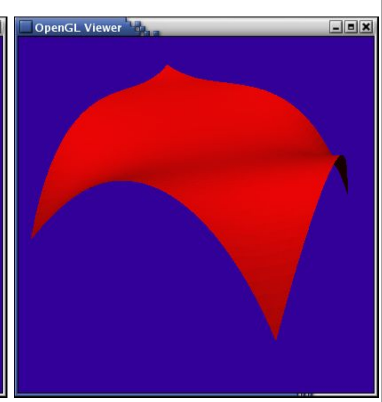
- Given 16 control points and a tessellation resolution, we can create a triangle mesh



resolution:
5x5 vertices



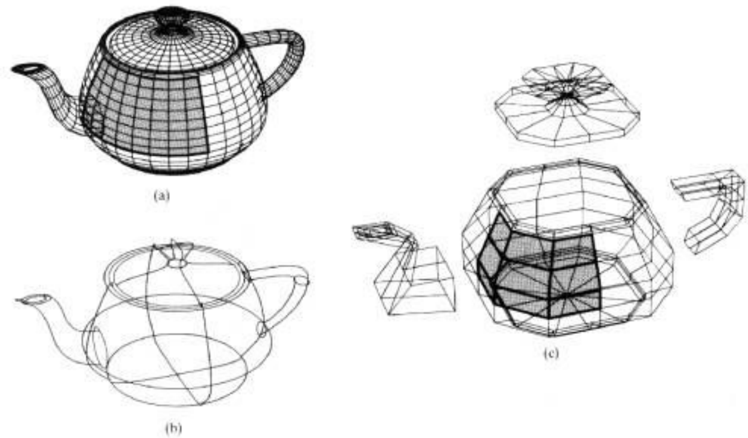
resolution:
11x11 vertices



resolution:
41x41 vertices

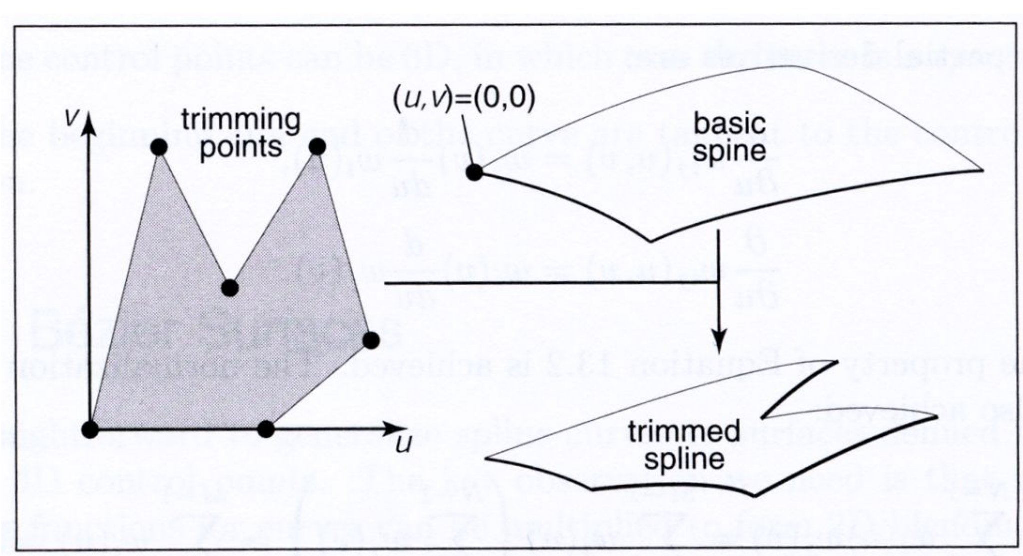
Modeling with Bicubic Bezier Patches

- Original Teapot specified with Bezier Patches



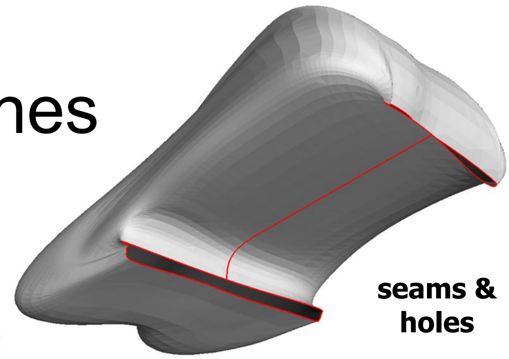
- But it’s not "watertight": it has intersecting surfaces at spout & handle, no bottom, a hole at the spout tip, a gap between lid & base

Trimming Curves for Patches

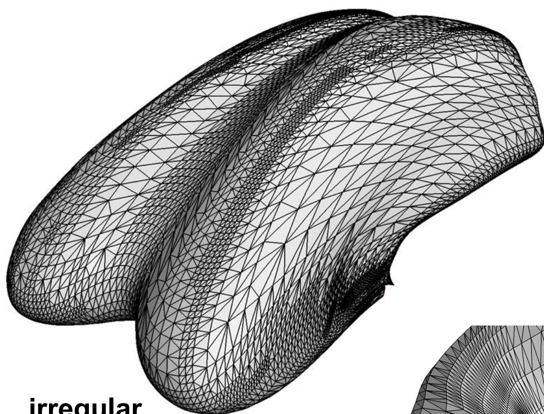


Shirley, Fundamentals of Computer Graphics

Spline-Based Modeling Headaches

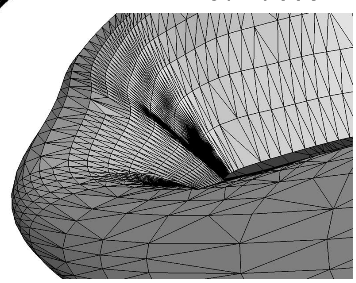


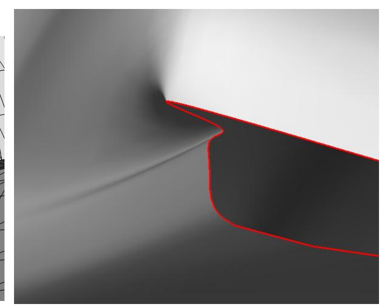
seams & holes



irregular sampling

"pinched" surfaces





Questions?

- Bezier Patches?

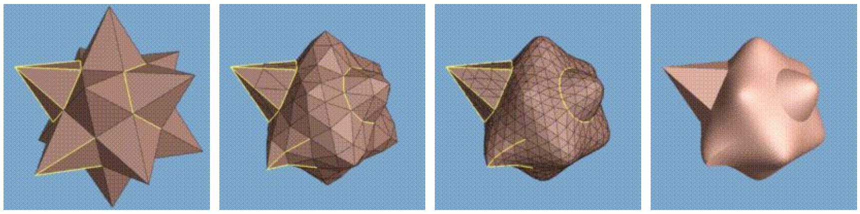
  or

- Triangle Mesh?

Readings for Next Time *(pick one)*

- Hoppe et al., “Piecewise Smooth Surface Reconstruction” SIGGRAPH 1994

Triangle meshes
directly applies to HW1!



- DeRose, Kass, & Truong, "Subdivision Surfaces in Character Animation", SIGGRAPH 1998

Quad Meshes
more common in artistic practice (e.g. Pixar’s Geri’s Game)

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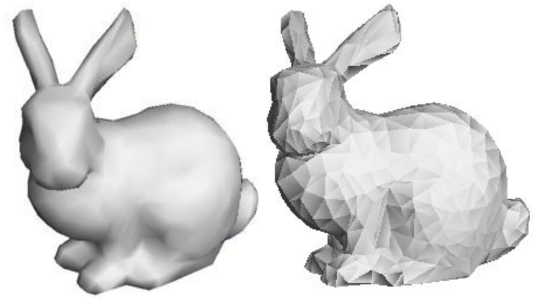
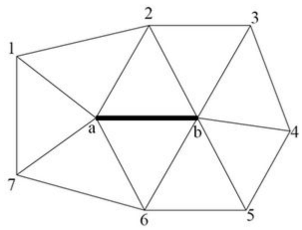
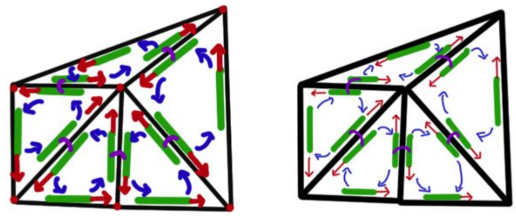
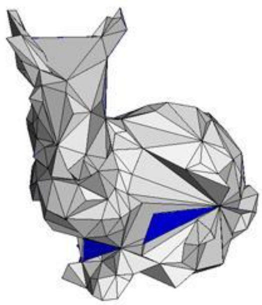
Homework 1:

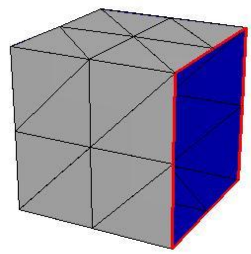
Expectations for
“Progress Posts”?

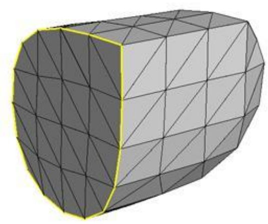












Pop Worksheet!

Breakout rooms, teams of 2 or 3

- Perform a sequence of 3 edge collapses, one-at-a-time
- Always collapse the shortest edge that does not result in a zero area or “flipped”/upside-down triangle
- Replacement vertex should be at the midpoint of the edge