Mass-Spring Systems

High Fashion in Equations

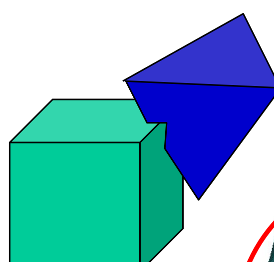
MIRALab, University of Geneva, SIGGRAPH 2007

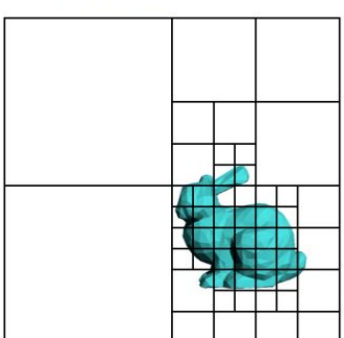
Simulating Knitted Cloth at the Yarn Level

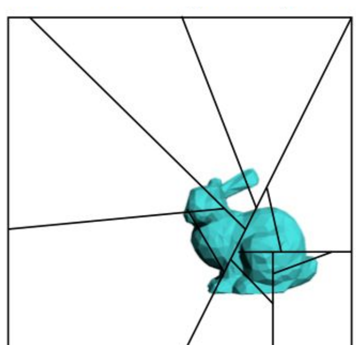
Kaldor, James, & Marshner, SIGGRAPH 2008

Last Time?

- Collision Detection & Conservative Bounding Regions
- Spatial Acceleration Data Structures
  - Octree, k-d tree, BSF tree







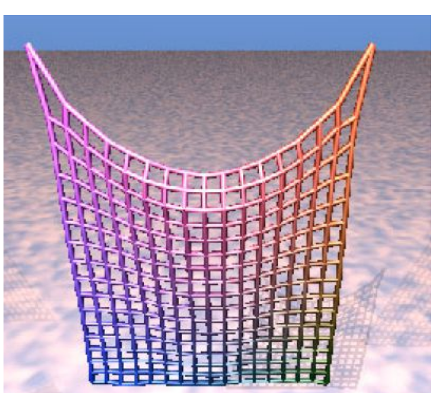
Today

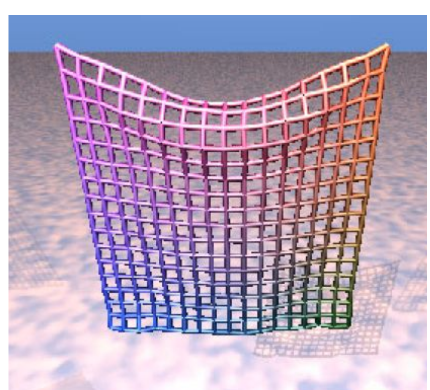
- Papers for Today
- Particle Systems
  - Equations of Motion (Physics)
  - Forces: Gravity, Spatial, Damping
  - Numerical Integration (Euler, Midpoint, etc.)
- Mass Spring System Examples
  - String, Hair, Cloth
- Stiffness
- Discretization
- Papers for Tuesday
- Worksheet on Volumetric Structures

Reading for Today:

Everyone should read this (simple cloth model used in HW2)

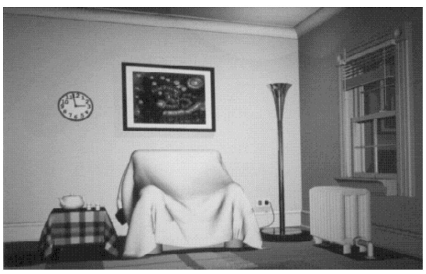
- "Deformation Constraints in a Mass-Spring Model to Describe Rigid Cloth Behavior", Provot, 1995.

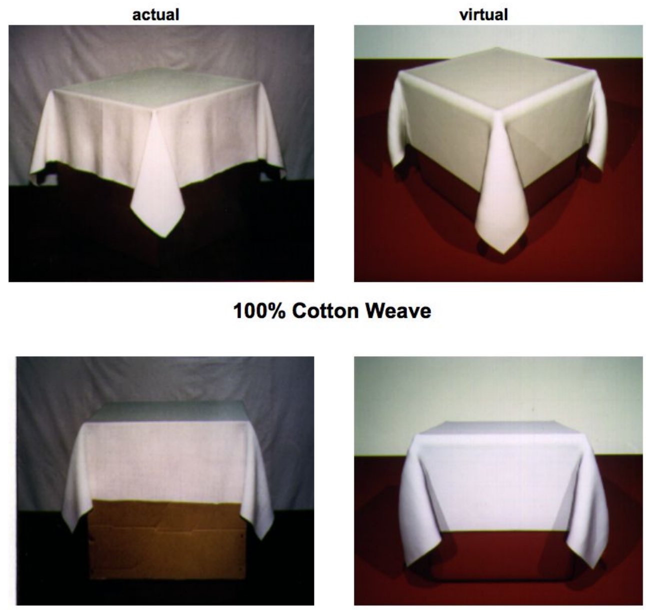

Simple mass-spring system


Improved solution

“Predicting the Drape of Woven Cloth Using Interacting Particles”

- Breen, House, and Wozny
- SIGGRAPH 1994

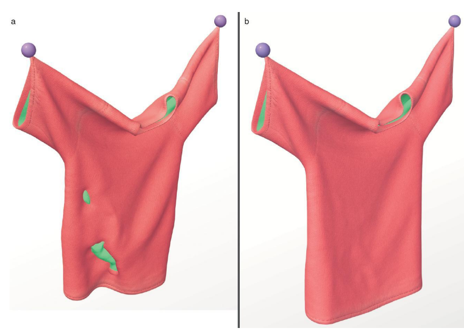


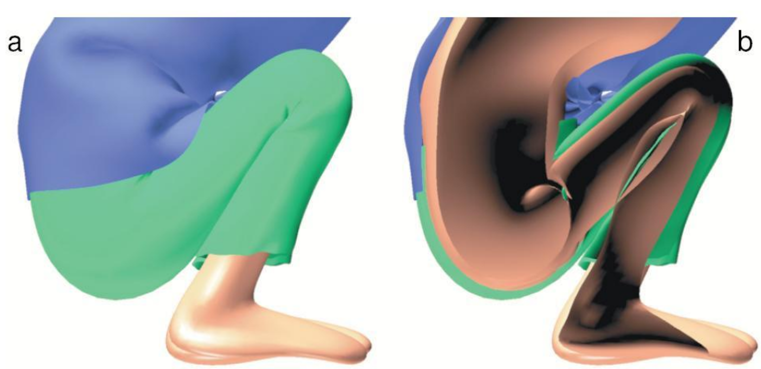


Cloth in Practice (w/ Animation)

OPTIONAL READING

- Baraff, Witkin & Kass
  Untangling Cloth
  SIGGRAPH 2003





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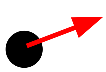
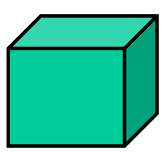
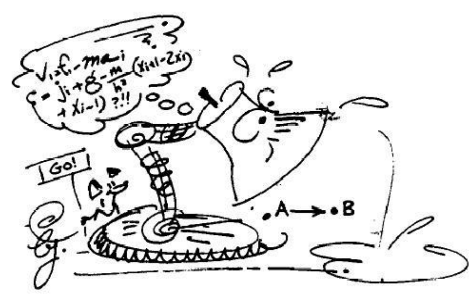
Types of Dynamics

- Point 
- Rigid body 



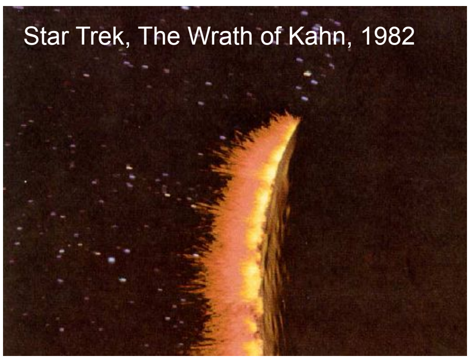
Witkin & Kass, “Spacetime Constraints”, 1988.

- Deformable body
  (include clothes, fluids, smoke, etc.)

Carlson, Mucha, Van Horn, & Turk 2002

What is a Particle System?

- Collection of many small simple particles that maintain *state* (position, velocity, color, etc.)
- Particle motion influenced by external *force fields*
- *Integrate* the laws of mechanics (ODE Solvers)
- To model: sand, dust, smoke, sparks, flame, water, etc.



Particle Motion

- mass *m*, position *x*, velocity *v*
- equations of motion:

$$\frac{d}{dt}x(t) = v(t)$$

$$\frac{d}{dt}v(t) = \frac{1}{m}F(x, v, t) \qquad F = ma$$

- Analytic solutions can be found for some classes of differential equations, but most can't be solved analytically
- Instead, we will numerically approximate a solution to our *initial value problem*

Higher Order ODEs

- Basic mechanics is a 2nd order ODE:

$$\frac{d^2}{dt^2}x = \frac{1}{m}F$$

- Express as 1st order ODE by defining *v*(*t*):

$$\frac{d}{dt}x(t) = v(t)$$
$$\frac{d}{dt}v(t) = \frac{1}{m}F(x,v,t)$$

$$\mathbf{X} = \begin{pmatrix} x \\ v \end{pmatrix} \qquad f(X,t) = \begin{pmatrix} v \\ \frac{1}{m}F(x,v,t) \end{pmatrix}$$

X is a vector storing the *current state* of the particle

f (*X,t*) describes how to *update* the state of the particle

Path Through a Field

- *f* (**X**,*t*) is a vector field defined everywhere
  - E.g. a velocity field which may change over time

Note: In the simplest particle systems, the particles do *not* interact with each other, only with external force fields

- ***X***(*t*) is a path through the field

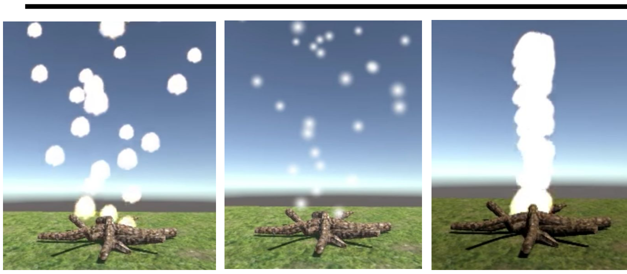
For a Collection of 3D particles…

$$\mathbf{X} = \begin{pmatrix} p_x^{(1)} \\ p_y^{(1)} \\ p_z^{(1)} \\ v_x^{(1)} \\ v_y^{(1)} \\ v_z^{(1)} \\ p_x^{(2)} \\ p_y^{(2)} \\ p_z^{(2)} \\ v_x^{(2)} \\ v_y^{(2)} \\ v_z^{(2)} \\ \vdots \end{pmatrix} \qquad f(\mathbf{X},t) = \begin{pmatrix} v_x^{(1)} \\ v_y^{(1)} \\ v_z^{(1)} \\ \frac{1}{m_1} F_x^{(1)}(\mathbf{X},t) \\ \frac{1}{m_1} F_y^{(1)}(\mathbf{X},t) \\ \frac{1}{m_1} F_z^{(1)}(\mathbf{X},t) \\ v_x^{(2)} \\ v_y^{(2)} \\ v_z^{(2)} \\ \frac{1}{m_2} F_x^{(2)}(\mathbf{X},t) \\ \frac{1}{m_2} F_y^{(2)}(\mathbf{X},t) \\ \frac{1}{m_2} F_z^{(2)}(\mathbf{X},t) \\ \vdots \end{pmatrix}$$

more generally, we can define *X* as a huge vector storing the current state of *all* particles in a system

Questions?

https://www.youtube.com/watch?v=M-Hz9Za5mCE
MixPixVisuals, Mikael Bellander



Note: current state *X* can also include color & transparency. And *f(X,t)* can animate changes in these values over time!

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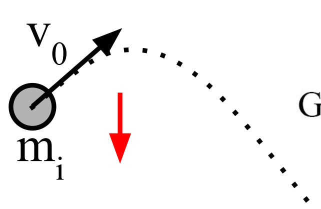
Forces: Gravity

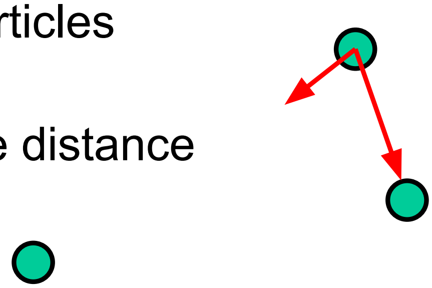
For smoke, flame: make "gravity" point up!

- Simple gravity: depends only on particle mass



$$\text{Gravity: } f^{(i)} = \begin{pmatrix} 0 \\ 0 \\ -m_i G \end{pmatrix}$$

- N-body problem: depends on all other particles
  - Magnitude inversely proportional to square distance
  - $F_{ij} = G\, m_i m_j / r^2$



Quickly gets impractical to compute analytically, and expensive to numerically approximate too!

Forces: Spatial Fields

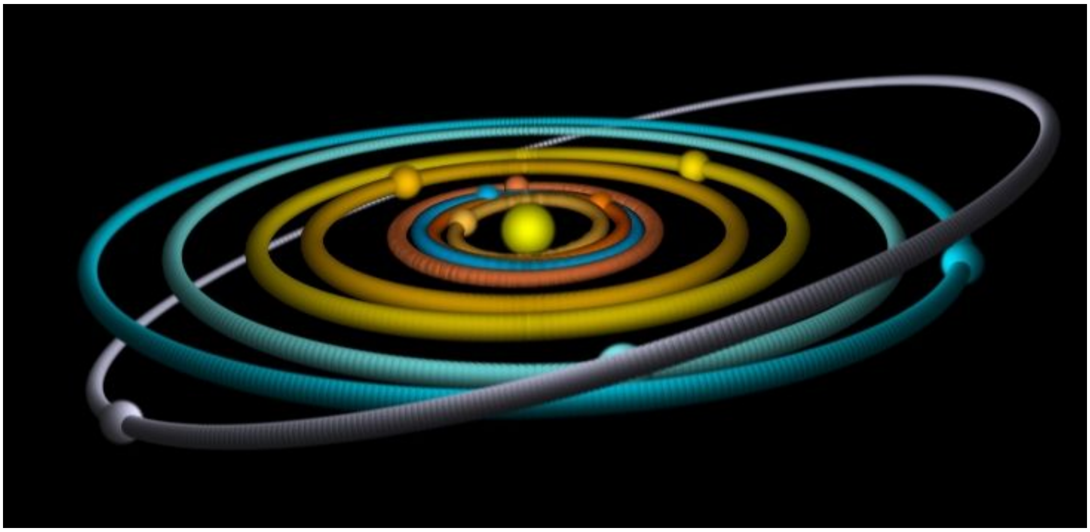
- Force on particle *i* depends only on position of *i*
  - wind
  - attractors
  - repulsers
  - vortices
- Can depend on time (e.g., wind gusts)
- Note: these forces will generally add energy to the system, and thus may need damping…

Forces: Damping

$$f^{(i)} = -dv^{(i)}$$

- Force on particle *i* depends only on velocity of *i*
- Force opposes motion
  - A hack mimicking real-world friction/drag
- Removes energy, so system can settle
- Small amount of damping can stabilize solver
- Too much damping makes motion too glue-like

Questions?



http://www.lactamme.polytechnique.fr/Mosaic/images/NCOR.U1.2048.D/display.html

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Euler's Method

- Examine *f* (**X**,*t*) at (or near) current state
- Take a step of size *h* to new value of **X**:

$$t_1 = t_0 + h$$
$$\mathbf{X}_1 = \mathbf{X}_0 + h\, f(\mathbf{X}_0, t_0)$$

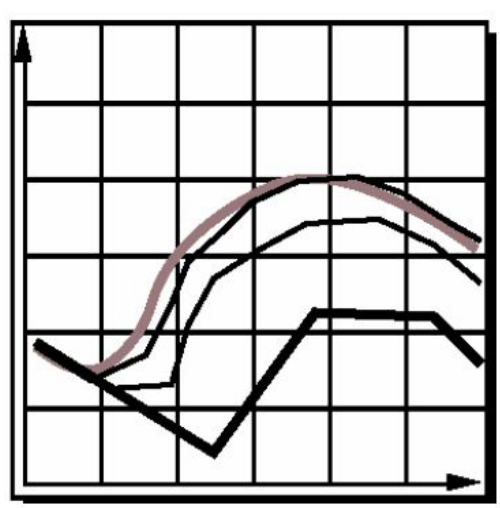
$$\mathbf{X} = \begin{pmatrix} x \\ v \end{pmatrix} \quad f(X,t) = \begin{pmatrix} v \\ \frac{1}{m} F(x,v,t) \end{pmatrix}$$

update the position by adding a little bit of the *current* velocity & update the velocity by adding a little bit of the *current* acceleration

- Piecewise-linear approximation to the curve

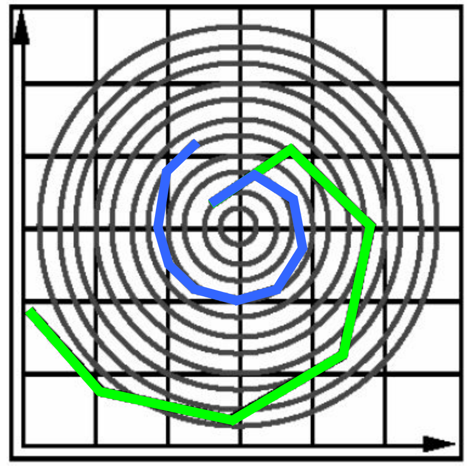
Effect of Step Size

- Step size controls accuracy
- Smaller steps more closely follow curve
- For animation, we may want to take *many* small steps per frame
  - How many frames per second for animation?
  - How many steps per frame?



Euler's Method: Inaccurate

- Simple example: particle in stable circular orbit around planet (origin)
- Current velocity is always tangent to circle
- Force is perpendicular to circle
- Euler method will spiral outward no matter how small *h* is

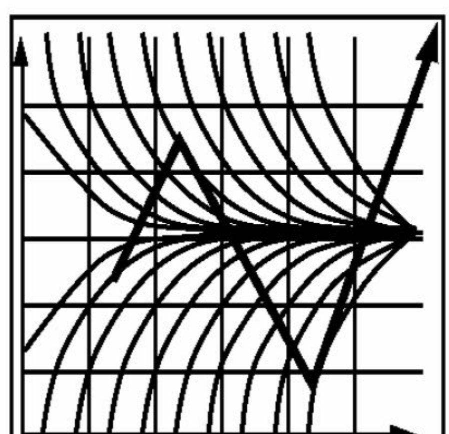


Euler's Method: Unstable

- Problem: $f(x,t) = -kx$
- Solution: $x(t) = x_0 e^{-kt}$
- Limited step size:

$$x_1 = x_0(1 - hk)$$

$$\begin{cases} h \le 1/k & \text{ok} \\ h > 1/k & \text{oscillates} \pm \\ h > 2/k & \text{explodes} \end{cases}$$



- If *k* is big, *h* must be small

Analysis using Taylor Series

- Expand exact solution **X***(t)*

$$\mathbf{X}(t_0+h)=\mathbf{X}(t_0)+h\left(\tfrac{d}{dt}\mathbf{X}(t)\right)\Big|_{t_0}+\tfrac{h^2}{2!}\left(\tfrac{d^2}{dt^2}\mathbf{X}(t)\right)\Big|_{t_0}+\tfrac{h^3}{3!}(\cdots)+\cdots$$

- Euler's method:

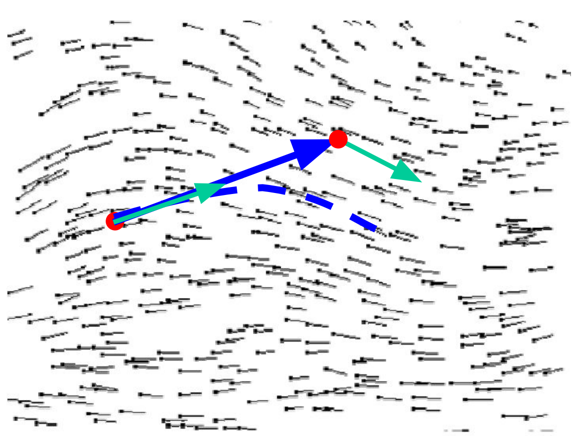
$$\mathbf{X}(t_0+h)=\mathbf{X}_0+h\,f(\mathbf{X}_0,t_0)\qquad \ldots+O(h^2)\,\text{error}$$

$$h\to h/2\ \Rightarrow\ error\to error/4\,\text{per step}\times\text{twice as many steps}$$
$$\to error/2$$

- First-order method: Accuracy varies with *h*
  - To get 100x better accuracy need 100x more steps

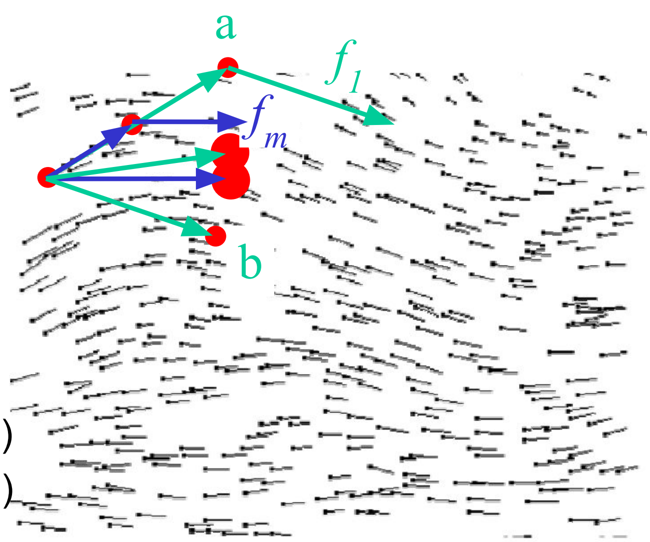
Can we do better than Euler's Method?

- Problem: *f* has varied along the step
- Idea: look at *f* at the arrival of the step and compensate for variation



2nd-Order Methods

- Midpoint:
  - ½ Euler step
  - evaluate f_m
  - full step using f_m
- Trapezoid:
  - Euler step (a)
  - evaluate f_1
  - full step using f_1 (b)
  - average (a) and (b)

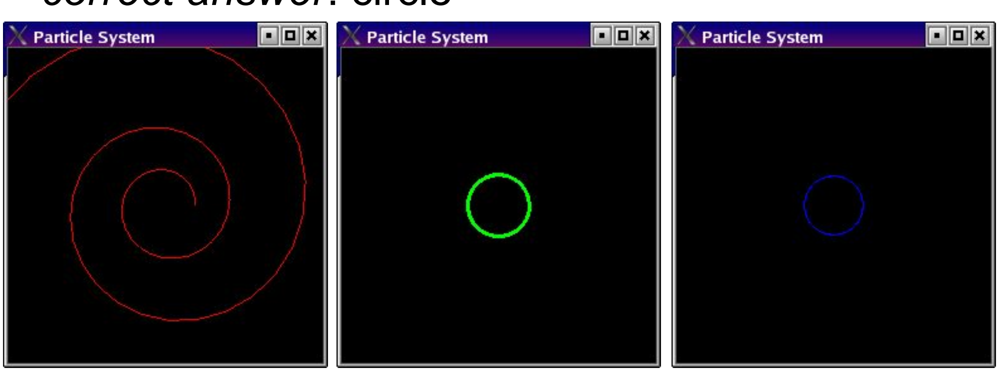


- Midpoint & trapezoid do not yield exactly the same result, but they have same order of accuracy

Comparison: Euler, Midpoint, Runge-Kutta

A 4th order method!

- *initial position*: (1,0,0)
- *initial velocity*: (0,5,0)
- *force field*: pulls particles to origin with magnitude proportional to distance from origin
- *correct answer*: circle

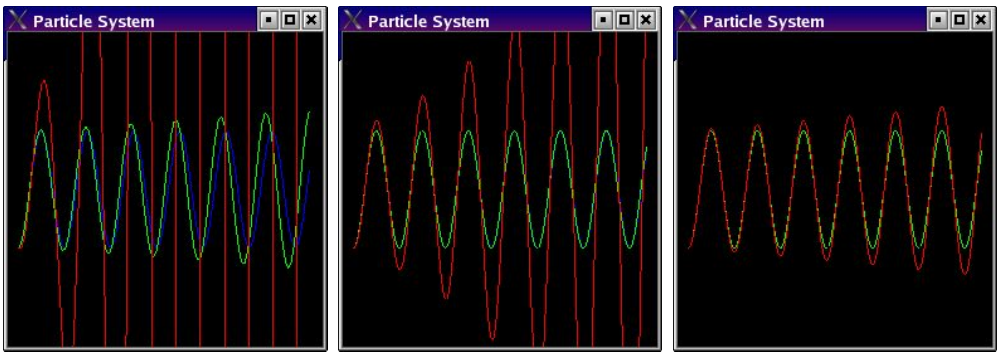


Euler will always diverge (even with small dt)

Comparison: Euler, Midpoint, Runge-Kutta

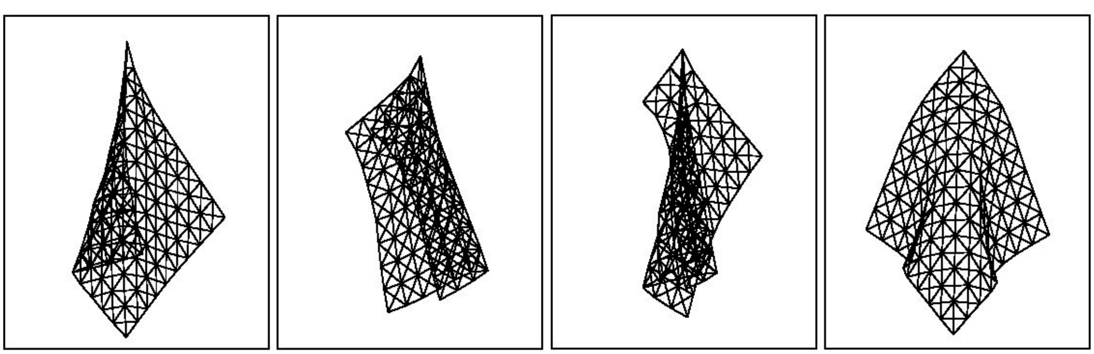
A 4^{th} order method!

- *initial position*: (0,-2,0)
- *initial velocity*: (1,0,0)
- *force field*: pulls particles to line y=0 with magnitude proportional to distance from line
- *correct answer*: sine wave



Decreasing the timestep (dt) improves the accuracy

Questions?



Interactive Animation of Structured Deformable Objects
Desbrun, Schröder, & Barr 1999

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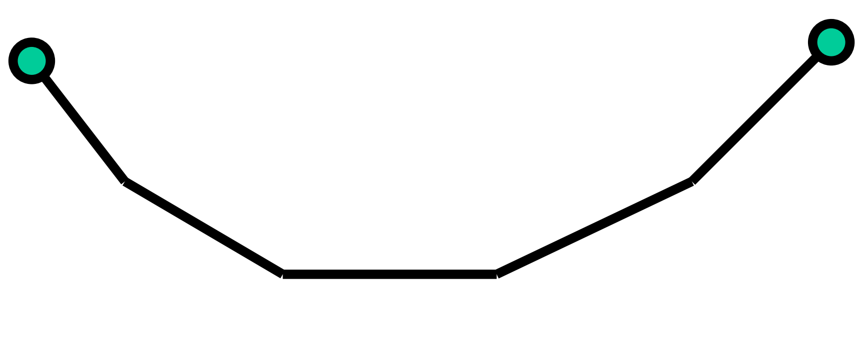
How would you simulate a string?

- Each particle is linked to two particles
- Forces try to keep the distance between particles constant
- What force?

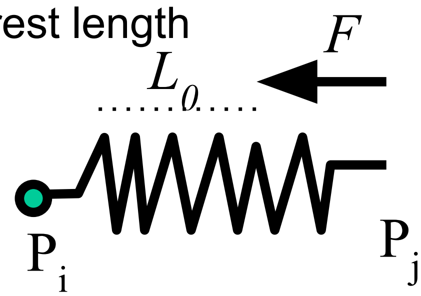


Spring Forces

- Force in the direction of the spring and proportional to difference with rest length L_0

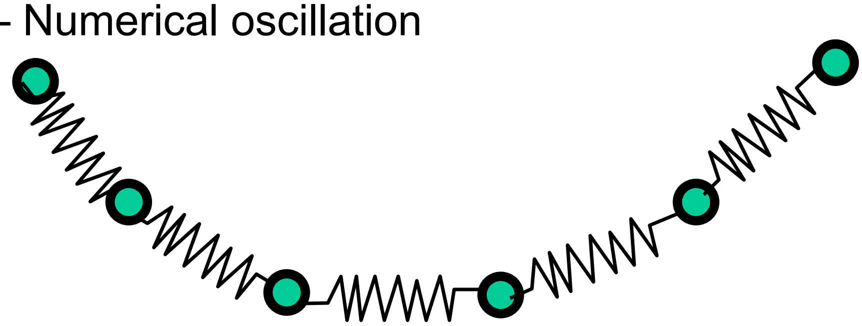
$$F(P_i, P_j) = K(L_0 - ||\vec{P_iP_j}||)\frac{\vec{P_iP_j}}{||\vec{P_iP_j}||}$$

- K is the stiffness of the spring
  - When K gets bigger, the spring really wants to keep its rest length



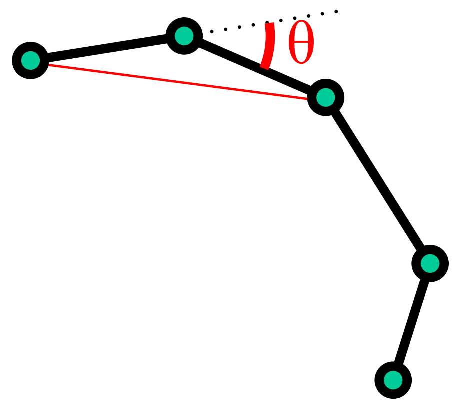
How would you simulate a string?

- Springs link the particles
- Springs try to keep their rest lengths and preserve the length of the string
- Problems?
  - Stretch, actual length will be greater than rest length
  - Numerical oscillation



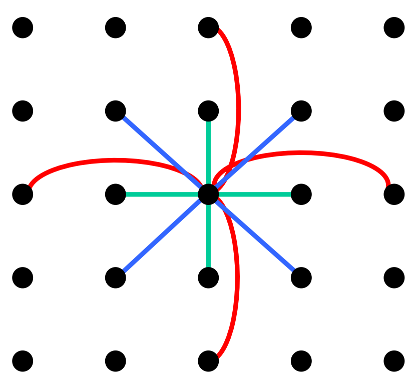
How would you simulate hair?

- Similar to string…
- Also… to keep hair straight or curly
  - Add forces based on the angle between segments
  - Add additional springs/constraints stretching between the non-immediate neighbors



Cloth Modeled with Mass-Spring

- Network of masses and springs
- Structural springs:
  - link (i, j) & (i+1, j) and (i, j) & (i, j+1)
- Shear springs
  - link (i, j) & (i+1, j+1) and (i+1, j) & (i, j+1)
- Flexion (Bend) springs
  - link (i, j) & (i+2, j) and (i, j) & (i, j+2)
- Be careful not to index out of bounds on the cloth edges!



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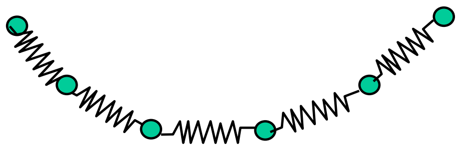
The Stiffness Issue

- What relative stiffness do we want for the different springs in the network?
- Cloth is barely elastic, shouldn't stretch so much!
- Inverse relationship between stiffness & Δt
- We really want constraints (not springs)
- Many numerical solutions
  - reduce Δt
  - use constraints
  - implicit integration
  - …

How would you simulate a string?

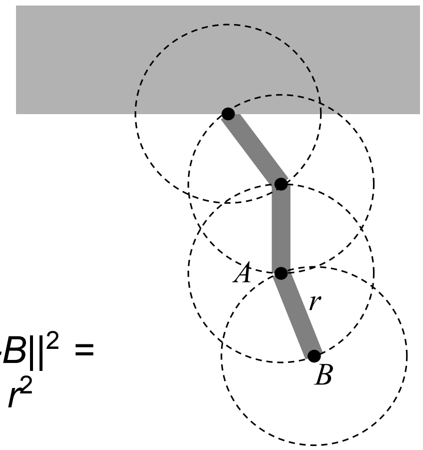
- Springs link the particles. Problems?
  - Stretch, actual length will be greater than rest length
  - Numerical oscillation



- Rigid, fixed-length bars link the particles
  - Dynamics &
  - Constraints

(must be solved simultaneously non-trivial, even for tiny systems)

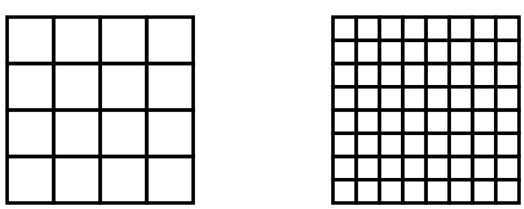
$\|A-B\|^2 = r^2$



https://www.youtube.com/watch?v=AwT0k09w-jw

The Discretization Problem

- What happens if we discretize our cloth more finely, or with a different mesh structure?



- Do we get the same behavior?
  - Usually not! It takes a lot of effort to design a scheme that does not depend on the discretization.
- Using (explicit) Euler, how many timesteps before a force propagates across the mesh?

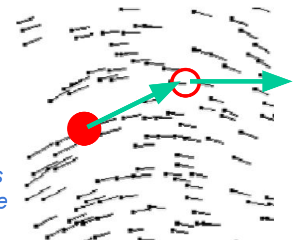
Explicit vs. Implicit Integration

- With an explicit/forward integration scheme:

  $$\mathbf{y}_{k+1} = \mathbf{y}_k + h\,\mathbf{g}(y_k)$$

  The future state (position & velocity) of this particle is a function of the current state of the particle.

  we must use a very small timestep to simulate *stable, stiff* cloth.



- Alternatively we can use an implicit/backwards scheme:

  $$\mathbf{y}_{k+1} = \mathbf{y}_k + h\,\mathbf{g}(y_{k+1})$$
  $$\mathbf{y}_k = \mathbf{y}_{k+1} - h\,\mathbf{g}(y_{k+1})$$

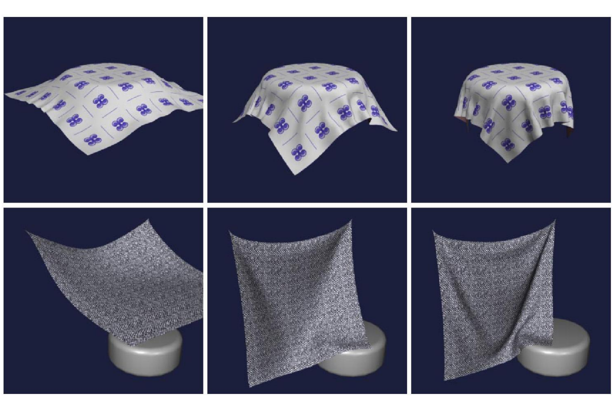
  The future state of this particle depends on the current state AND the future state.

  Solving one step is much more expensive (Newton's Method, Conjugate Gradients, …) but overall faster than the thousands of explicit timesteps required for very stiff springs.
- Larger timesteps are possible with implicit methods.

Questions?

David Baraff & Andrew Witkin
Large Steps in Cloth Simulation
SIGGRAPH 1998

- Dynamic motion driven by animation



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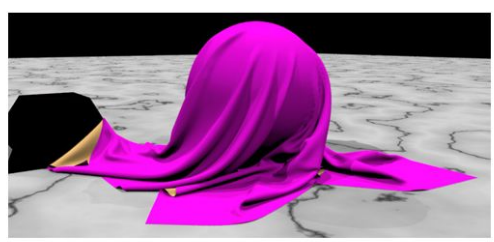
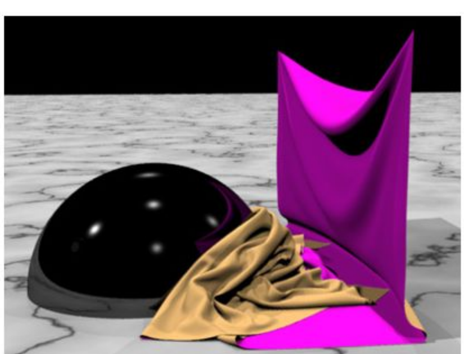
Cloth Collision

Robert Bridson, Ronald Fedkiw & John Anderson
Robust Treatment of Collisions, Contact and Friction for Cloth Animation
SIGGRAPH 2002



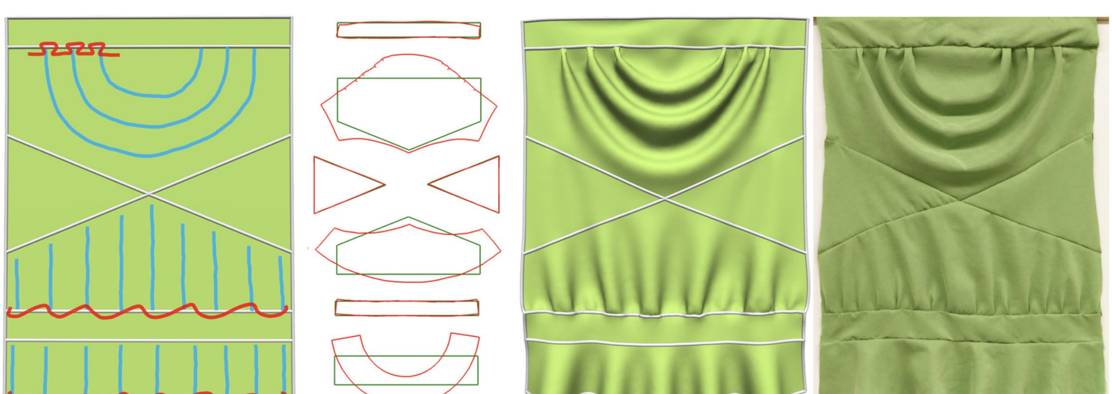
- A cloth has many points of contact
- Often stays in contact
- Requires
  - Efficient collision detection
  - Efficient numerical treatment (stability)



Artistic Simulation of Curly Hair

Iben, Meyer, Petrovic, Soares, Anderson, and Witkin
Symposium on Computer Animation 2013

"FoldSketch: Enriching Garments with Physically Reproducible Folds"
Li, Sheffer, Grinspun, & Vining, SIGGRAPH 2018.

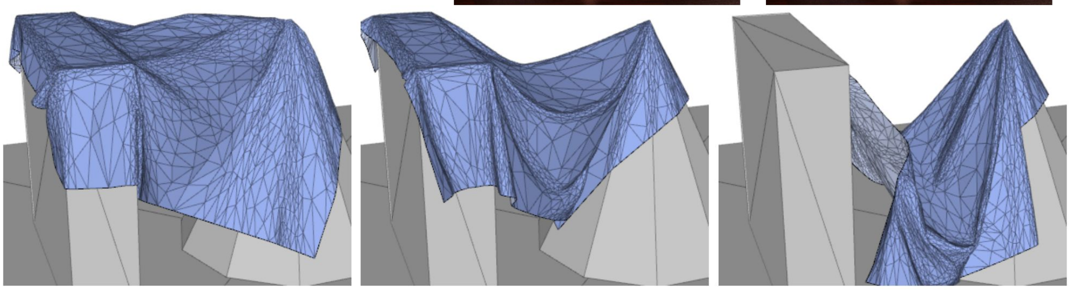


(a) input (b) patterns (c) output (d) manufactured flag

"An Implicit Frictional Contact Solver for Adaptive Cloth Simulation"

Li, Daviet, Narain, Bertails-Descoubes, Overby, Brown, & Boissieux, SIGGRAPH 2018.

$\mu = 0.0$ $\mu = 0.3$



Today

- Papers for Today
- Particle Systems
  - Equations of Motion (Physics)
  - Forces: Gravity, Spatial, Damping
  - Numerical Integration (Euler, Midpoint, etc.)
- Mass Spring System Examples
  - String, Hair, Cloth
- Stiffness
- Discretization
- Papers for Tuesday
- Worksheet on Volumetric Structures

Pop Worksheet!

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