Curves & Surfaces

https://www.moillusions.com/glass-water-optical-illusion/

One Line Bison, Tyler Foust, 2020

https://www.tylerfoust.com/
One Line Bison, Tyler Foust, 2020

https://www.tylerfoust.com/

Squaring the Circle, Troika, 2013

http://troika.uk.com/work/troika-squaring-the-circle/
Squaring the Circle, Troika, 2013

Herbstlaub

Oliver Vogel, Siggraph 2007
Last Time?

• Adjacency Data Structures
  – Geometric & topologic information
  – Dynamic allocation
  – Efficiency of access

• Mesh Simplification
  – edge collapse/vertex split
  – geomorphs
  – progressive transmission
  – view-dependent refinement
Today

- Worksheet: Shortest Edge Collapse
- Reading: “Teddy: A Sketching Interface for 3D Freeform Design”
- Limitations of Polygonal Models
- What's a Spline?
- Bézier Spline
- BSpline (NURBS)
- Extending to Surfaces & Paper for Friday

Pop Worksheet!

- Perform a sequence of 3 edge collapses, one-at-a-time
- Always collapse the shortest edge that does not result in a zero area or “flipped”/upside-down triangle
- Replacement vertex should be at the midpoint of the edge
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## Reading for Today

- "Teddy: A Sketching Interface for 3D Freeform Design", Igarashi et al., SIGGRAPH 1999

- How do we represent objects that don’t have flat polygonal faces & sharp corners? What are the right tools to design/construct digital models of blobby, round, or soft things? What makes a user interface intuitive, quick, and easy-to-use for beginners?
• Attention to UI, lowering the barrier to entry for novices
• Simple algorithm
• Limitation: does not work for non-spherical base shapes
• Challenge: making 3D shape with 2D input
• Tradeoff: simplicity vs. fully-featured modeling software
• “Direct manipulation” – draw contours on screen rather than typing numbers into boxes physically separated from visual result
• Has Teddy made an impact on modeling software? If not, why not?

What NOT to write about the assigned reading:

• "There was alot of math in the paper. Math is hard. I didn't understand the math."
• "This paper was published in the dark ages using slow computers. I wonder how fast it would be with a GPU."
• "The pictures were pretty. I liked watching the video."
• “Now that we have AI/ML, the results will be much better.”
Today

- Worksheet: Shortest Edge Collapse
- Reading: “Teddy: A Sketching Interface for 3D Freeform Design”
- Limitations of Polygonal Models
  - Interpolating Color & Normals in OpenGL
  - Some Modeling Tools & Definitions
- What's a Spline?
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Limitations of Polygonal Meshes

- Planar facets (& silhouettes)
- Fixed resolution
- Deformation is difficult
- No natural parameterization (for texture mapping)
- Incorrect collision detection
- Solid texturing problems
Color & Normal Interpolation
• It’s easy in OpenGL to specify different colors and/or normals at the vertices of triangles:
• Why is this useful?

What is Gouraud Shading?
• Instead of shading with the normal of the triangle, we’ll shade the vertices with the average normal and interpolate the shaded color across each face
  – This gives the illusion of a smooth surface with smoothly varying normals

• How do we compute Average Normals? Is it expensive??
**Phong Normal Interpolation**  
(Not Phong Shading)

- *Interpolate the average vertex normals* across the face and compute *per-pixel shading*
  - Normals should be re-normalized (ensure length=1)

Before shaders, per-pixel shading was not possible in hardware (Gouraud shading is actually a decent substitute!)

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**Some Non-Polygonal Modeling Tools**

- **Extrusion**
- **Surface of Revolution**
- **Spline Surfaces/Patches**
- **Quadrics and other implicit polynomials**
Continuity definitions:

- \( C^0 \) continuous
  - curve/surface has no breaks/gaps/holes
- \( G^1 \) continuous
  - tangent at joint has same direction
- \( C^1 \) continuous
  - curve/surface derivative is continuous
  - tangent at joint has same direction \textit{and} magnitude
- \( C^n \) continuous
  - curve/surface through \( n^{th} \) derivative is continuous
  - important for shading

"Shape Optimization Using Reflection Lines", Tosun et al., 2007

Questions?
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- What's a Spline?
  - Interpolation Curves vs. Approximation Curves
  - Linear Interpolation
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Definition: What's a Spline?

- Smooth curve defined by some control points
- Moving the control points changes the curve

Interpolation

Bézier (approximation)

BSpline (approximation)
Interpolation Curves / Splines

- Curve is constrained to pass through all control points

- Given points $P_0$, $P_1$, ... $P_n$, find lowest degree polynomial which passes through the points

$$x(t) = a_{n-1}t^{n-1} + \ldots + a_2t^2 + a_1t + a_0$$

$$y(t) = b_{n-1}t^{n-1} + \ldots + b_2t^2 + b_1t + b_0$$
Linear Interpolation

• Simplest "curve" between two points

\[ Q(t) = (1 - t) P_0 + t P_1 \]

Splines Basis Functions

Spline Basis Functions

a.k.a. Blending Functions

\[
Q(t) = \begin{pmatrix} Q_x(t) \\ Q_y(t) \\ Q_z(t) \end{pmatrix} = \begin{pmatrix} P_0 \\ P_1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix}
\]

\[ Q(t) = GBT(t) = \text{Geometry } G \cdot \text{Spline Basis } B \cdot \text{Power Basis } T(t) \]

Interpolation vs. Approximation Curves

Interpolation

curve must pass through control points

Approximation

curve is influenced by control points
Interpolation vs. Approximation Curves

- Interpolation Curve – over constrained → lots of (undesirable?) oscillations

- Approximation Curve – more reasonable?

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Cubic Bézier Curve

- 4 control points
- Curve passes through first & last control point
- Curve is tangent at $P_1$ to $(P_2 - P_1)$ and at $P_4$ to $(P_4 - P_3)$

A Bézier curve is bounded by the convex hull of its control points.
Cubic Bézier Curve

- de Casteljau's algorithm for constructing Bézier curves

\[ Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4 \]

\[ B(t) = GBT(t) \]

Bernstein Polynomials

\[ B_1(t) = (1-t)^3; B_2(t) = 3t(1-t)^2; B_3(t) = 3t^2(1-t); B_4(t) = t^3 \]
Connecting Cubic Bézier Curves

- How can we guarantee $C^0$ continuity?
- How can we guarantee $G^1$ continuity?
- How can we guarantee $C^1$ continuity?
- Can’t guarantee higher $C^2$ or higher continuity

Asymmetric: Curve goes through some control points but misses others

Connecting Cubic Bézier Curves

- Where is this curve
  - $C^0$ continuous?
  - $G^1$ continuous?
  - $C^1$ continuous?
- What’s the relationship between:
  - the # of control points, and
  - the # of cubic Bézier subcurves?
Higher-Order Bézier Curves

• > 4 control points
• Bernstein Polynomials as the basis functions

\[ B^n_i(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \quad 0 \leq i \leq n \]

• Every control point affects the entire curve
  – Not simply a local effect
  – More difficult to control for modeling

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Cubic BSplines

- ≥ 4 control points
- Locally cubic
- Curve is not constrained to pass through any control points

A BSpline curve is also bounded by the convex hull of its control points.
Cubic BSplines

- Iterative method for constructing BSplines

Shirley, Fundamentals of Computer Graphics

Cubic BSplines

\[ Q(t) = \frac{(1-t)^3}{6} P_{t-3} + \frac{3t^3 - 6t^2 + 4}{6} P_{t-2} + \frac{-3t^3 + 3t^2 + t + 1}{6} P_{t-1} + \frac{t^3}{6} P_t \]

\[ Q(t) = GBT(t) \]

\[ B_{B-Spline} = \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]
Connecting Cubic BSpline Curves

- Can be chained together
- Better control locally (windowing)

What’s the relationship between
- the # of control points,
- and
- the # of cubic BSpline subcurves?
BSpline Curve Control Points

Default BSpline
BSpline with Discontinuity
BSpline which passes through end points
Repeat interior control point
Repeat end points

Bézier is not the same as BSpline

Bézier
BSpline
Bézier is not the same as BSpline

- Relationship to the control points is different

Converting between Bézier & BSpline

original control points as Bézier

new BSpline control points to match Bézier

new Bézier control points to match BSpline

original control points as BSpline
Converting between Bézier & BSpline

Using the basis functions:

\[ Q(t) = GBT(t) = \text{Geometry } G \cdot \text{Spline Basis } B \cdot \text{Power Basis } T(t) \]

\[ G_{\text{Bezier}} \cdot B_{\text{Bezier}} \cdot T = G_{\text{BSpline}} \cdot B_{\text{BSpline}} \cdot T \]

\[ G_{\text{Bezier}} = \frac{G_{\text{BSpline}} \cdot B_{\text{BSpline}} \cdot T}{B_{\text{Bezier}} \cdot T} \]

\[ B_{\text{Bezier}} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]

\[ B_{\text{BSpline}} = \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]

NURBS (generalized BSplines)

- BSpline: uniform cubic BSpline
- NURBS: Non-Uniform Rational BSpline
  - non-uniform = different spacing between the blending functions, a.k.a. knots
  - rational = ratio of polynomials (instead of cubic)
Neat Bezier Spline Trick

- A Bezier curve with 4 control points:
  - $P_0 \ P_1 \ P_2 \ P_3$
- Can be split into 2 new Bezier curves:
  - $P_0 \ P'_1 \ P'_2 \ P'_3$
  - $P'_3 \ P'_4 \ P'_5 \ P_3$

A Bézier curve is bounded by the convex hull of its control points.

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Spline Surface via Tensor Product

- Of two vectors:

\[
\begin{bmatrix}
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3 & b_4
\end{bmatrix} \otimes \begin{bmatrix}
    a_1 b_1 & a_2 b_1 & a_3 b_1 \\
    a_1 b_2 & a_2 b_2 & a_3 b_2 \\
    a_1 b_3 & a_2 b_3 & a_3 b_3 \\
    a_1 b_4 & a_2 b_4 & a_3 b_4
\end{bmatrix}
\]

- Similarly, we can define a surface as the tensor product of two curves.

Bilinear Patch

Bi-lerp a (typically non-planar) quadrilateral

Notation: \( L(P_1, P_2, \alpha) \equiv (1 - \alpha)P_1 + \alpha P_2 \)

\[
Q(s, t) = L(L(P_1, P_2, t), L(P_3, P_4, t), s)
\]
Bilinear Patch

- Smooth version of quadrilateral with non-planar vertices...

- But will this help us model smooth surfaces?
- Do we have control of the derivative at the edges?

Ruled Surfaces in Art & Architecture

http://www.bergenwood.no/wp-content/media/images/frozenmusic.jpg

Chiras Iulia
Astri Isabella
Matiss Shteinerts

Antoni Gaudi
Children’s School
Barcelona

http://www.lonelyplanetimages.com/images/399954
Bicubic Bezier Patch

Notation: $\text{CB}(P_1, P_2, P_3, P_4, \alpha)$ is Bézier curve with control points $P_i$ evaluated at $\alpha$

Define “Tensor-product” Bézier surface

$$Q(s, t) = \text{CB}( \text{CB}(P_{00}, P_{01}, P_{02}, P_{03}, t), \text{CB}(P_{10}, P_{11}, P_{12}, P_{13}, t), \text{CB}(P_{20}, P_{21}, P_{22}, P_{23}, t), \text{CB}(P_{30}, P_{31}, P_{32}, P_{33}, t), s)$$

Editing Bicubic Bezier Patches

Curve Basis Functions

Surface Basis Functions
Bicubic Bezier Patch Tessellation

- Given 16 control points and a tessellation resolution, we can create a triangle mesh:
  - resolution: 5x5 vertices
  - resolution: 11x11 vertices
  - resolution: 41x41 vertices

Modeling with Bicubic Bezier Patches

- Original Teapot specified with Bezier Patches

- But it’s not "watertight": it has intersecting surfaces at spout & handle, no bottom, a hole at the spout tip, a gap between lid & base
Trimming Curves for Patches

Shirley, Fundamentals of Computer Graphics

Spline-Based Modeling Headaches

irregular sampling

“pinched” surfaces

seams & holes
Questions?

• Bezier Patches?
  or

• Triangle Mesh?

Henrik Wann Jensen

Readings for Next Time *(pick one)*

• Hoppe et al., “Piecewise Smooth Surface Reconstruction” SIGGRAPH 1994

  Triangle meshes directly applies to HW1!

• DeRose, Kass, & Truong, "Subdivision Surfaces in Character Animation", SIGGRAPH 1998

  Quad Meshes more common in artistic practice (e.g. Pixar’s Geri’s Game)
Homework 1:

Expectations for “Progress Posts”?