CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/F23/

Lecture 9: Point Location & Trapezoidal Maps

Outline for Today

- Homework 4 Questions?
- Last Time: kD Trees & Range Trees
- Motivating Application: Point Location
- Motivating Application: 2D/3D Mouse “Picking” for Graphics
- Brute Force Point Location
- Point Location by Vertical Slab
- Trapezoidal Map & Adjacency Structure
- Trapezoidal Map Analysis & Construction
- Think-Outside-of-the-Box Graphics Picking Algorithm
- Next Time: Voronoi Diagram

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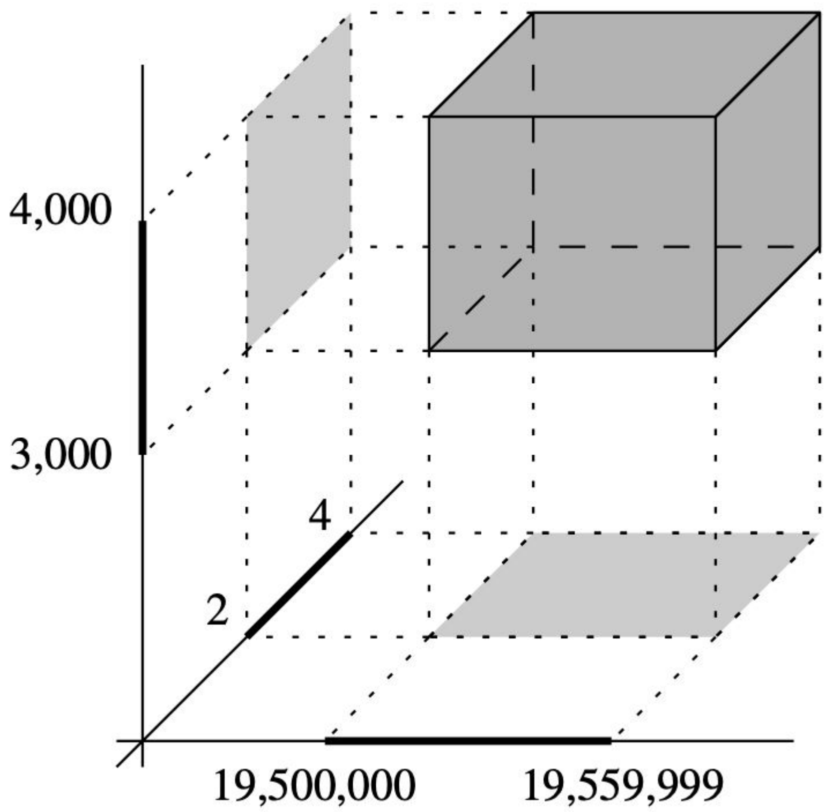
Higher Dimensional Database Queries

- Return all data points with x value in range $[x_0, x_1]$ and y value in range $[y_0, y_1]$ and z value in range $[z_0, z_1]$ and …

Select all people born 1950-1960, with salary 3,000-4,000, who have 2-4 children

Find all values in an axis parallel box:
a “*rectangular range query*”
a.k.a. “*orthogonal range query*”



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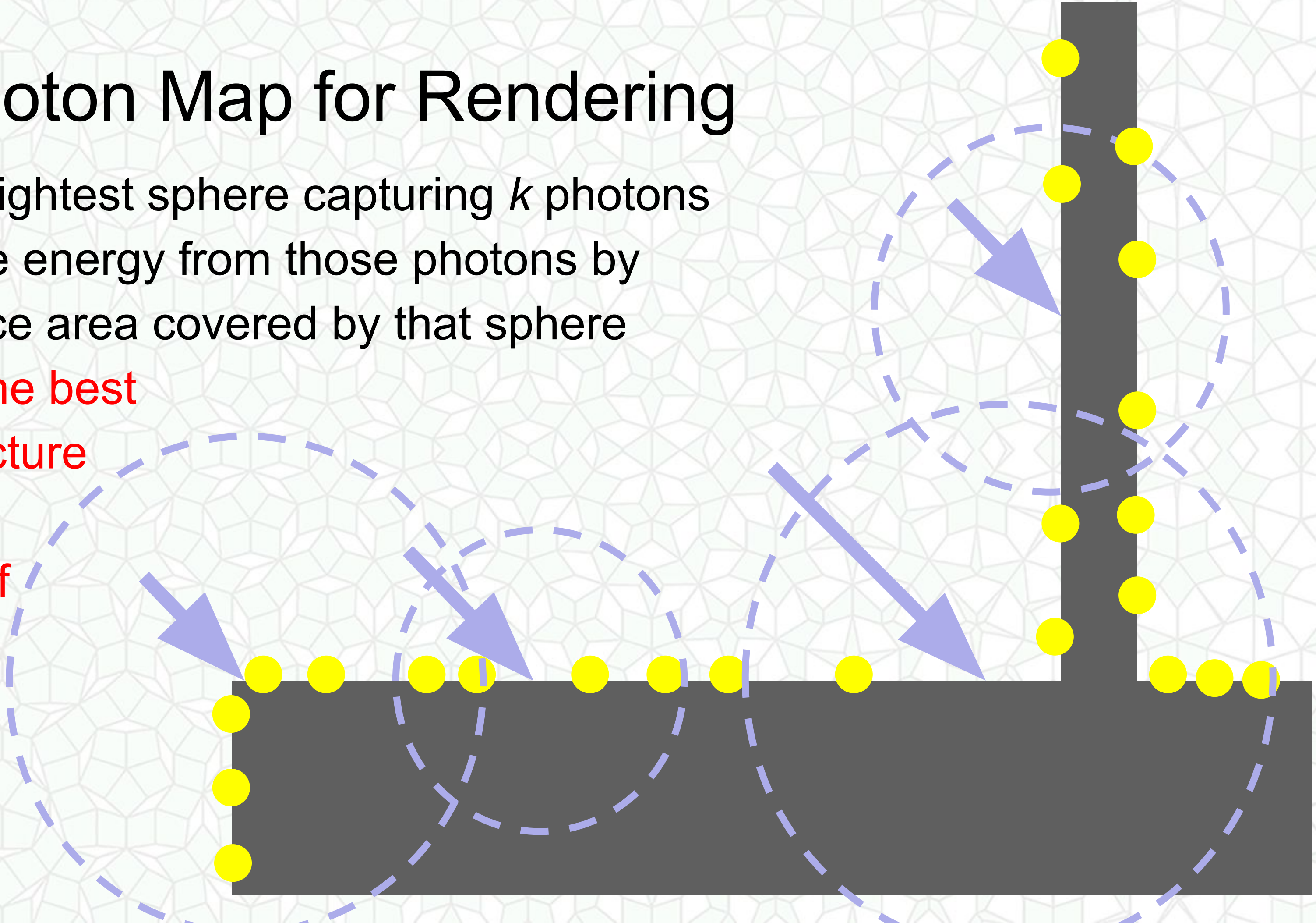
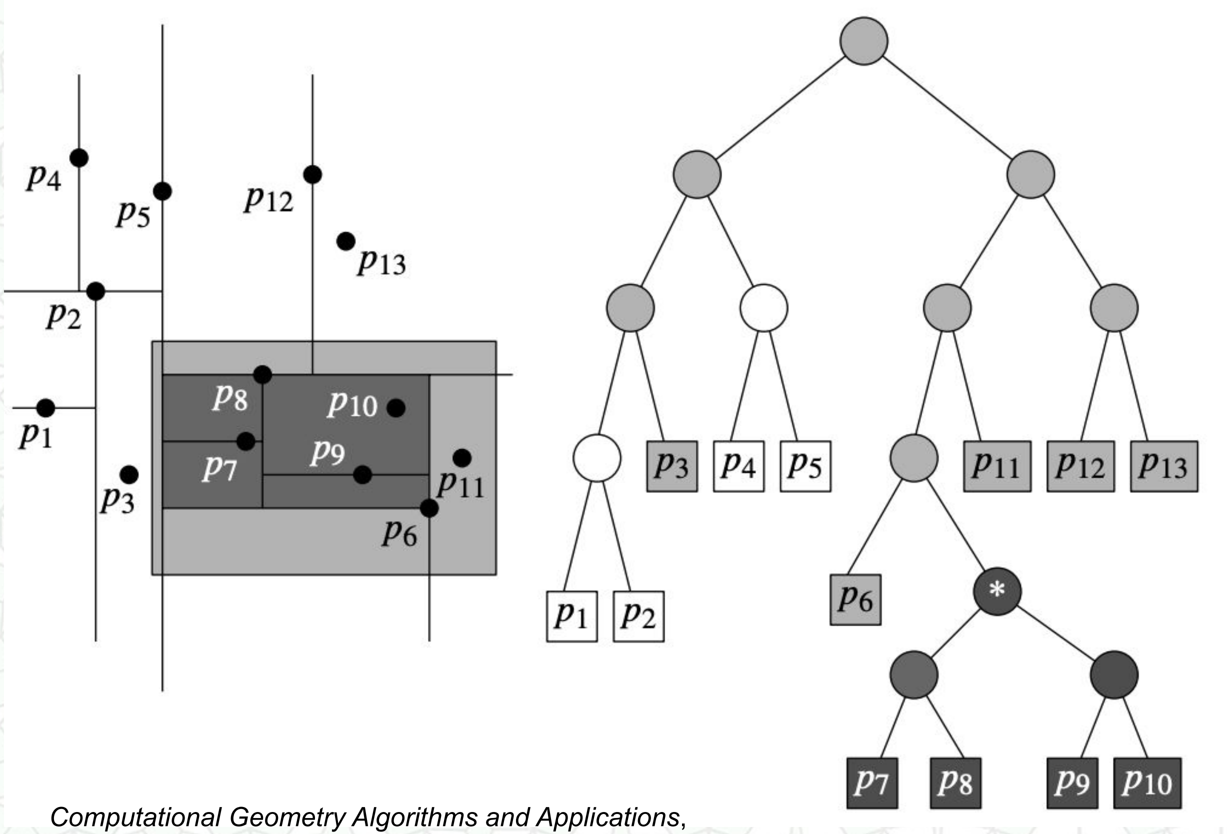
Using Photon Map for Rendering

- Find the tightest sphere capturing *k* photons
- Divide the energy from those photons by the surface area covered by that sphere
- What is the best data structure to store millions of photons?



2D kd Tree Query Algorithm

- At each split point
- Determine if the query box overlaps the split line
- Recurse down one or both branches
- If a subtree lies complete inside the box, return all items in that subtree
- Perform filtering in the leaves as necessary



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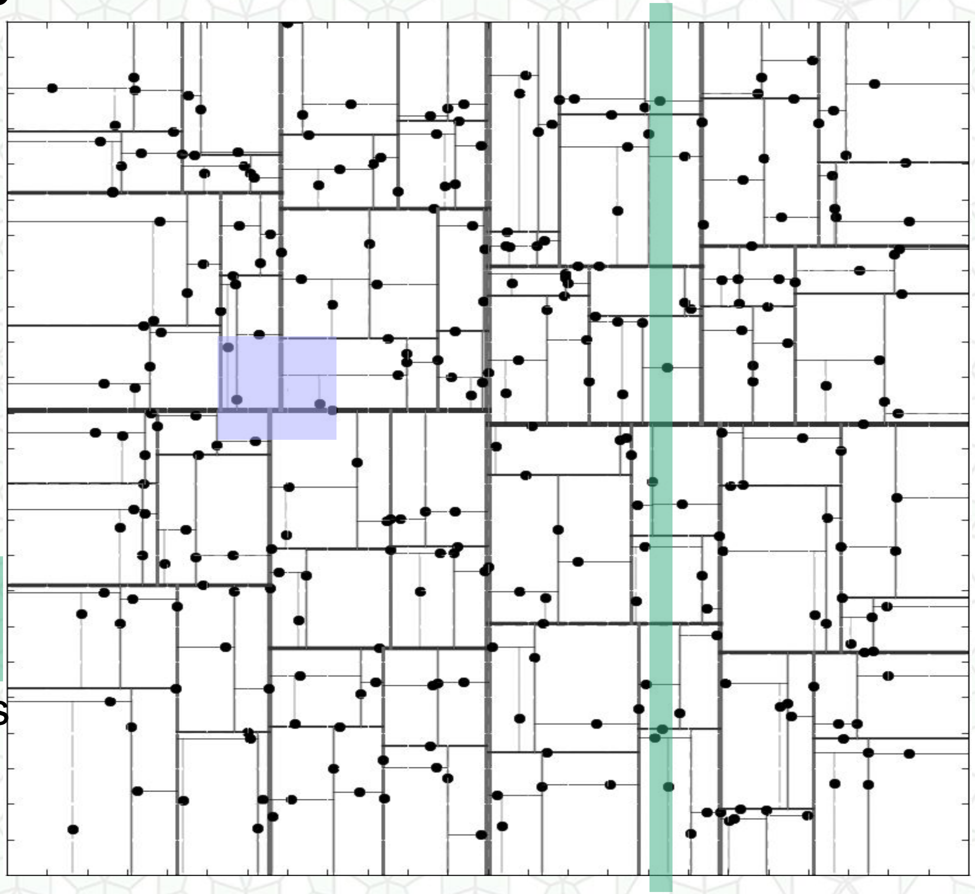
2D kd Tree Query Analysis

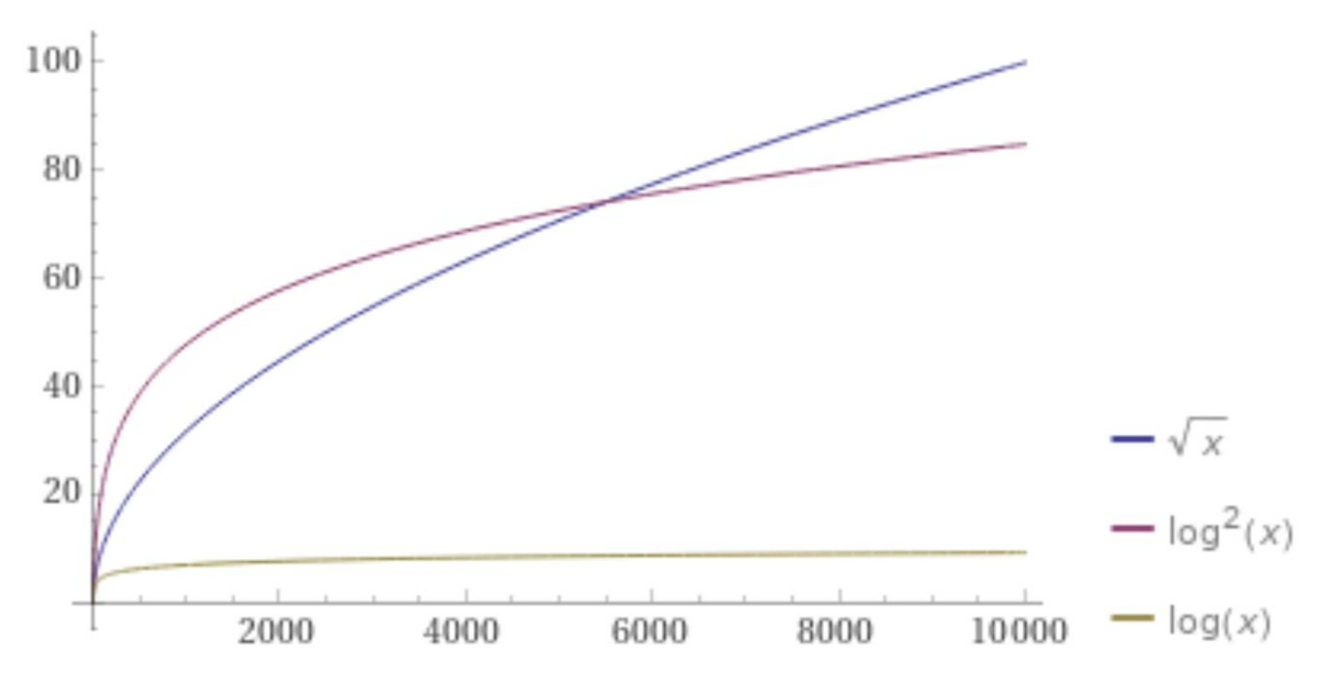
https://salzis.wordpress.com/2014/06/28/kd-tree-and-nearest-neighbor-nn-search-2d-case/

- 1 item is stored per leaf node
- For a query that will collect *k* items
- Best/Average(?) Case:
  An approximately square query
  (equal width & height)
  - touches/overlaps *O(k)* leaves
  - gathering leaves *O(log n + k)*
  - *Overall* → O(log n + k)
- Worst Case Query:
  For a skinny / lopsided query box
  - touches/overlaps - √*n* +*k* leaves
  - gathering leaves O(√*n* +*k)*
  - *Overall* → O(√*n* + k)



Is Query Time = $O(\sqrt{n} + k)$ *a problem?*

- $O(1) < O(\log n) < O(\log^2 n) < O(\sqrt{n}) < O(n)$

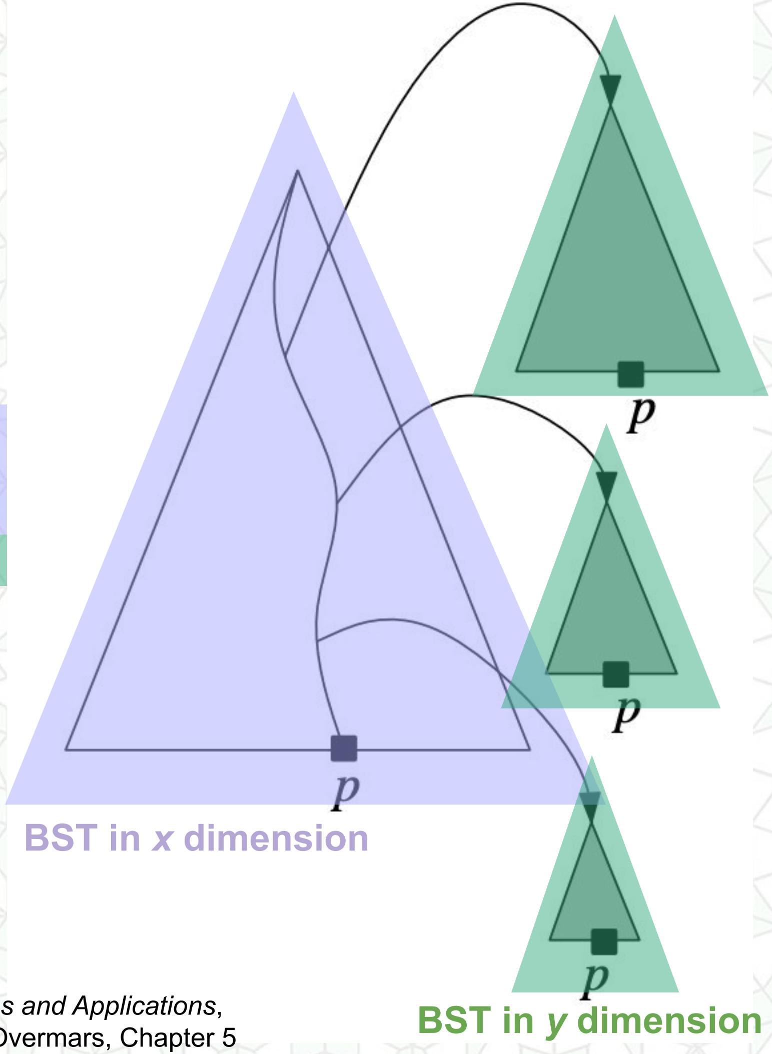


2D Range Tree
(and higher dimension!)

How much memory does it use?

- Each point p is stored once in the level 1 (organized by x) tree
- And many times in level 2 (organized by y) trees
- How many level 2 trees? And how big are they?
  - 1 tree with n values
  - 2 trees with n/2 values
  - 4 trees with n/4 values
  - …
  - *n* trees with 1 values

→ *O(n log n)* memory



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Summary Comparison

- For *n* points, dimension *d*, with query to collect *k* items
- *k*d tree
  - Construction time: → $O(n \log n)$
  - Memory: → $O(n)$
  - Query time
    - Square(ish) box: → $O(\log n + k)$
    - Worst case (long, skinny box): → $O(n^{(1-1/d)} + k)$
- Range tree
  - Construction time → $O(n \log^{d-1} n)$
  - Memory → $O(n \log^{d-1} n)$
  - Query time → $O(\log^{d} n + k)$

Tradeoff:
Use more memory
Faster runtime

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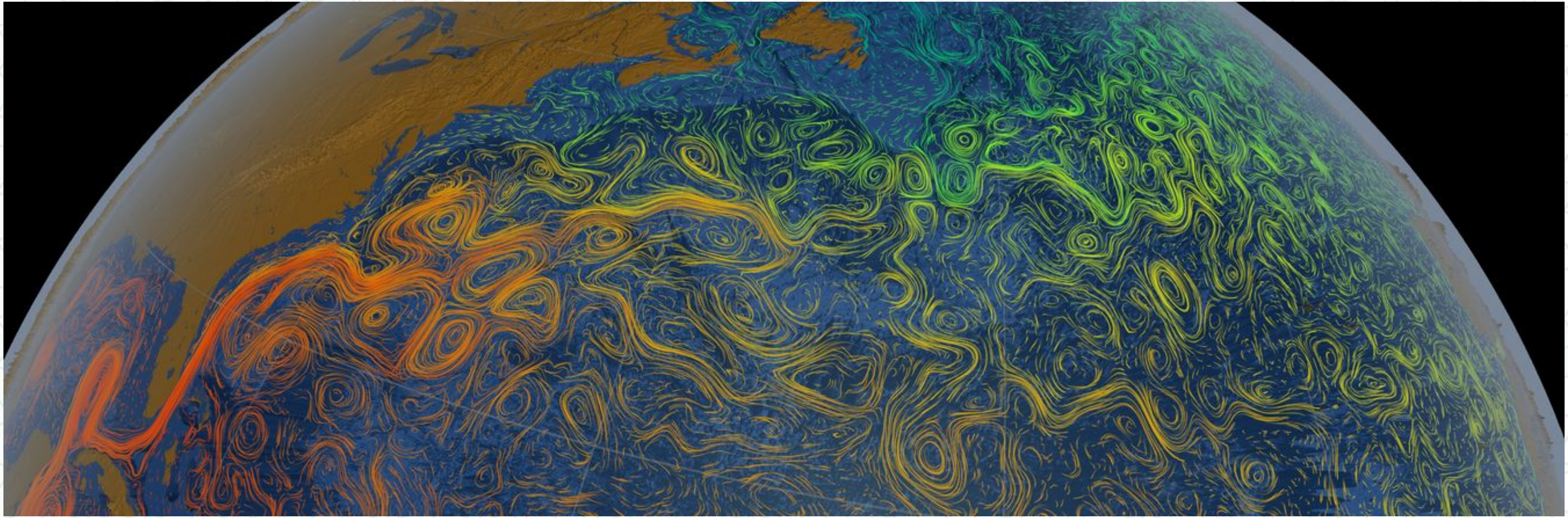
Motivation Application: GPS Point Localization

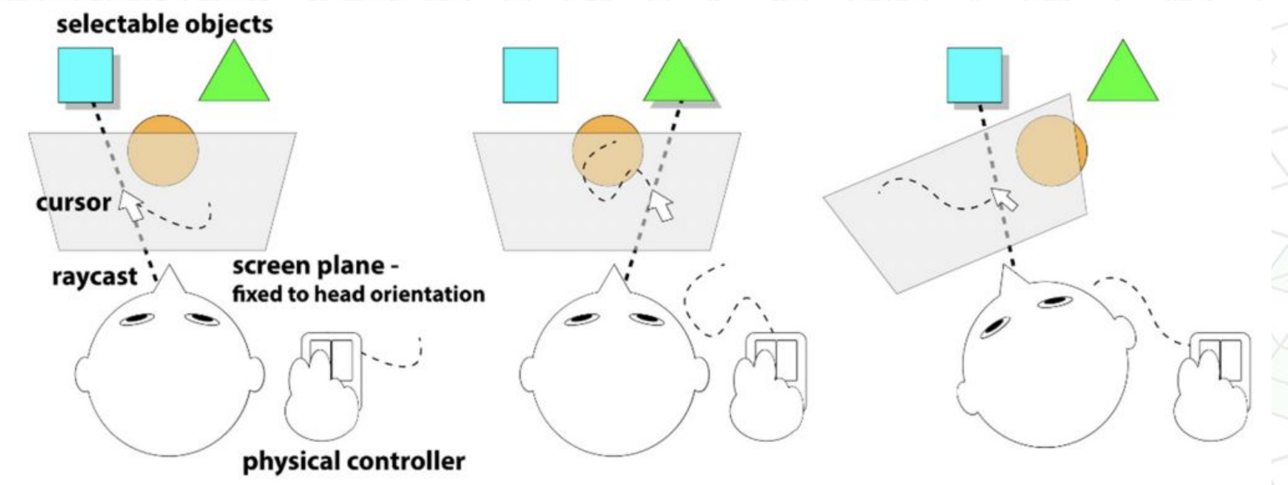
- Given a 2D coordinate, e.g., a latitude & longitude
- What region of the ocean contains this point?
  - Access currents, weather, etc.

NASA Scientific Visualization Studio
https://svs.gsfc.nasa.gov/



Graphics / Virtual Reality: What is "Picking"?

- Get the (3D) world coordinates of a (2D) mouse click
- Identify which object was selected and
  the point on the object closest to the click
- *Do we as users take this for granted??*
  - *What are the performance bottlenecks?*
  - *What are the usability concerns?*



https://www.csit.carleton.ca/~rteather/pdfs/GI_2018_EZCursorVR.pdf

Graphics Application: 3D Painting

http://www-ui.is.s.u-tokyo.ac.jp/~takeo/gallery/chameleon.png

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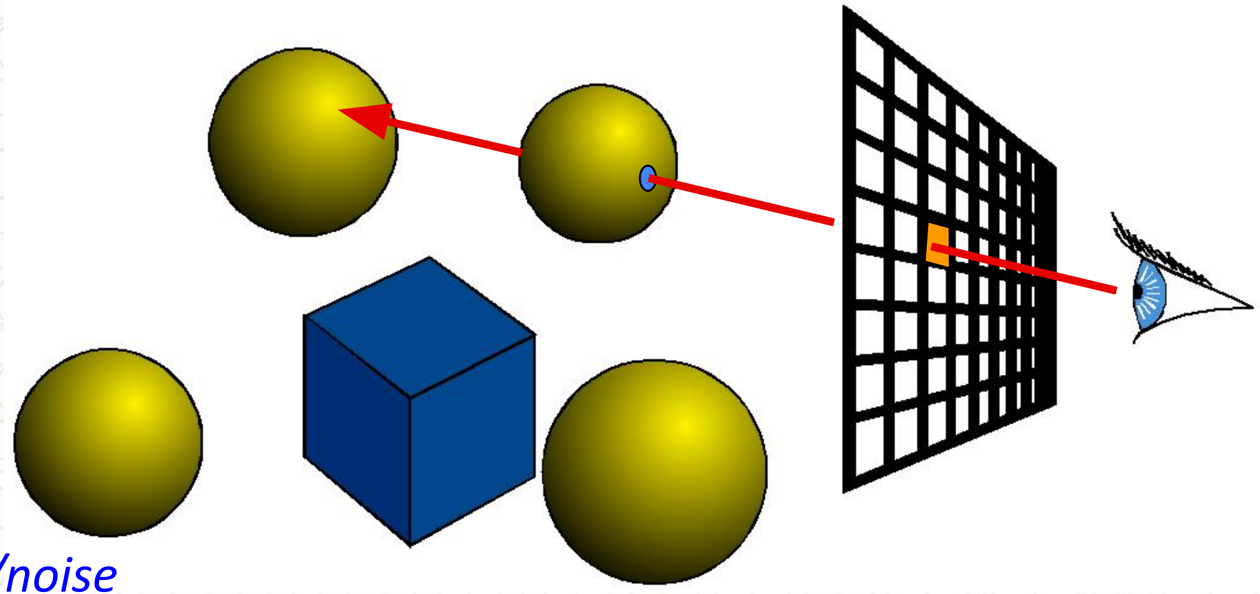
"Picking" by Ray Casting

- Construct a ray from the eye through the image plane into the scene
- Intersect with all objects in the scene
- Keep the closest



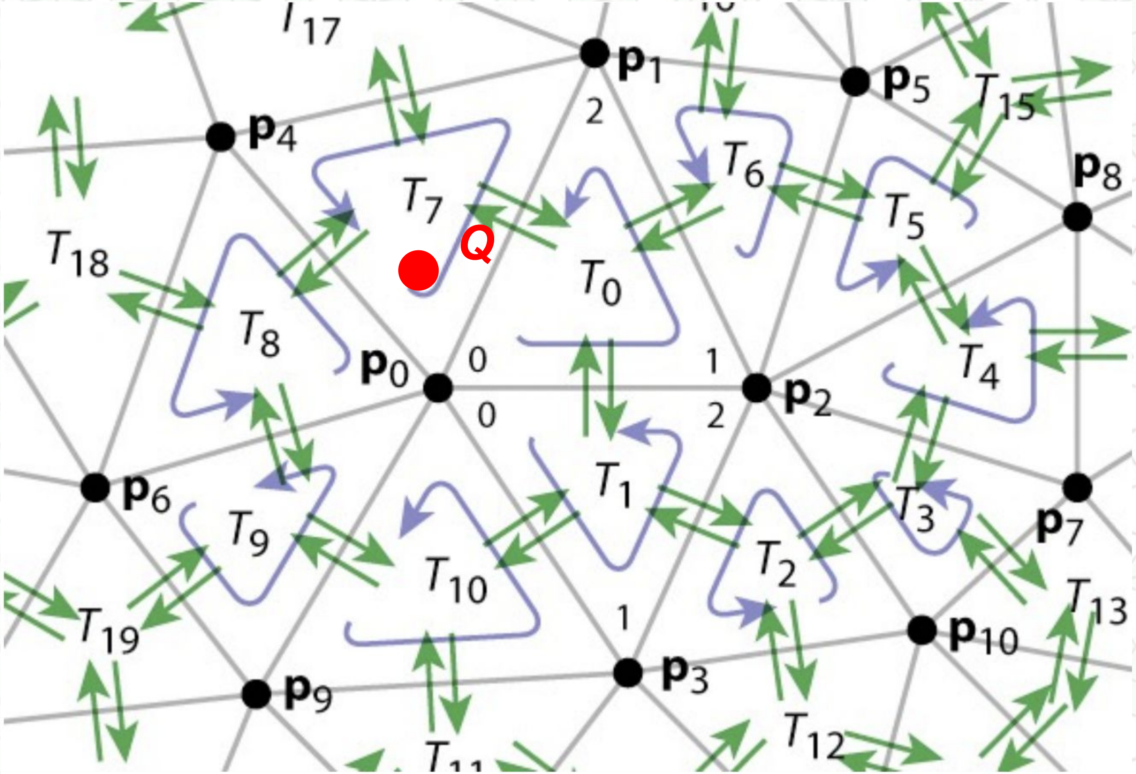
Concerns:

- *Cost of intersection*
- *How often are you asking?*
  - *on click*
  - *continuously*
- *Position imprecision/noise*

Brute Force Picking Algorithm

- Given a planar subdivision
  - *E.g., a collection of non-overlapping triangles (or polygons) that cover the plane*
- And a query point Q

- Which triangle/polygon is Q inside of?
  - E.g., T_7

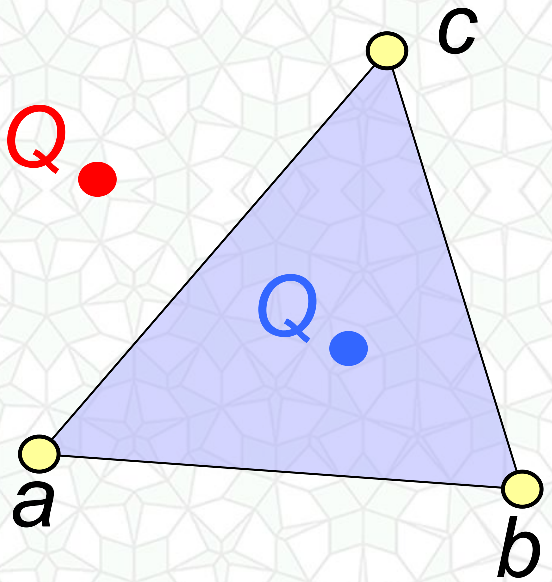


Steve Marschner
http://www.cs.cornell.edu/courses/cs4620

Is Query Point *inside* a specific Triangle?

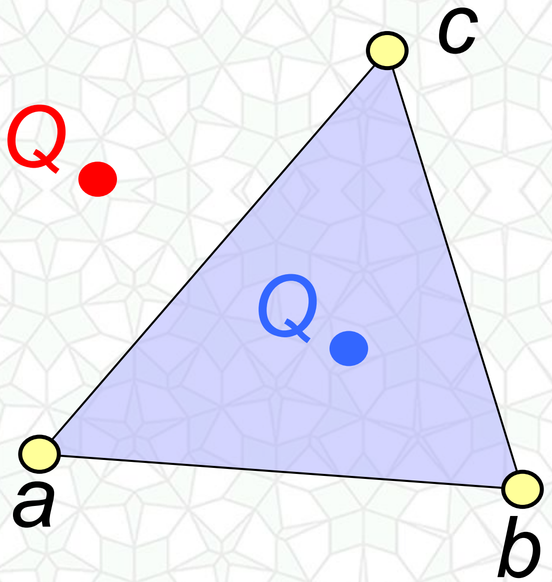
- Compare the point to each line segment
- Are you on the "right side" of all three line segments?
- Are you on the "wrong side" of one or two segments?

- Use cross product! (more on this later…)



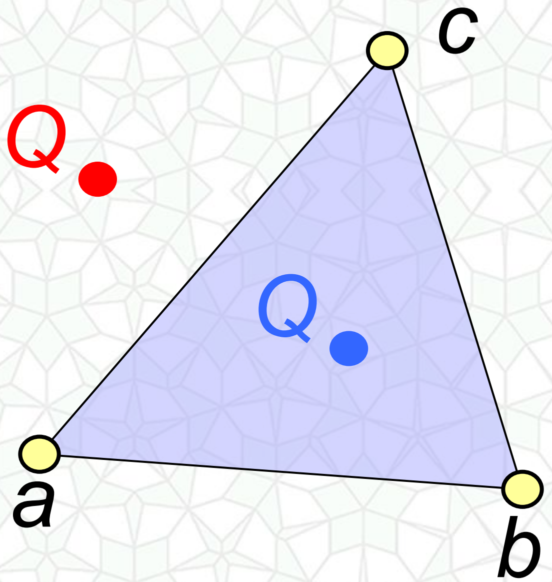
Is Query Point *inside* a specific Triangle?

- Does the half edge adjacency data structure accelerate this query?



Is Query Point *inside* a specific Triangle?

- Does the half edge adjacency data structure accelerate this query?
- *Unfortunately… NO!*
- *While we can navigate to the adjacent neighbors, we can NOT do better than a O(n) linear floodfill to find the correct triangle.*



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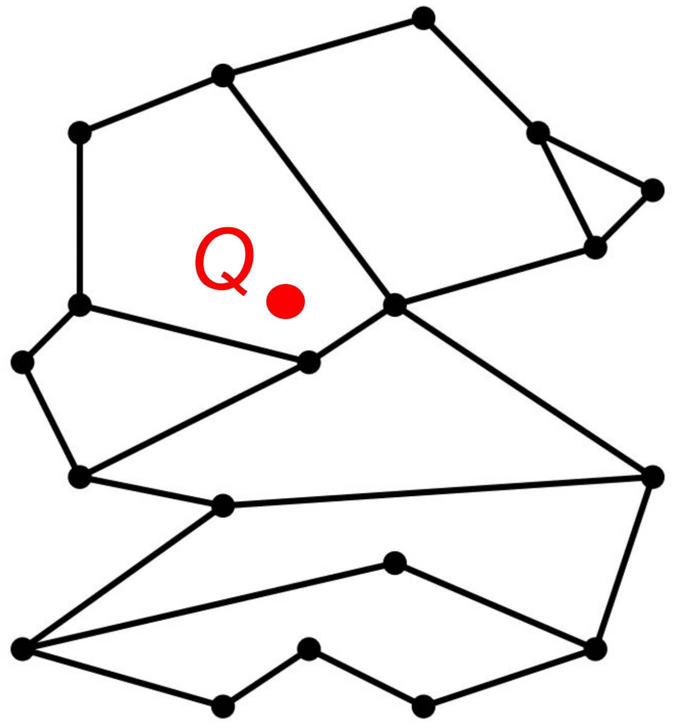
Point Location in Planar Subdivision

- Given *v* vertices, *n* edges, and *f* polygonal faces
- Which polygonal region contains the query point Q?



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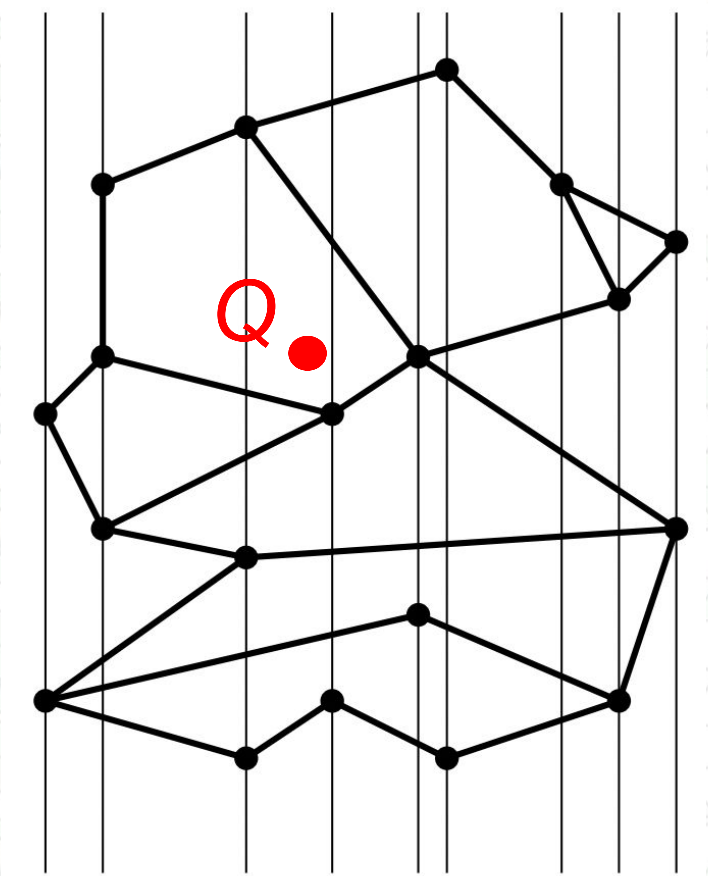
Point Location in Planar Subdivision

- Given *v* vertices, *n* edges, and *f* polygonal faces
- Which polygonal region contains the query point Q?
- Let's slice the plane into vertical "slabs"
- Draw a vertical line through every point



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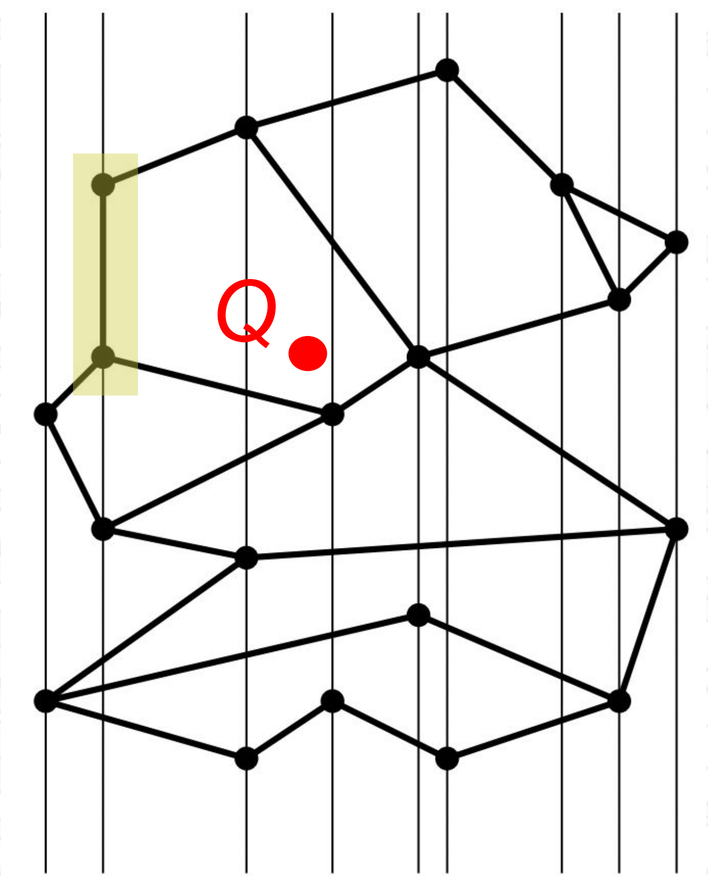
Point Location in Planar Subdivision

Let's assume "General Position":

- No two points have same x coordinate
- There will be no vertical segments!

- The query point will not be on a vertical segment or on a vertex.

- *Workaround is to have a tie breaker, rotate/shear the diagram a tiny amount*



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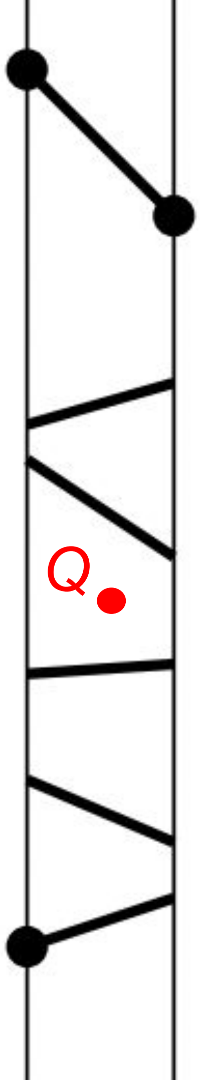
Point Location in a Vertical Slab?

- Within this slab, the line segments:
  - Do not cross
    (guaranteed by planar subdivision construction)
  - Do not start or stop
    (we've split at every vertex)
- We can sort the line segments vertically
  (by left endpoint's *y* coordinate)
- Which trapezoid is *Q* located within?
  - Each trapezoid is mapped back
    to the original polygonal face



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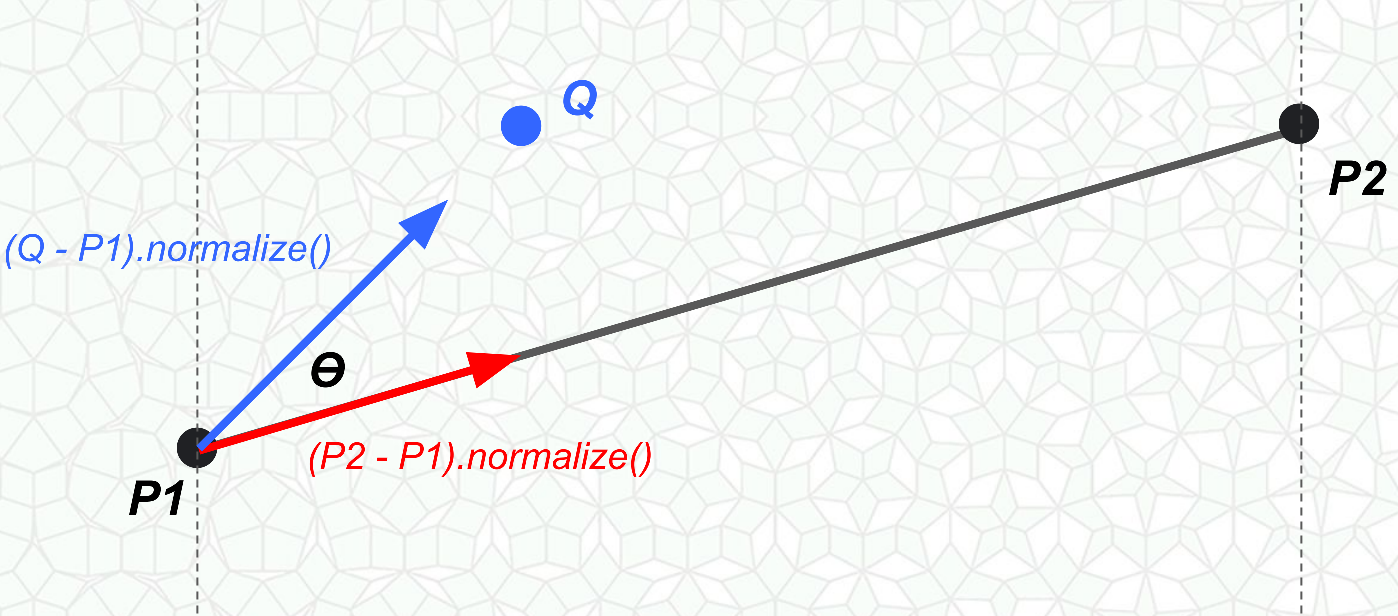
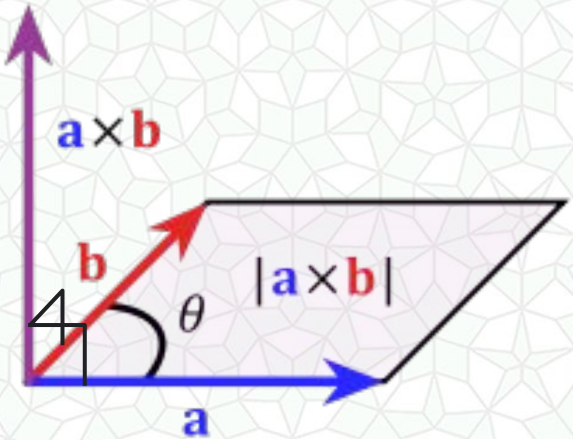
Is Query Point above (or below) Line Segment?

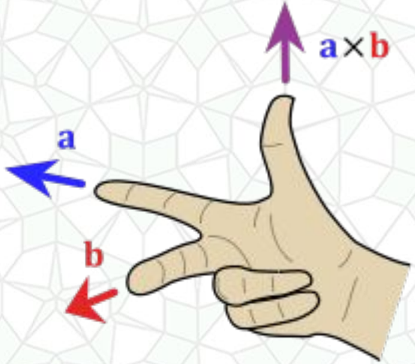
- $P1_x < Q_x < P2_x$
- Is $0° < \Theta < 180°$



Cross Product

- If the *Θ > 0°* & *Θ < 180°*, then *a* x *b* will be positive in the *z* axis.
- If the *Θ > 180°* & *Θ < 360°*, then *a* x *b* will be negative in the *z* axis.
- If a is parallel to b (*Θ = 0°* or *Θ = 180°*), then *a* x *b* will have zero magnitude.
- | *a* x *b* | = *sin Θ*





$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

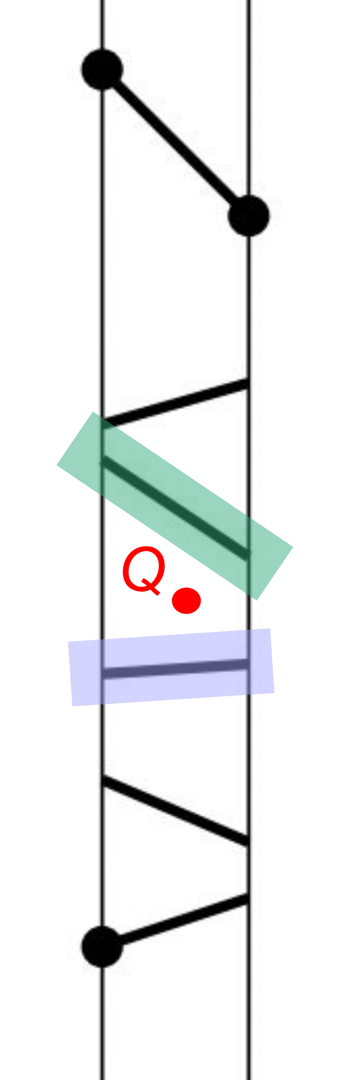
$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

https://en.wikipedia.org/wiki/Cross_product

Analysis: Running Time

- Algorithm Preprocess

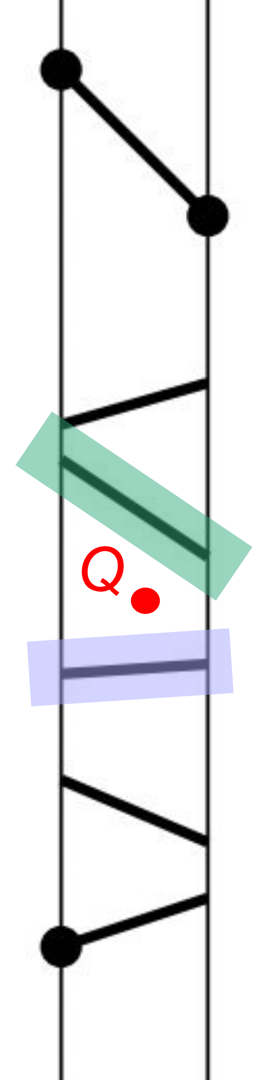
- Point Location Algorithm



Analysis: Running Time

- Algorithm Preprocess
  - Sort slabs left to right
  - Within each slab, sort trapezoids from top to bottom

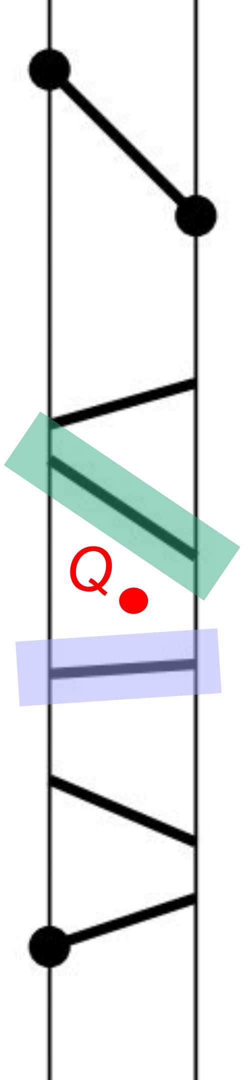
- Point Location Algorithm
  - Binary search to locate the correct slab between two points
    - Left vertical $_x$ < Q_x < right vertical $_x$
  - Binary search to locate correct trapezoid
    - Q is below the upper segment and above the lower segment



Analysis: Running Time

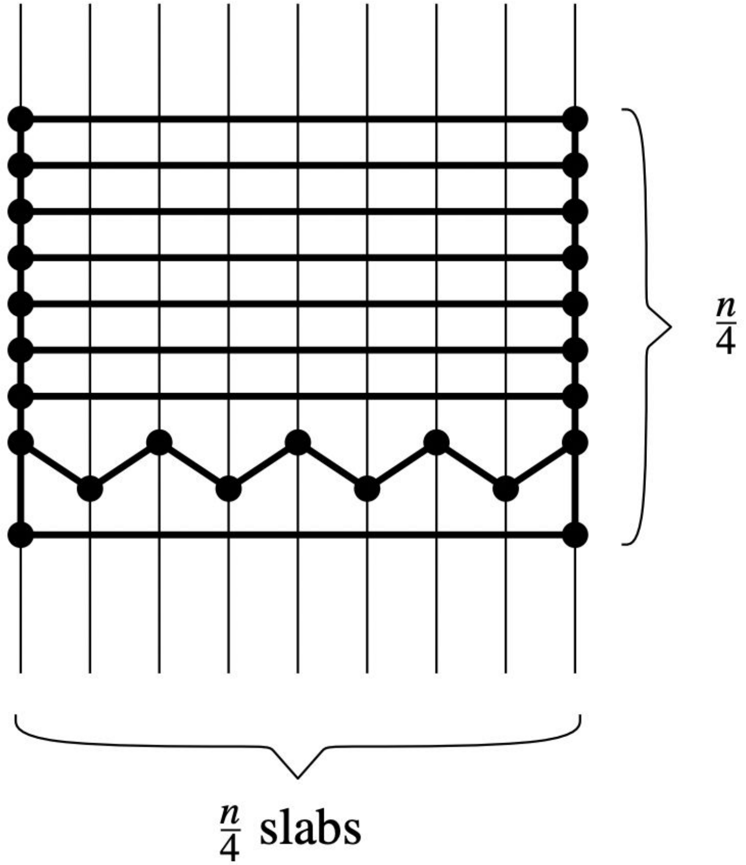
- Algorithm Preprocess
  - Sort slabs left to right → *O(n log n)*
  - Within each slab, sort trapezoids from top to bottom → *O(n log n)*
- Point Location Algorithm   Overall: → *O(log n)*
  - Binary search to locate the correct slab between two points
    - Left vertical $_x$ < Q_x < right vertical $_x$ → *O(log n)*
  - Binary search to locate correct trapezoid
    - Q is below the upper segment and above the lower segment → *O(log n)*

Where n is the # of edges



Analysis: Memory Usage

- Unfortunately, this representation is very costly
- It is redundantly storing many faces in many slabs
- In the worst case:



Where n is the # of edges

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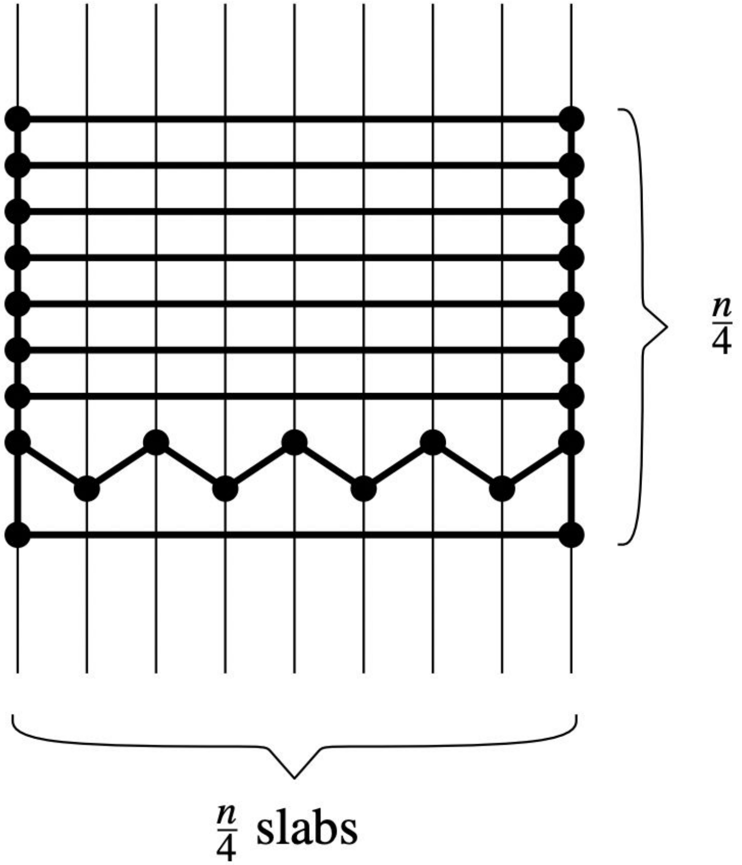
Analysis: Memory Usage

- Unfortunately, this representation is very costly
- It is redundantly storing many faces in many slabs
- In the worst case:

Euler: faces + vertices - edges = 2

9 faces + 25 vertices - 32 edges = 2

*Created 10 * 8 = 80 trapezoids!!*



Where n is the # of edges

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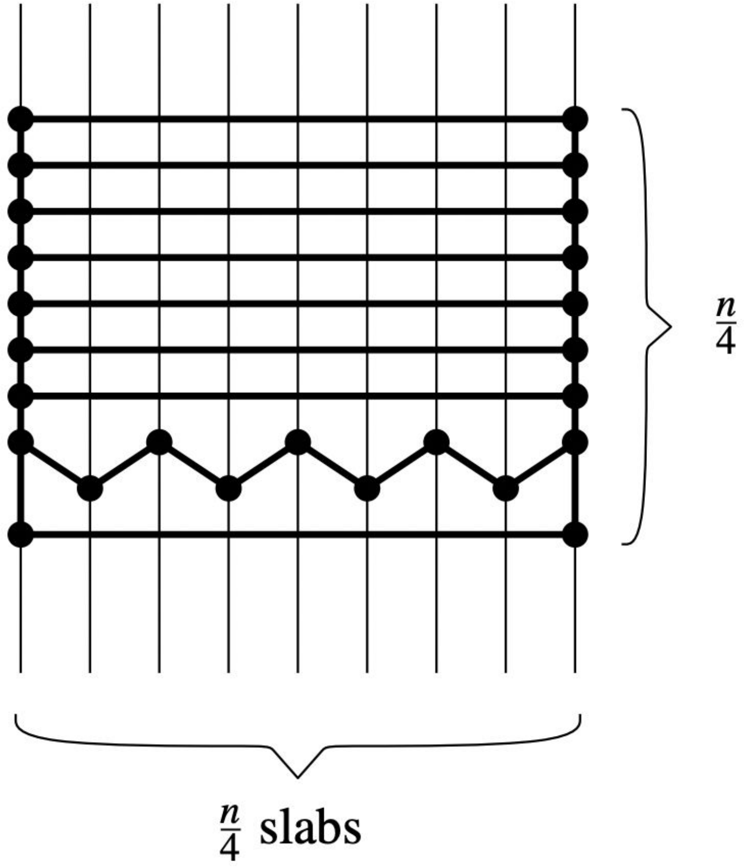
Analysis: Memory Usage

- Unfortunately, this representation is very costly
- It is redundantly storing many faces in many slabs
- In the worst case:
  - Every polygon appears in nearly every slab!
    $\rightarrow O(n^2)$
- Even average/expected case is unacceptable: $\rightarrow O(n\sqrt{n})$

Where n is the # of edges



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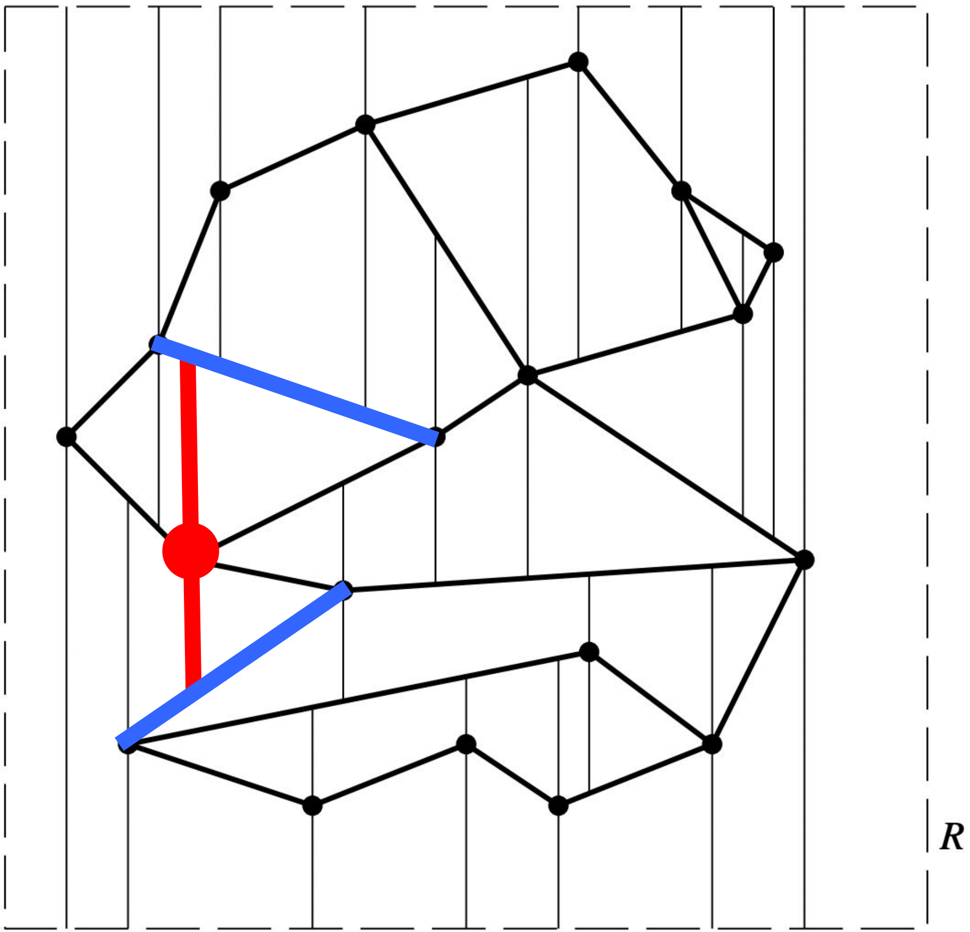
Idea: Reduce Redundant Storage

- Horizontally merge some of these cells
- Split vertically at every vertex
- But stop splitting when you reach the closest line segment above & below



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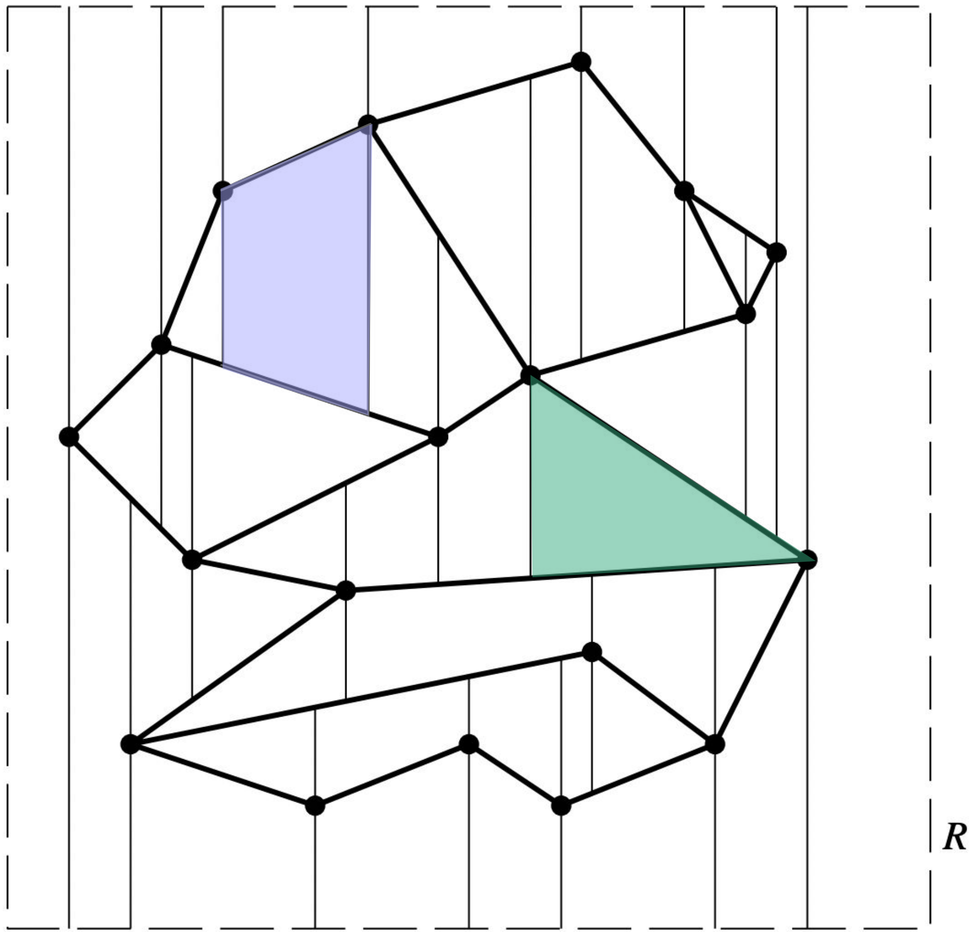
Create Convex Trapezoids & Triangles

- This defines a planar subdivision with full coverage of the plane by non-overlapping
  - **convex trapezoids** and
  - degenerate trapezoids: **triangles**



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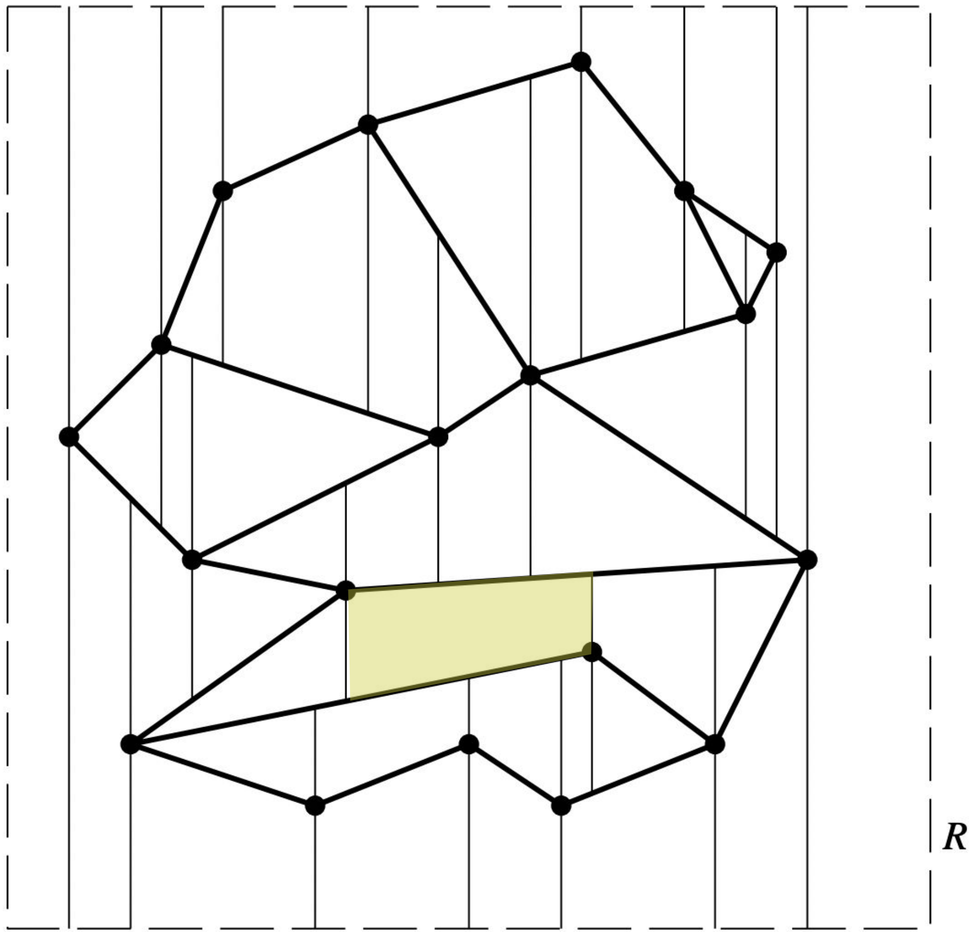
Adjacency Structure

- Can we connect these triangles and trapezoids with a classic half-edge adjacency data structure?



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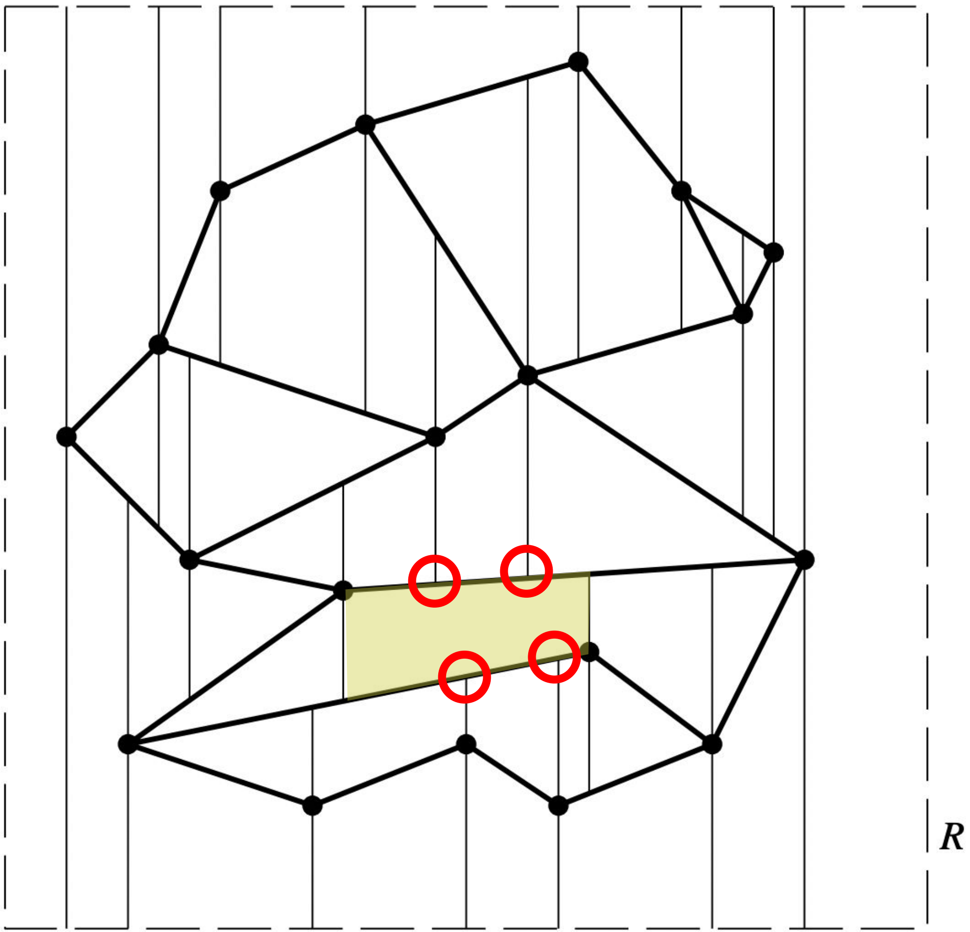
Adjacency Structure

- Can we connect these triangles and trapezoids with a classic half-edge adjacency data structure?
- *No!*
- Many of the faces have one or more “T junctions” on their top and/or bottom edges.
  - This is NOT ALLOWED with a traditional polygonal planar subdivision / halfedge data structure



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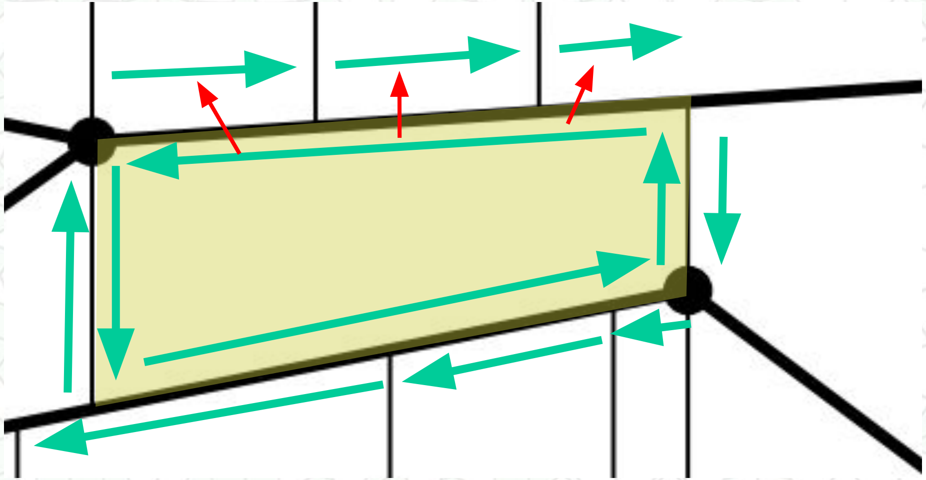
Classic Half-Edge Adjacency Structure

- Each face points to a half edge
- Each vertex points to a half edge
- Each half edge points:
  - Its opposite edge – only 1!
  - Its next edge
  - Its face
  - Its vertex
- A hacked modification would require an array of unknown size to point at all "opposite" edges

  This would be inefficient and an implementation nightmare!



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Trapezoid Map Adjacency Structure

Instead… each trapezoid (or triangle) points to:

- line segment ***top***, makes upper boundary
- line segment ***bottom***, makes lower boundary
- vertex ***leftp***, defines left vertical boundary
- vertex ***rightp***, defines right vertical boundary

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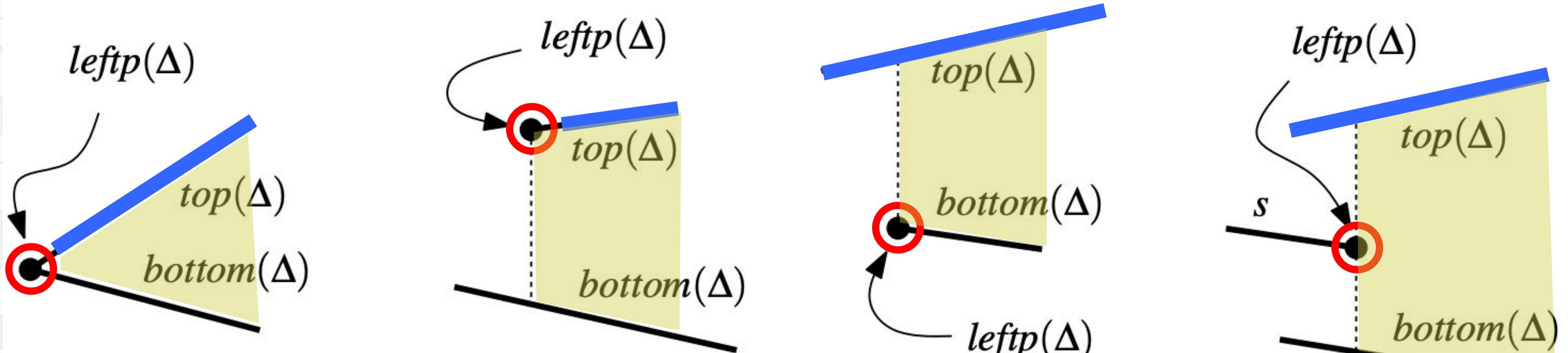


triangle case

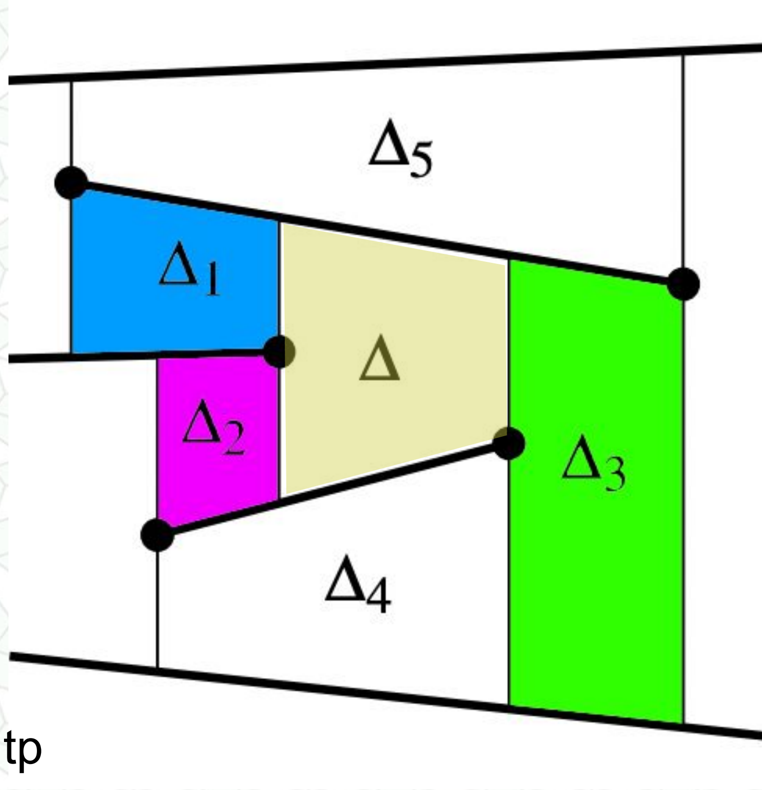
Trapezoid Map Adjacency Structure

Instead… each trapezoid (or triangle) points to:

- line segment ***top***, makes upper boundary
- line segment ***bottom***, makes lower boundary
- vertex ***leftp***, defines left vertical boundary
- vertex ***rightp***, defines right vertical boundary

Additionally… each trapezoid **Δ** may have up to 4 adjacent neighbors (or NULL if they do not exist)

- **upper left neighbor**, shares top and leftp
- **lower left neighbor**, shares bottom and leftp
- **upper right neighbor**, shares top and rightp
- **lower right neighbor**, shares bottom and rightp



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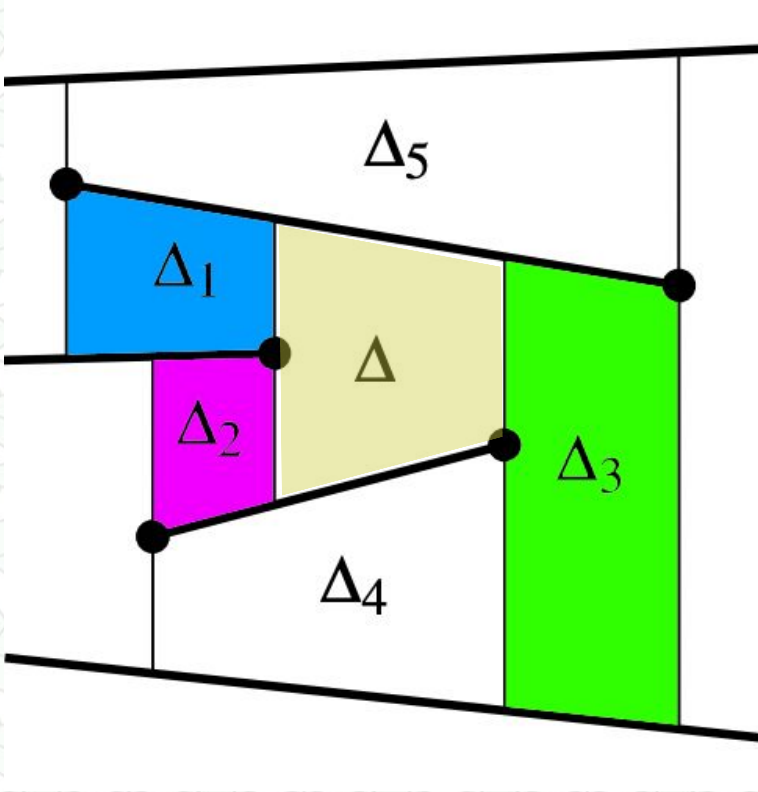
Trapezoid Map Adjacency Structure

- Does this new adjacency structure allow us to navigate through the structure more efficiently, faster than a O(n) floodfill for the classic polygon adjacency structure?



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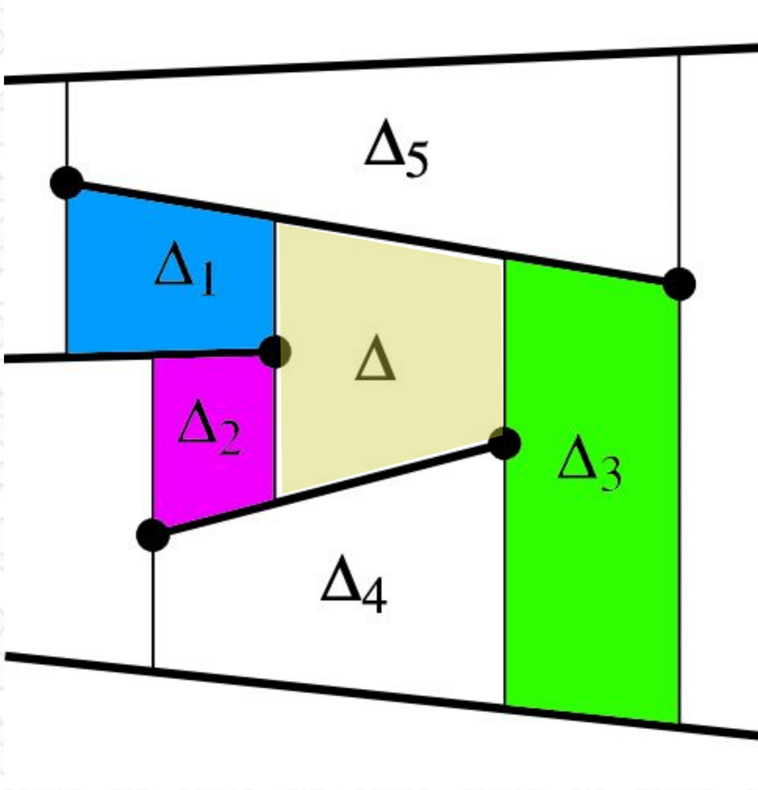
Trapezoid Map Adjacency Structure

- Does this new adjacency structure allow us to navigate through the structure more efficiently, faster than a O(n) floodfill for the classic polygon adjacency structure?

- *Unfortunately, no…*
- *But we can build a binary tree (actually a DAG) for this structure to perform these queries!*



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Outline for Today

- Homework 4 Questions?
- Last Time: kD Trees & Range Trees
- Motivating Application: Point Location
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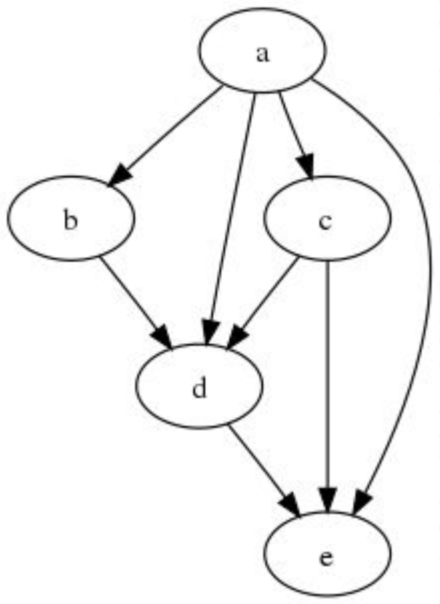
- Next Time: Voronoi Diagram

What is a "Directed Acyclic Graph" (DAG)?

- A graph (collection of nodes & edges)
  - with *directed edges*, and
  - with *no directed cycles*

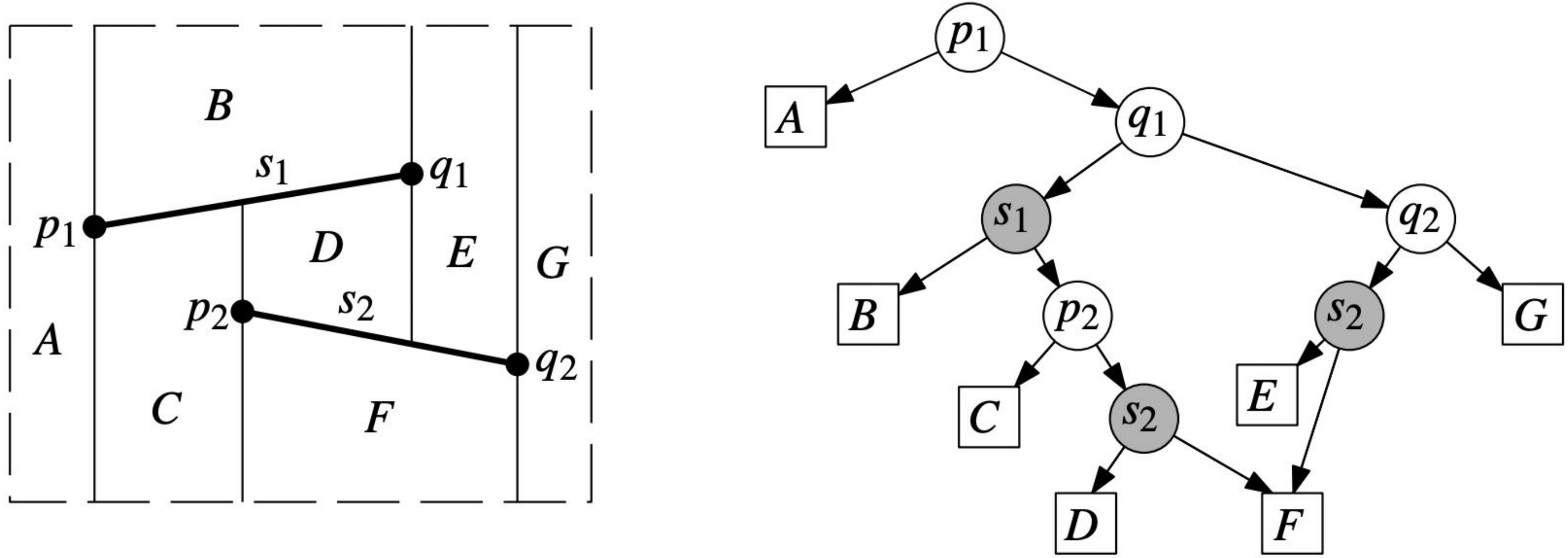
- A graph is a DAG if and only if
  - it can be topologically-ordered
    to arrange the vertices in a
    linear sequence such that
    all edges are oriented consistently
    (e.g., flowing from top to bottom)



https://en.wikipedia.org/wiki/Directed_acyclic_graph

Directed Acyclic Graph (DAG)

- Intermediate nodes are vertices (vertical lines) and line segments
- The leaves are the trapezoidal regions (which map back to original polygons)



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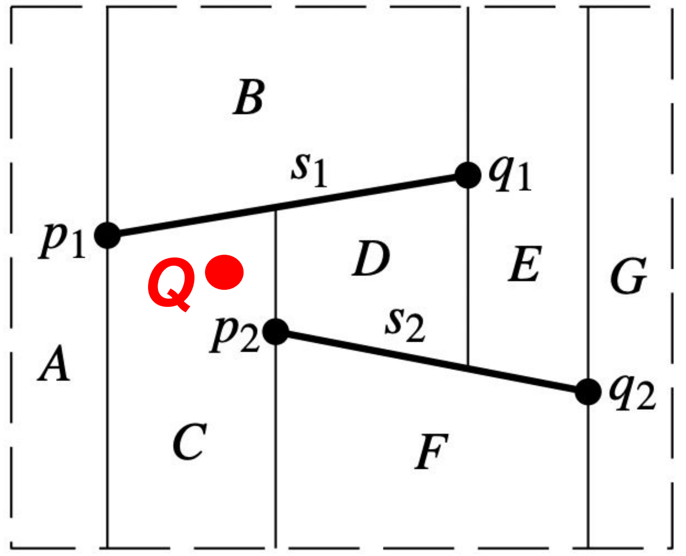

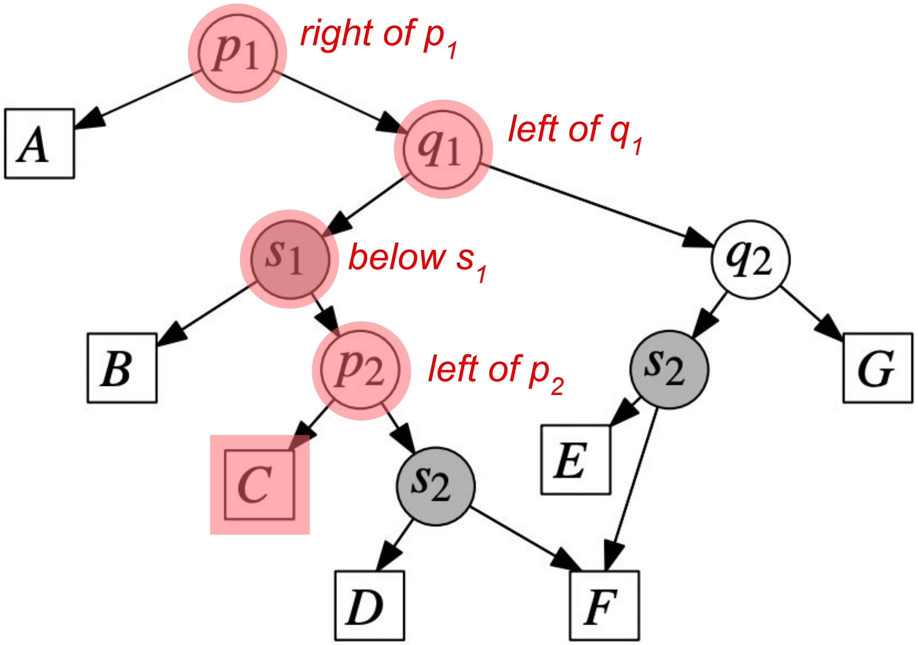

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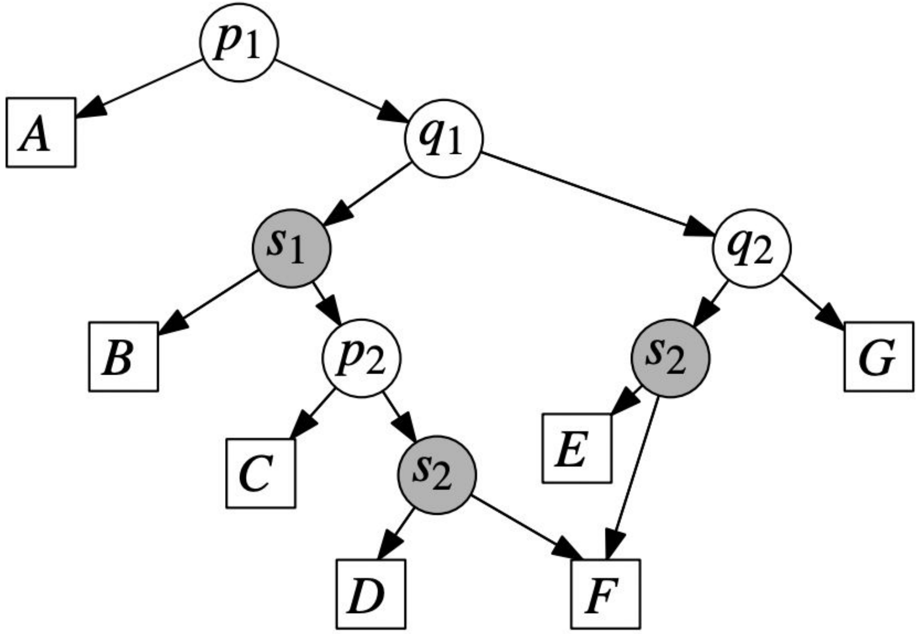
Analysis: Directed Acyclic Graph (DAG)

Size of the DAG?

- # of leaves = # of trapezoids

- # of intermediate nodes
  = # of vertices + # of line segments

- Height of DAG



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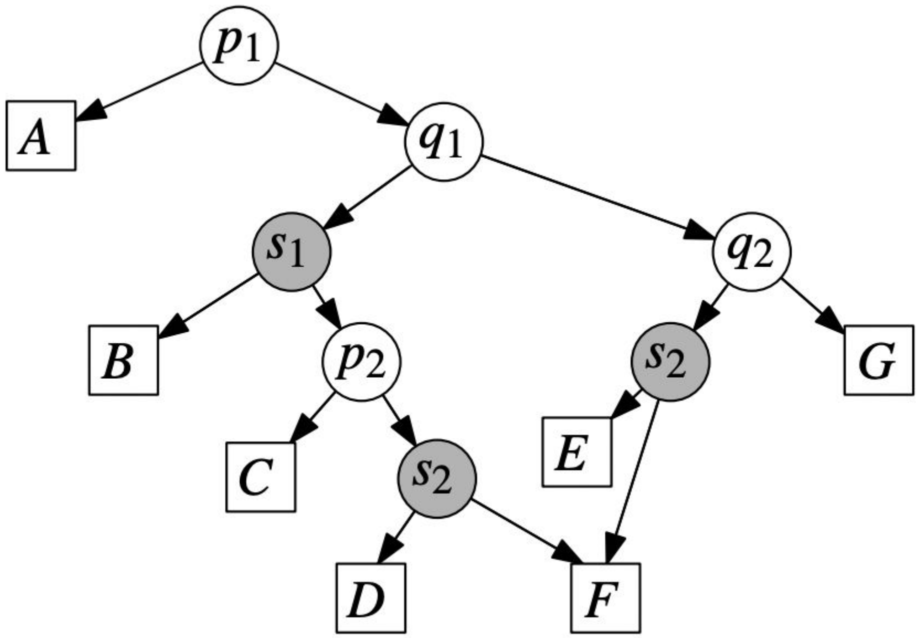
Analysis: Directed Acyclic Graph (DAG)

Size of the DAG?

- # of leaves = # of trapezoids
  → *O(n)*
- # of intermediate nodes
  = # of vertices + # of line segments
  → *O(n)*
- Height of DAG
  → *O(log n) best case*
  → *O(n) worst case*
- Use **Randomized Incremental Construction** to achieve height
  → *O(log n) expected case!*



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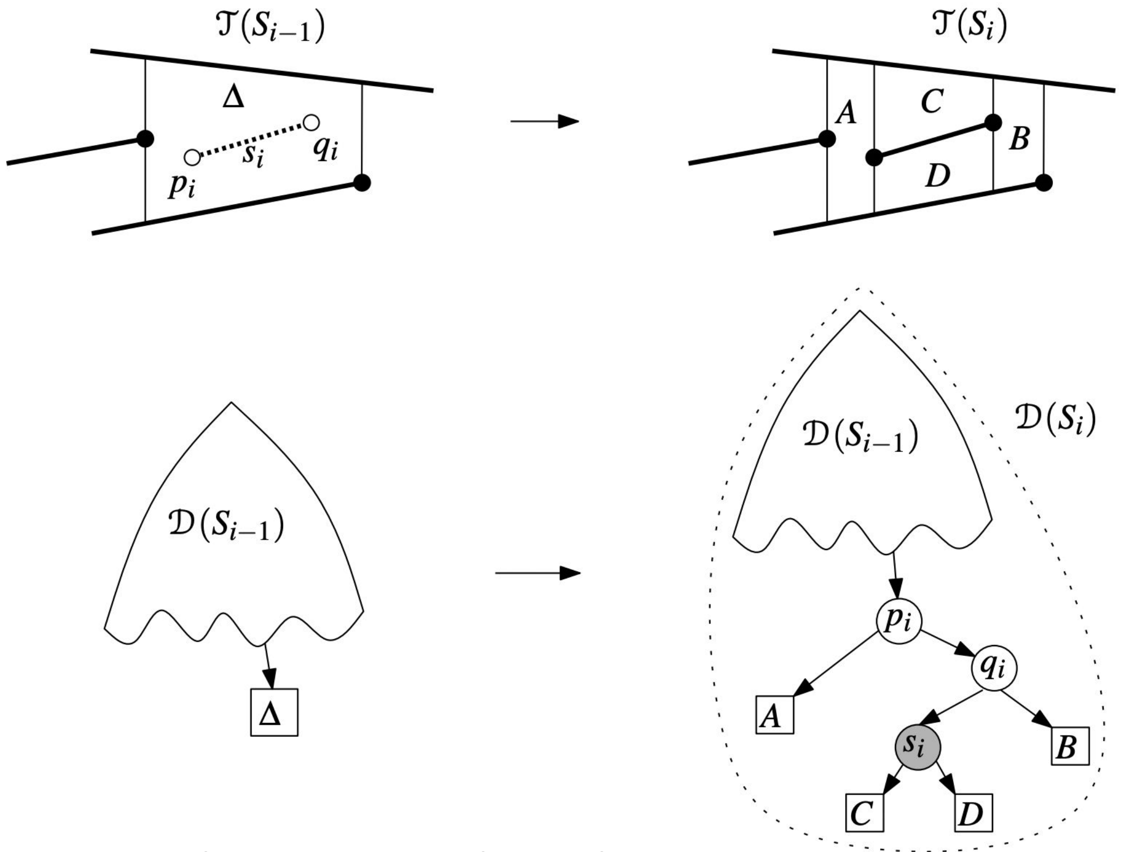
Randomized Incremental Construction

- Randomize the order of the line segments
- Inserting the segments one at a time
- Handle all of the cases



Book has lengthy description of the full algorithm & proof!

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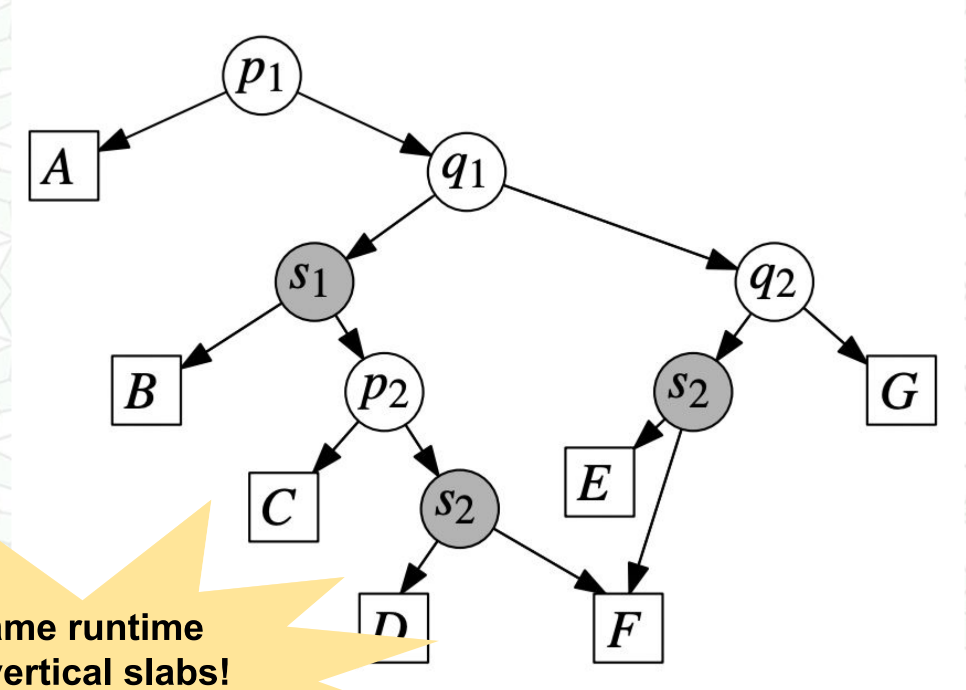
Analysis: Directed Acyclic Graph (DAG)

- Memory to store DAG?
  → *O(n)*
- Height of the DAG?
  → *O(log n) expected*
- Query time to locate the trapezoid/polygon containing point Q?
  → *O(log n) expected*
- Cost to construct?
  → *O(n log n) expected*

Book has lengthy description of the full algorithm & proof!



Same runtime as vertical slabs! Linear memory usage!

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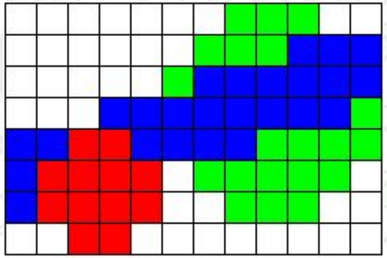
"Picking" by the Framebuffer

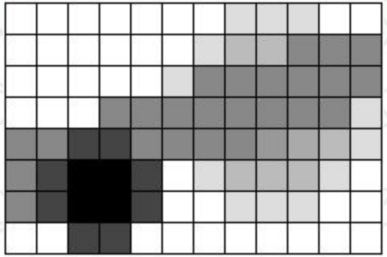
- Graphics "Hack"
- Take advantage of fast GPU hardware rendering
- Color each object a different, unique color (no lighting/shading)

- Grab the color of the pixel from the framebuffer (object id)
- Grab the z-value (depth) from the depth buffer

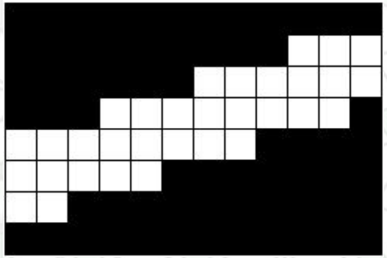
"Capturing and Animating Occluded Cloth"
White, Crane, & Forsyth, SIGGRAPH 2007



frame buffer



depth buffer



stencil buffer

“Picking” by the Framebuffer

- Are there enough colors?

- Screen Resolution

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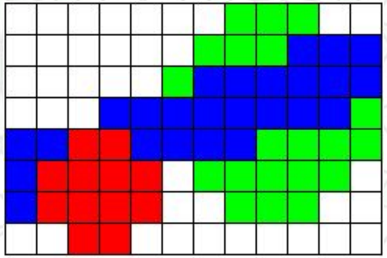

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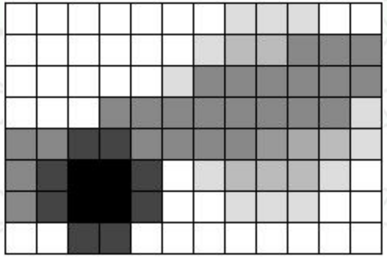

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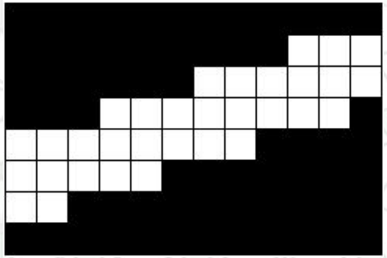

stencil buffer

“Picking” by the Framebuffer

- Are there enough colors?
  - *3 colors (RGB) w/ 8 bits each*
  - $2^8 2^8 2^8 = 2^{24} =$ *16 million*

- Screen Resolution
  - *“4k” = 4096 x 2160 = 9 million pixels*
  - *“8k” = 7680 x 4320 = 33 million pixels*

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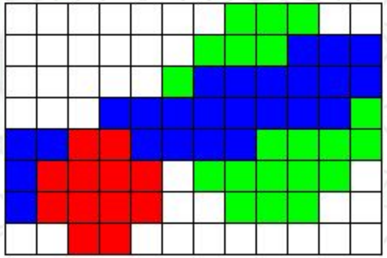

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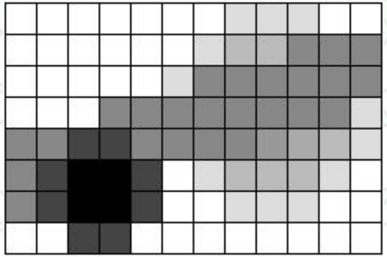

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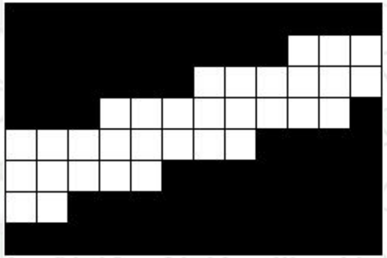

stencil buffer

Painting by “Picking” a Picket Fence?

2D → 3D & Usability:

- You “click” on a picket to start painting
- Move up and down, you stay on the picket
- Move left or right, you fall between the pickets.
    - Do you hover in the air between pickets?
    - Does your mouse *z* coordinate change? Do you start painting the ground?

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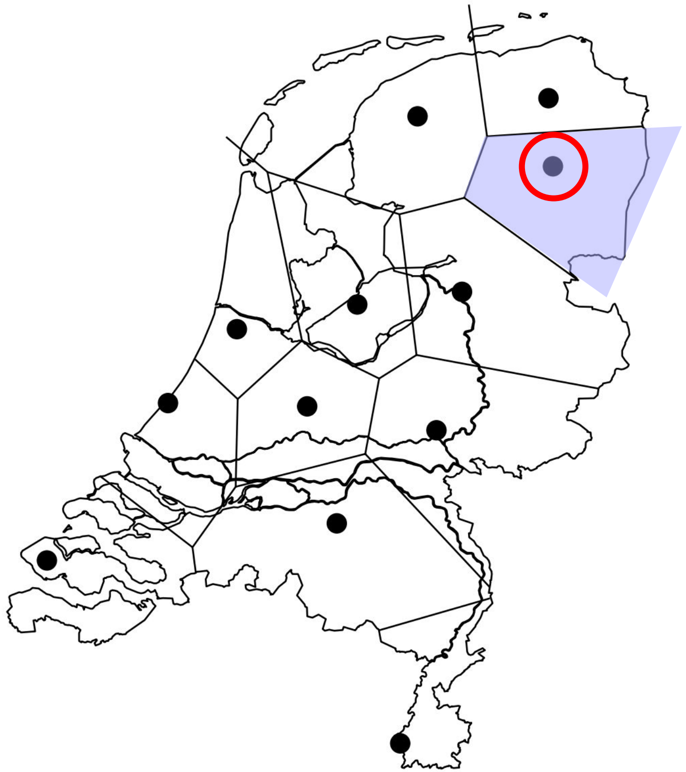
Voronoi Diagram - Social Geography

actually these are the capitals of each province in the Netherlands

- There are a bunch of grocery stores spread across a large city.
- You're planning to open another grocery store at a specific location.
- How many customers can you expect at the new store location?
  Customers will choose the new store if it is closer to their home than their current store.

- a.k.a. The "Post Office Problem"



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