CSCI 4560/6560 Computational Geometry

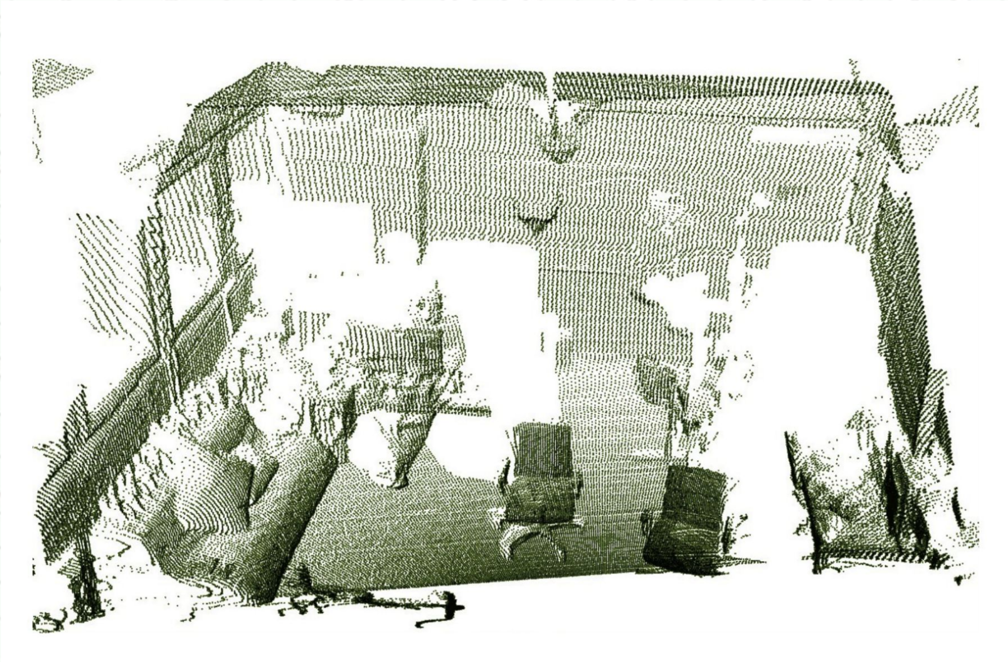
https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/F23/

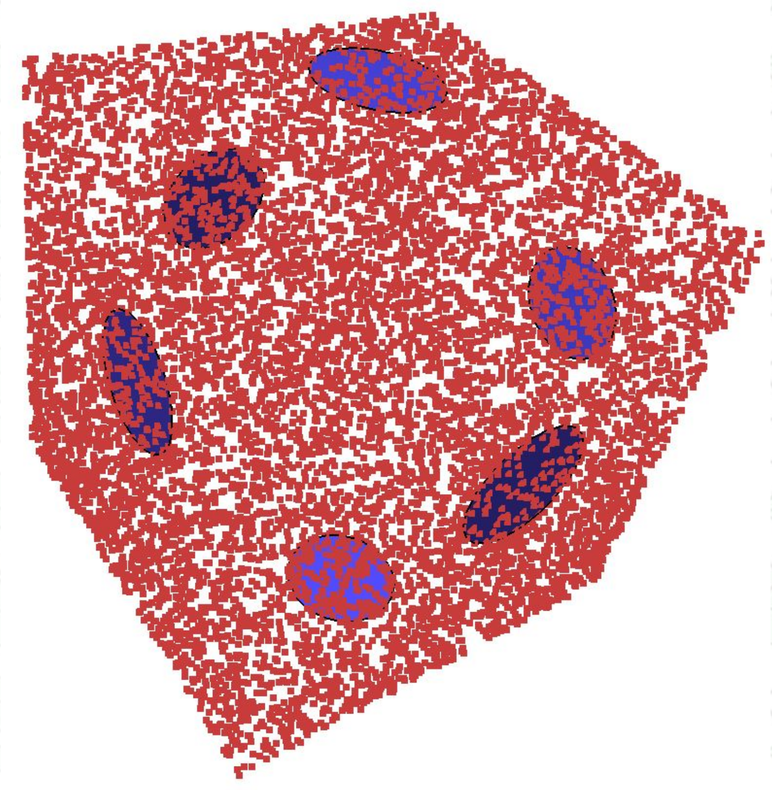
Lecture 10: Voronoi Diagrams, Part 1

Outline for Today

- Homework 5 Questions?
- Last Time: Point Location & Trapezoidal Maps
- Motivating Application: Social Geography
- Observations about Voronoi Diagrams
- Brute Force Constructions & Analysis & Complexity
- A History of the Names Voronoi/Dirichlet/Thiessen
- How to Graph A Parabola
- Sweep Line Algorithm to Construct a Voronoi Diagram
- Analysis of Sweep Line Algorithm
- Next Time: More Voronoi Diagrams!

Homework 5 - CGAL Programming Task





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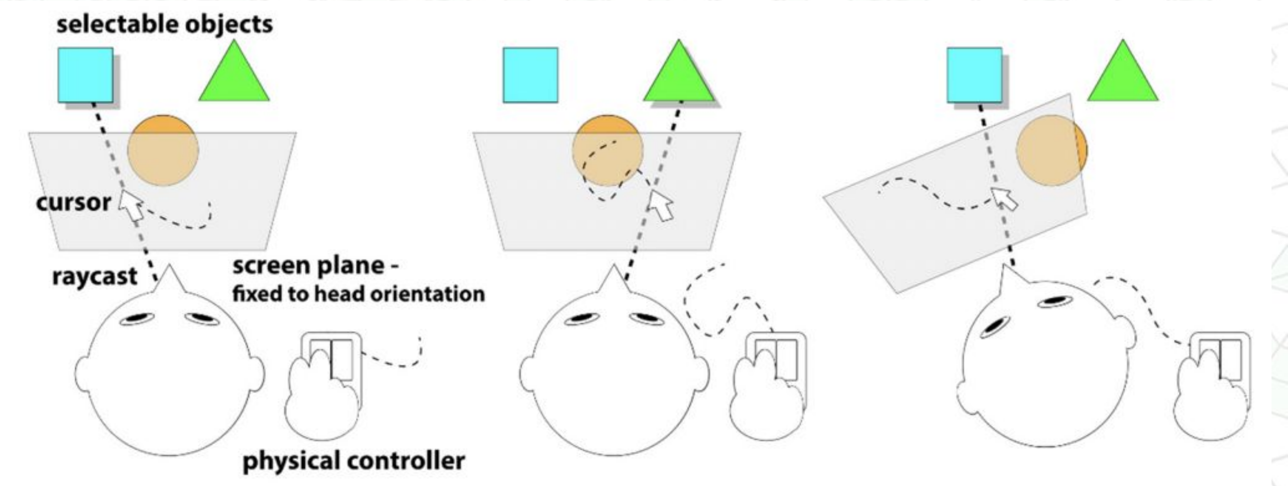
Graphics/VR Application: What is "Picking"?

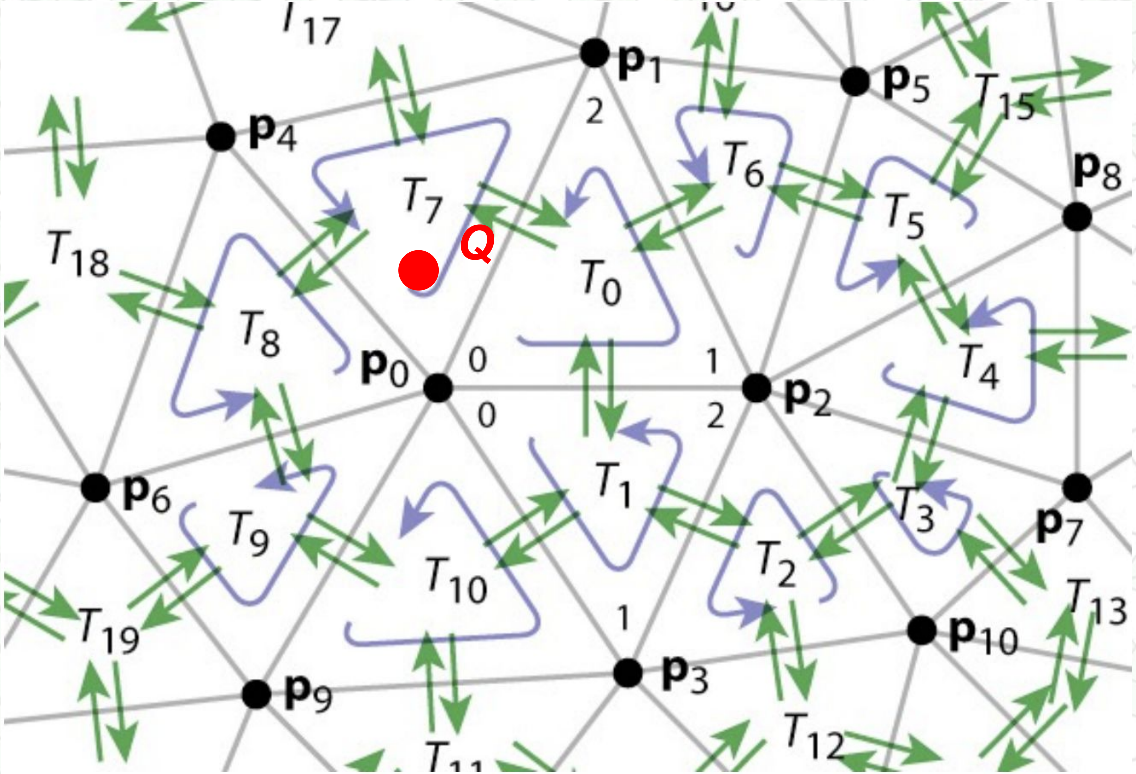
- Get the (3D) world coordinates of a (2D) mouse click
- Identify which object was selected and
  the point on the object closest to the click
- *Do we as users take this for granted??*
  - *What are the performance bottlenecks?*
  - *What are the usability concerns?*



https://www.csit.carleton.ca/~rteather/pdfs/GI_2018_EZCursorVR.pdf

Brute Force Picking Algorithm

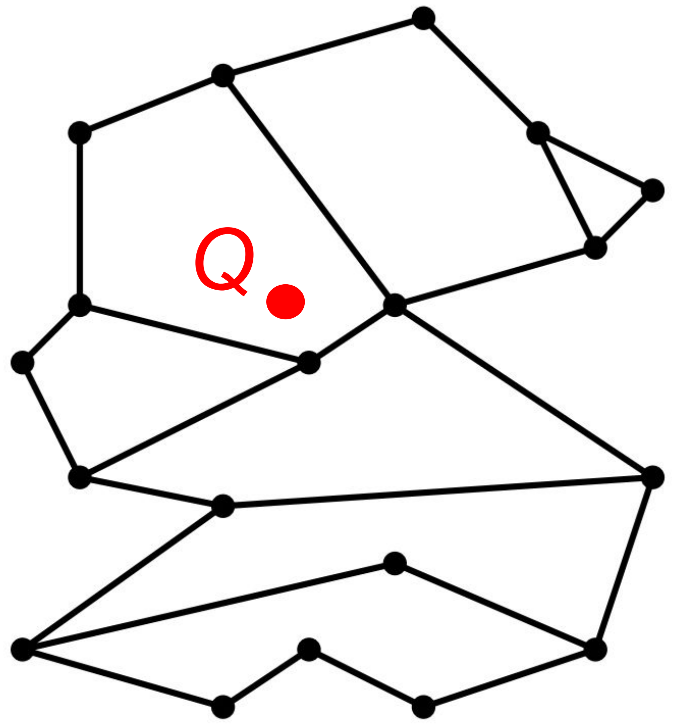
- Given a planar subdivision
  - *E.g., a collection of non-overlapping triangles (or polygons) that cover the plane*
- And a query point Q
- Which triangle/polygon is Q inside of?
  - E.g., T_7



Steve Marschner
http://www.cs.cornell.edu/courses/cs4620

Point Location in Planar Subdivision

- Given *v* vertices, *n* edges, and *f* polygonal faces
- Which polygonal region contains the query point Q?



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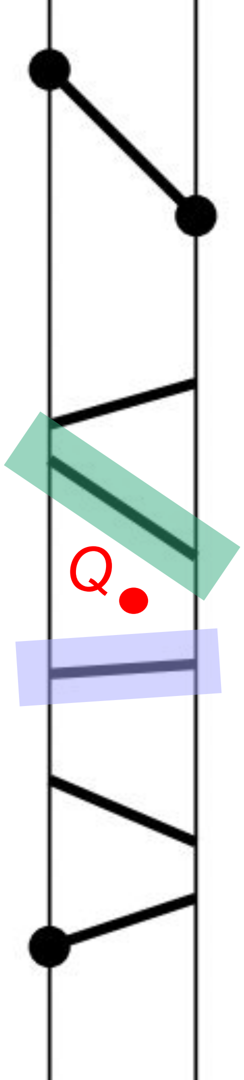
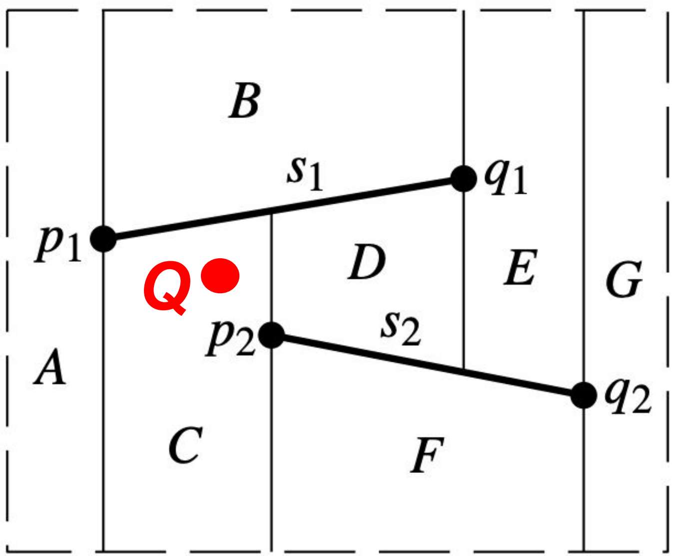
Analysis: Running Time

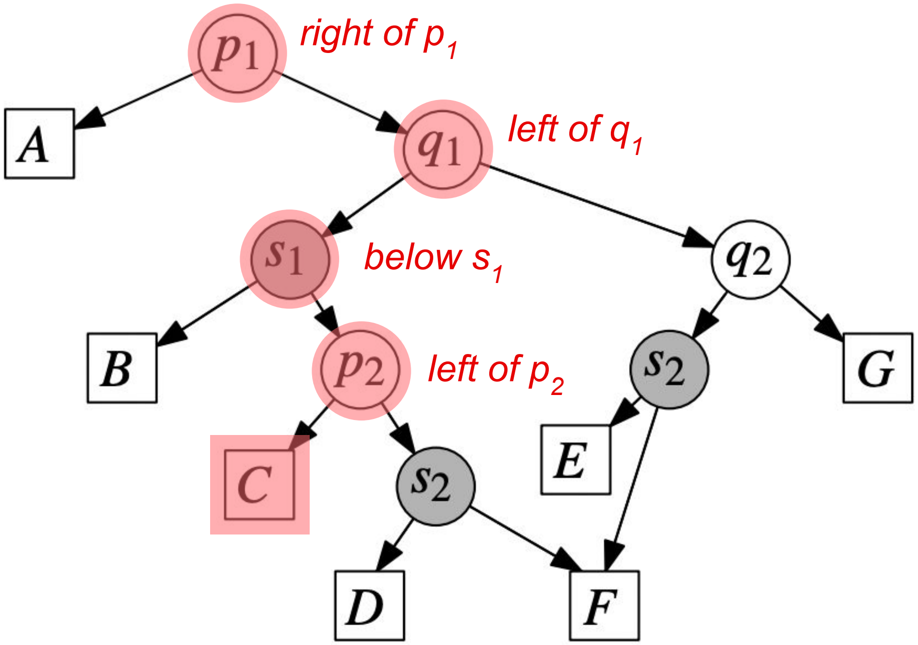
- Algorithm Preprocess
  - Sort slabs left to right → *O(n log n)*
  - Within each slab, sort trapezoids from top to bottom → *O(n log n)*
- Point Location Algorithm Overall: → *O(log n)*
  - Binary search to locate the correct slab between two points
    - Left vertical $_x$ < Q_x < right vertical $_x$ → *O(log n)*
  - Binary search to locate correct trapezoid
    - Q is below the upper segment and above the lower segment → *O(log n)*



Directed Acyclic Graph (DAG)

- Intermediate notes are vertices (vertical lines) and line segments
- The leaves are the trapezoidal regions (map back to original polygons)





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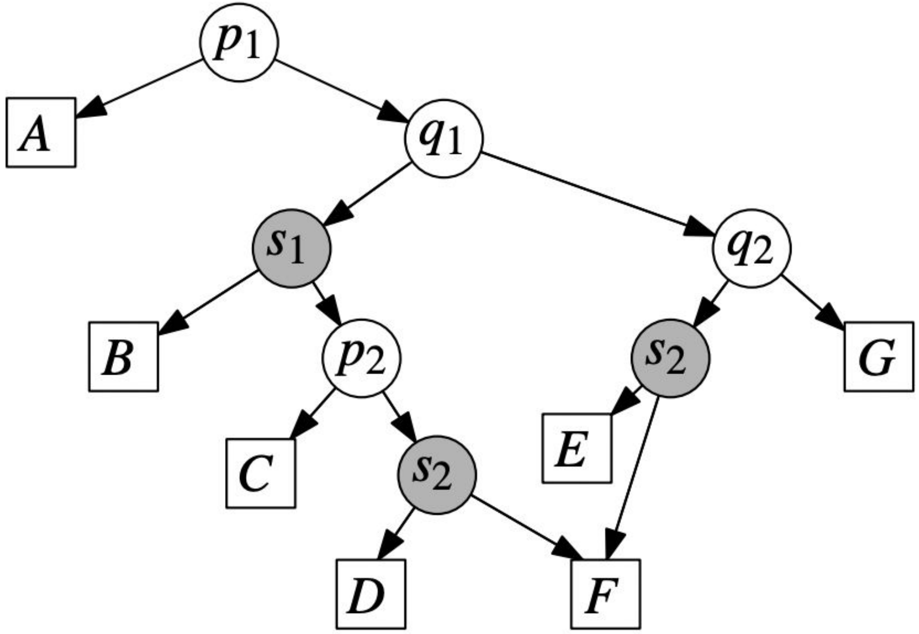
Analysis: Directed Acyclic Graph (DAG)

Size of the DAG?

- # of leaves = # of trapezoids
  → *O(n)*
- # of intermediate nodes
  = # of vertices + # of line segments
  → *O(n)*
- Height of DAG
  → *O(log n) best case*
  → *O(n) worst case*
- Use **Randomized Incremental Construction** to achieve height
  → *O(log n) expected case!*



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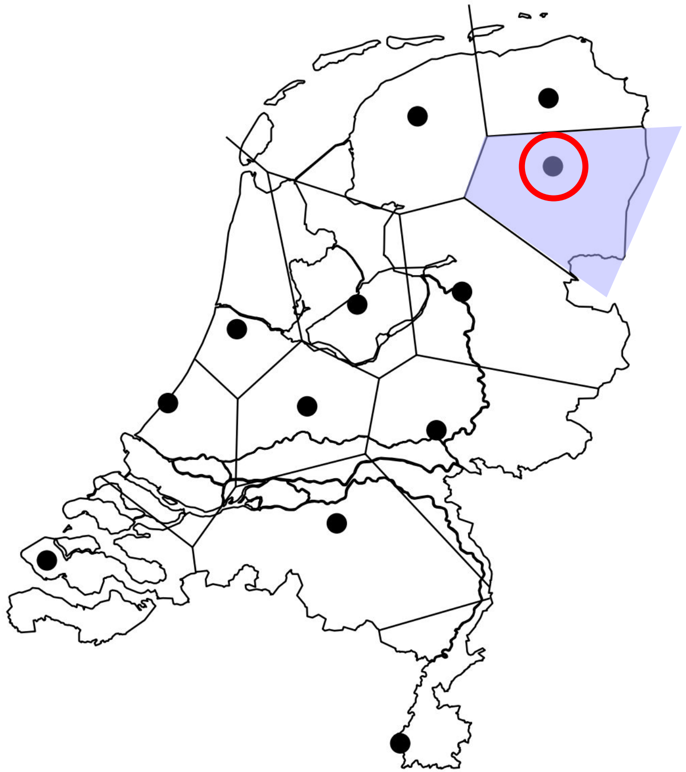
Voronoi Diagram - Social Geography

actually these are the capitals of each province in the Netherlands

- There are a bunch of grocery stores spread across a large city.
- You're planning to open another grocery store at a specific location.
- How many customers can you expect at the new store location?
  Customers will choose the new store if it is closer to their home than their current store.

- a.k.a. The "Post Office Problem"



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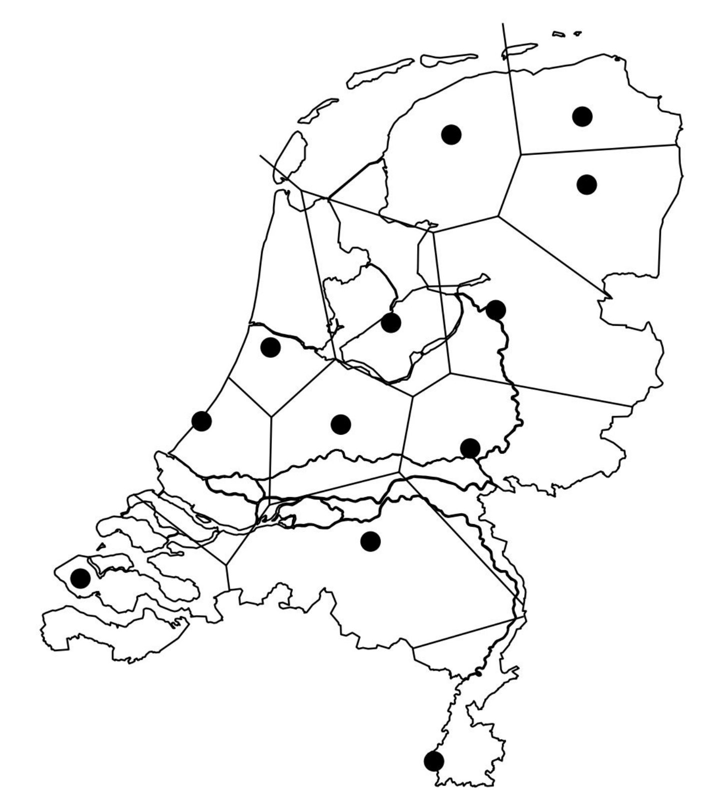
Voronoi Diagram - Social Geography

actually these are the capitals of each province in the Netherlands

Assumptions:

- The price of a good is the same at every site.
- The cost of acquiring the good is the price + transportation to the site.
- The cost of transportation is the Euclidean Distance * fixed price per unit distance.
- Consumers try to minimize cost of acquiring the good.



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Euclidean Distance

- **Euclidean Distance** - straight-line distance between two points
- *NOT* **Manhattan Distance** - only travel along axis aligned roads
- *NOT* **Geodesic Distance** - shortest distance between two cities on the globe – via a “great circle”

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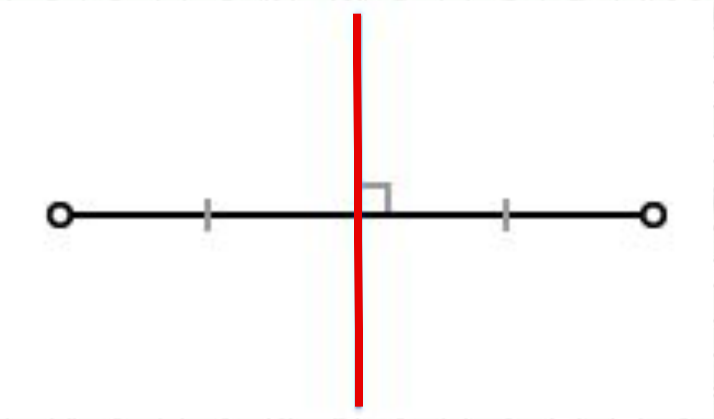
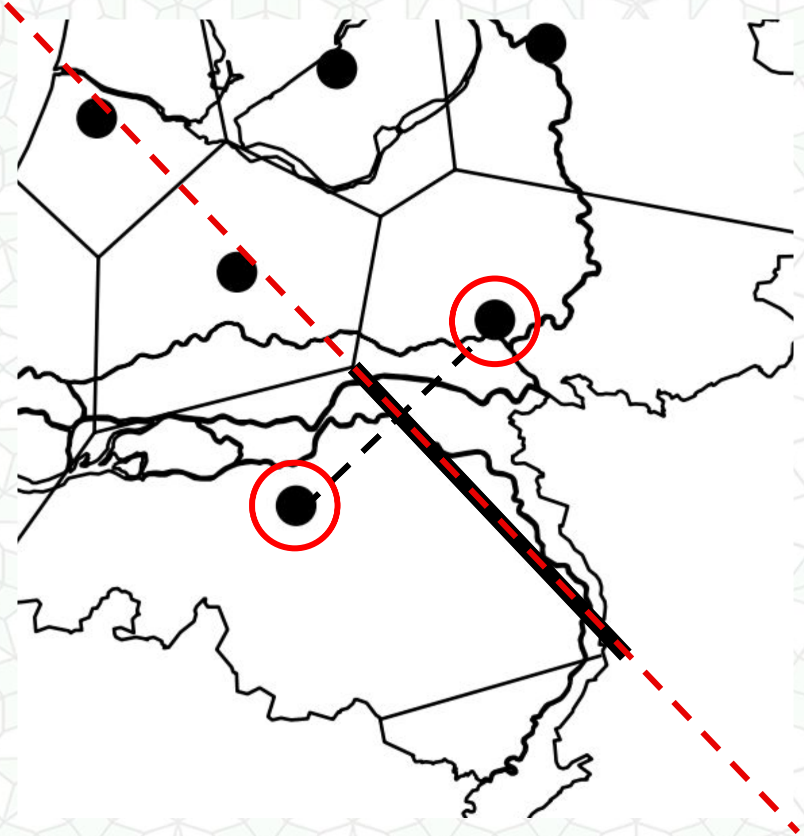
Observation: Perpendicular Bisector

- Points on the edge between two Voronoi cells are equidistant from two Voronoi sites.
- Edges of Voronoi cells are perpendicular bisectors of two Voronoi sites.



https://euclidea.fandom.com/wiki/Perpendicular_Bisector



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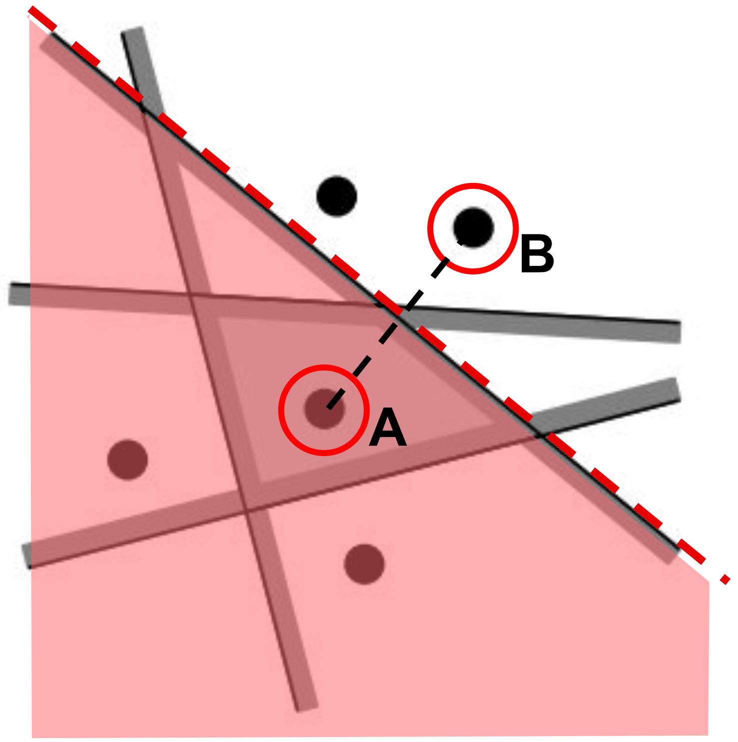
Observation: Intersection of Half Spaces

- All points that lie on one side of the perpendicular bisector,
- Are the half-space of points that will chose site A over site B because site A is closer than site B.

- This suggests a brute force construction algorithm…



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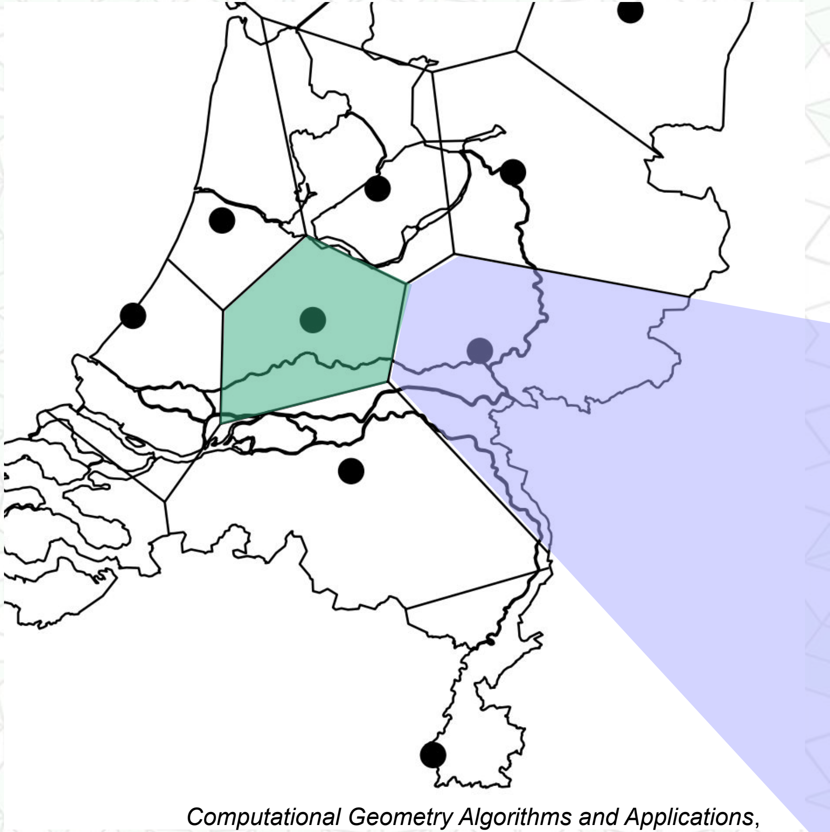
Observation: Voronoi Cells are Convex

- Because a Voronoi cell is the intersection of half-spaces…
- A Voronoi Cell must be convex
- *Note: Some Voronoi Cells are unbounded*



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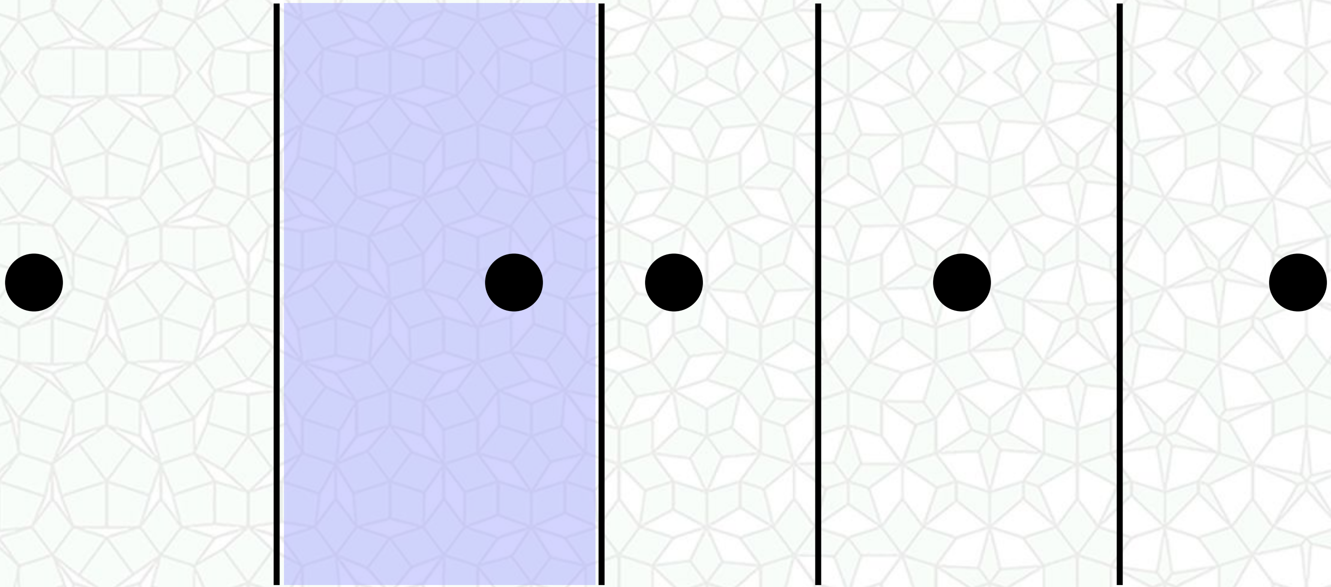
What about Collinear Voronoi Sites?

- If all Voronoi sites are collinear – a degenerate configuration…

What about Collinear Voronoi Sites?

- If all Voronoi sites are collinear – a degenerate configuration…
- All Voronoi edges will be parallel lines (unbounded in both directions)
- All Voronoi cells will be unbounded areas

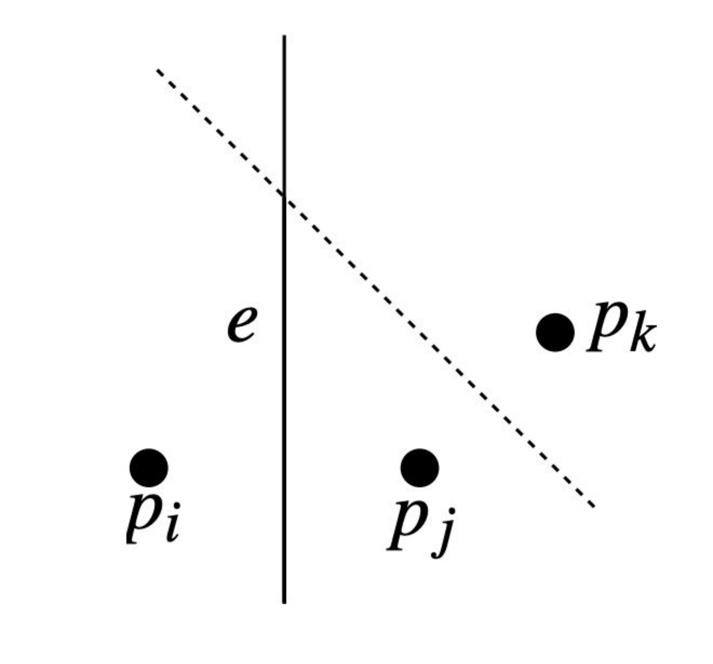


What about Collinear Voronoi Sites?

- If some Voronoi sites are *not* collinear…

- Every perpendicular bisector will intersect with at least one other perpendicular bisector

- Every Voronoi edge will be bounded on one or both directions



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Voronoi Diagram has Linear Complexity

- Given *n* Voronoi sites
- The diagram will have *n* Voronoi cells

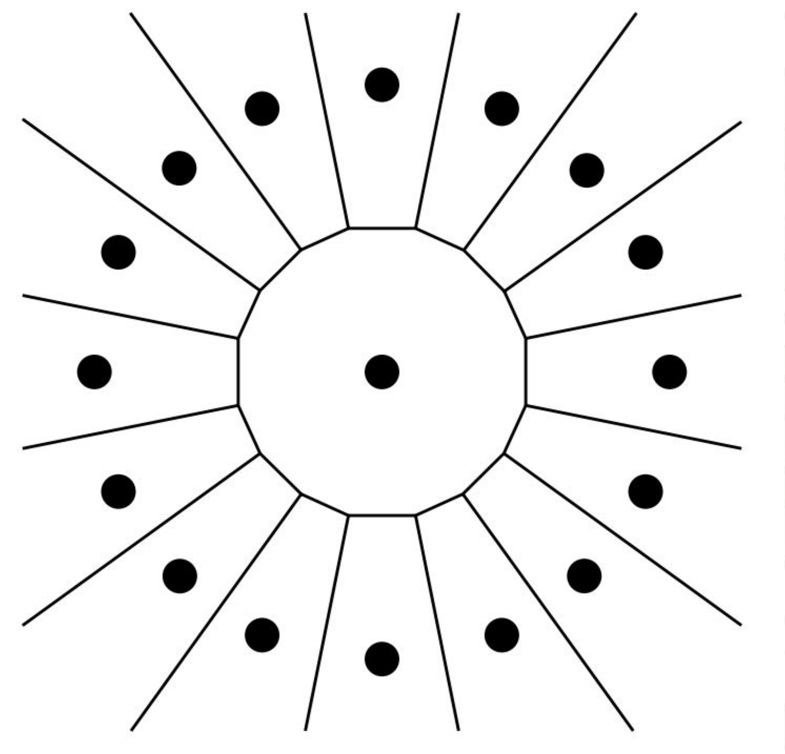
cells = faces

- A single Voronoi cell may have many edges – as many as *n-1*

- But by Euler's formula $F + V = E + 2$

sites ≠ vertices

  - # of vertices ≤ *2n-5*
  - # of edges ≤ *3n-6*



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Voronoi Diagram has Linear Complexity

- For *n* Voronoi sites = *n* cells/faces = F
- To apply Euler's formula,

  $F + V = E + 2$

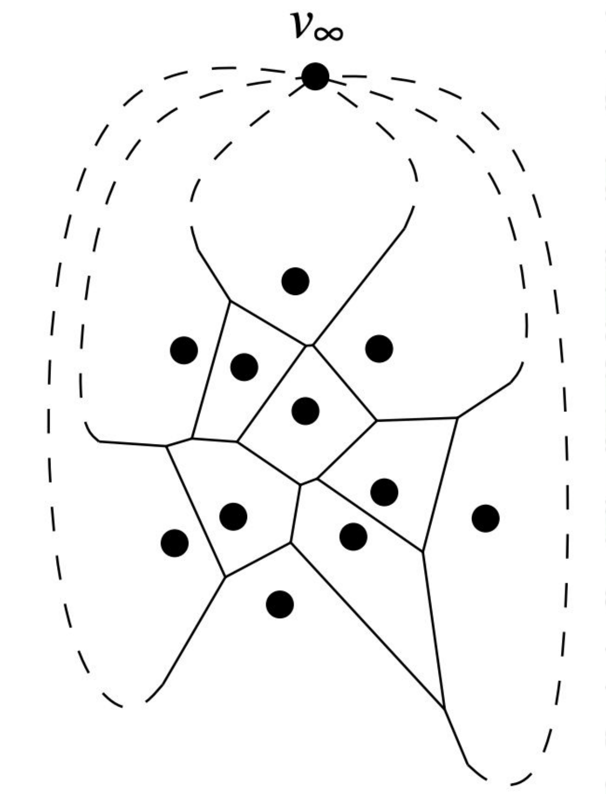
  we V_∞ to connecting all unbounded edges

- Σ all *vertex degrees* = $2 * E \leq 3 * (V+1)$
  - Every edge touches 2 vertices
  - Vertex degree = # edges that touch vertex
  - Minimum vertex degree = 3

- Substitute & algebra…

  # of vertices = V ≤ *2n-5*

  # of edges = E ≤ *3n-6*



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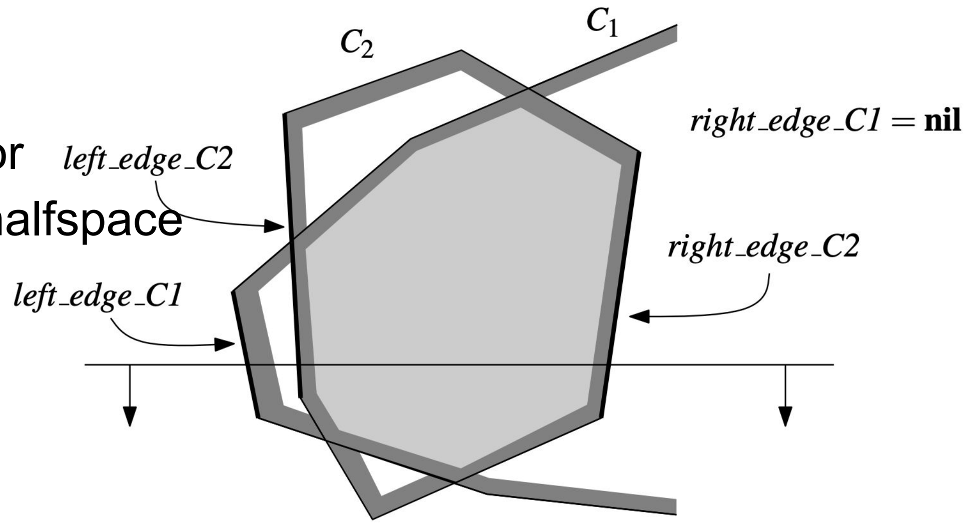
Brute Force Voronoi Algorithm - Analysis

- Given *n* Voronoi sites

Brute Force Voronoi Algorithm - Analysis

- Given *n* Voronoi sites
- For every point:
    - For every other point:
      Construct perpendicular bisector between two points, defines a halfspace
    - Intersect *n-1* half spaces
      Divide & conquer recursion, polygonal Sweep Line overlay

    - This is a Voronoi Cell
- Overall:



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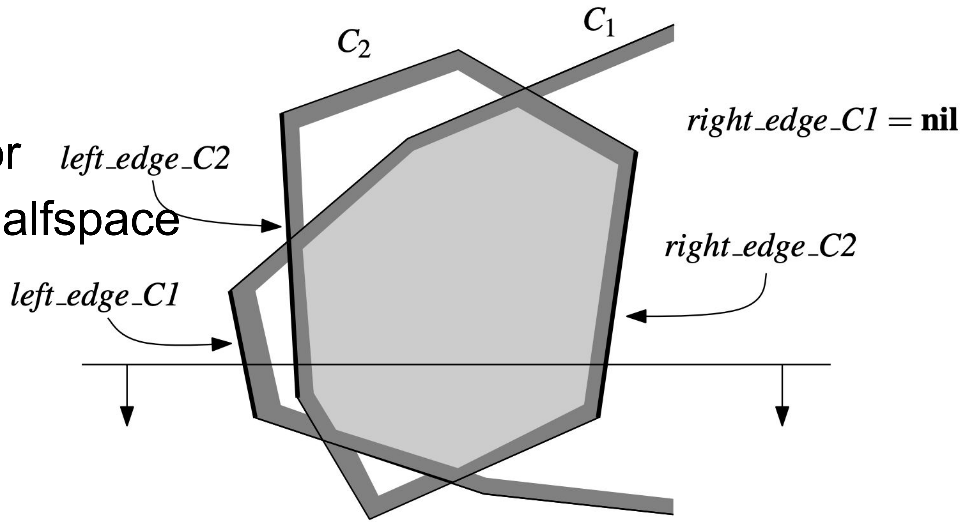
Brute Force Voronoi Algorithm - Analysis

- Given *n* Voronoi sites
- For every point: *O(n)*
  - For every other point: *O(n)*
    Construct perpendicular bisector between two points, defines a halfspace
  - Intersect *n-1* half spaces
    Divide & conquer recursion, polygonal Sweep Line overlay
    → *O(n log n)*
  - This is a Voronoi Cell
- Overall: → $O(n^2 \log n)$



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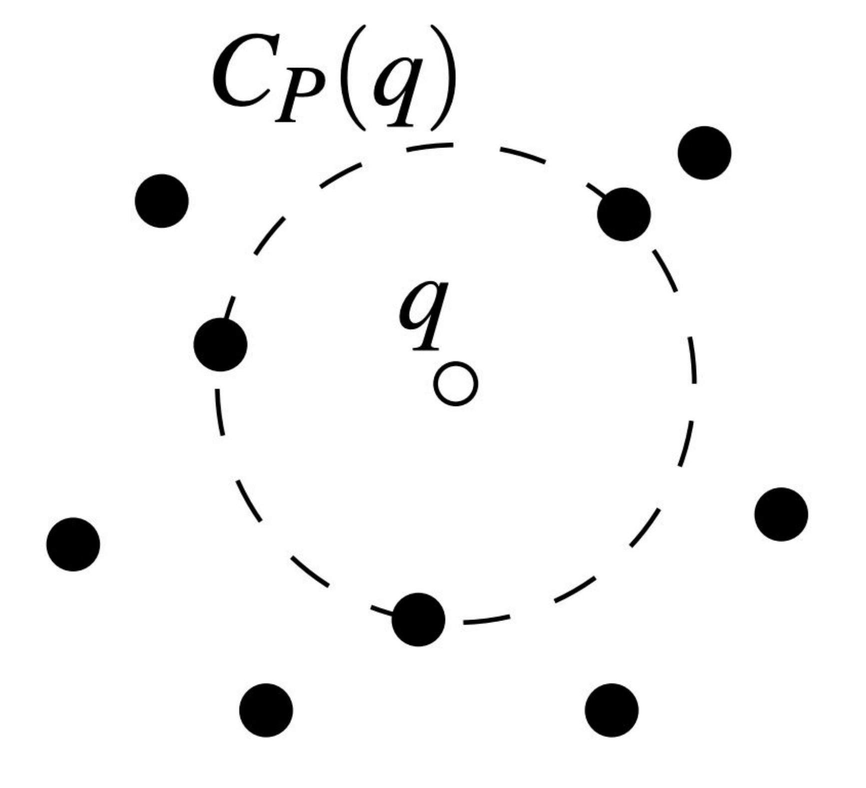
Voronoi Vertices

- A Voronoi vertex, *q*, is the center of a circle that passes through 3 (or more) Voronoi sites and does not contain any other sites inside of it.



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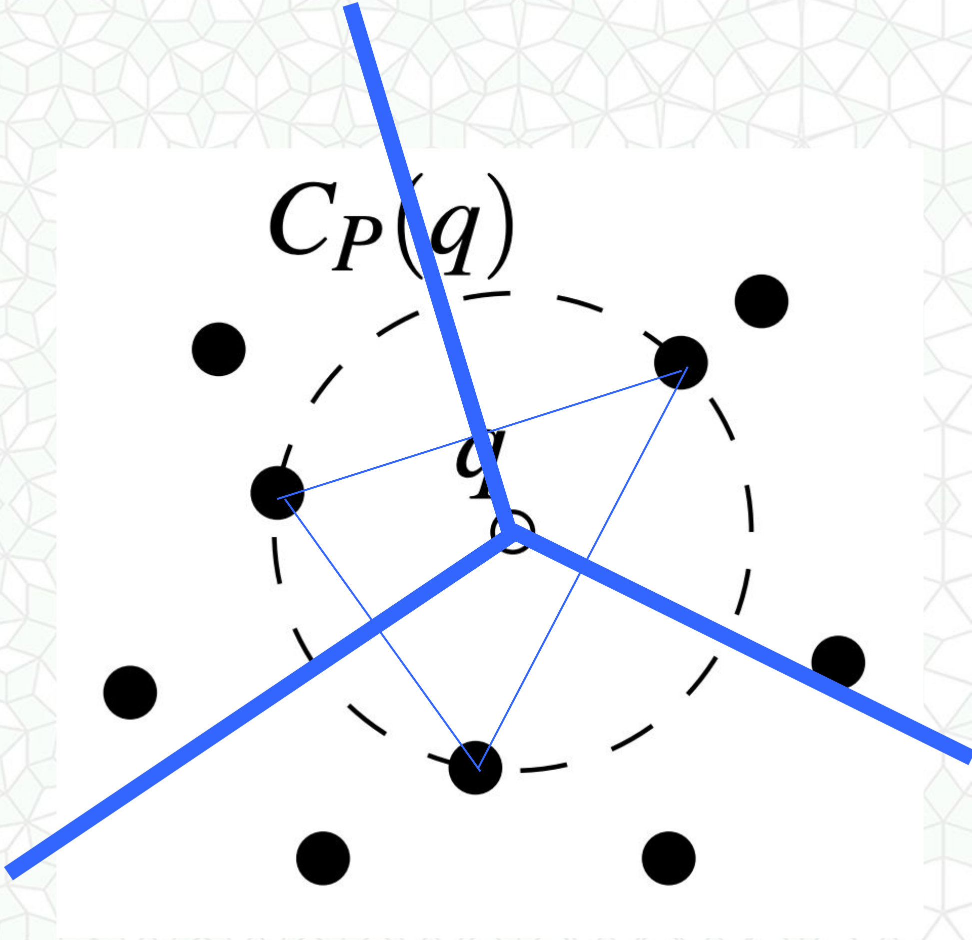
Voronoi Vertices

- A Voronoi vertex, *q*, is the center of a circle that passes through 3 (or more) Voronoi sites and does not contain any other sites inside of it.

- It is also the intersection of 3 (or more) perpendicular bisectors of those Voronoi sites.



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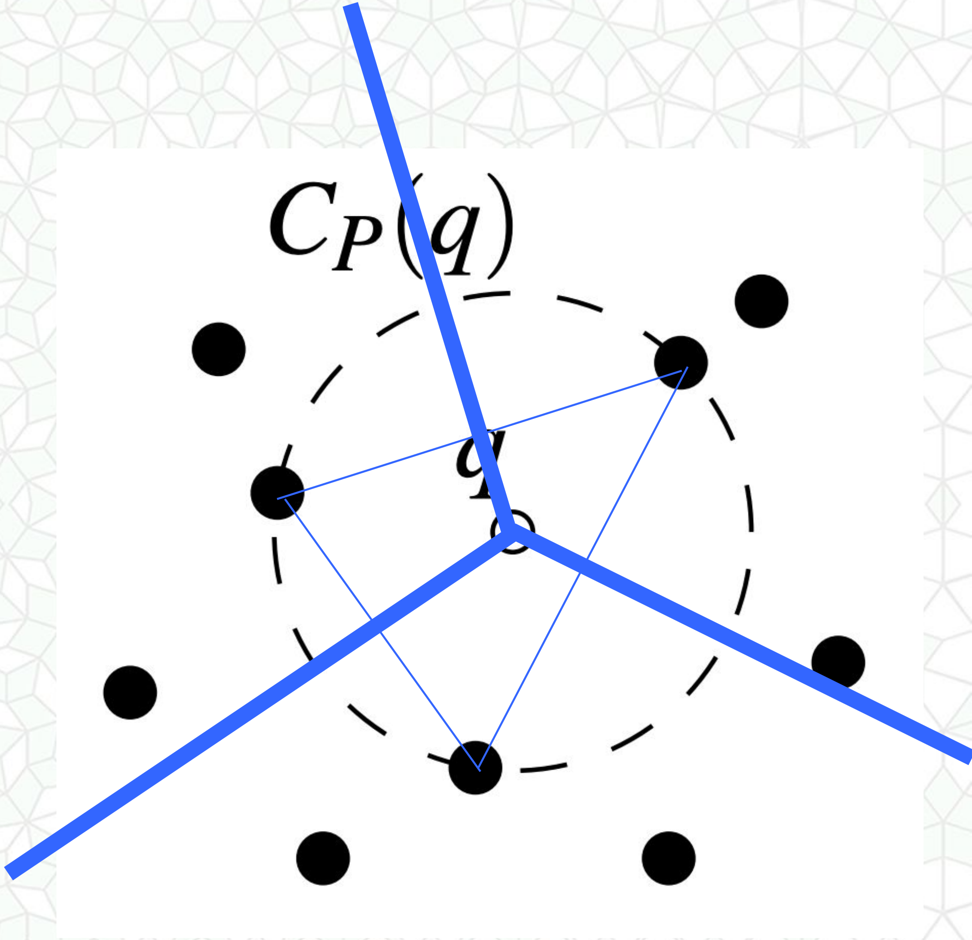
Voronoi Vertices

- Alternate Brute Force Construction algorithm:

- For all triples of Voronoi sites
  - If they are not collinear
  - Construct the circle



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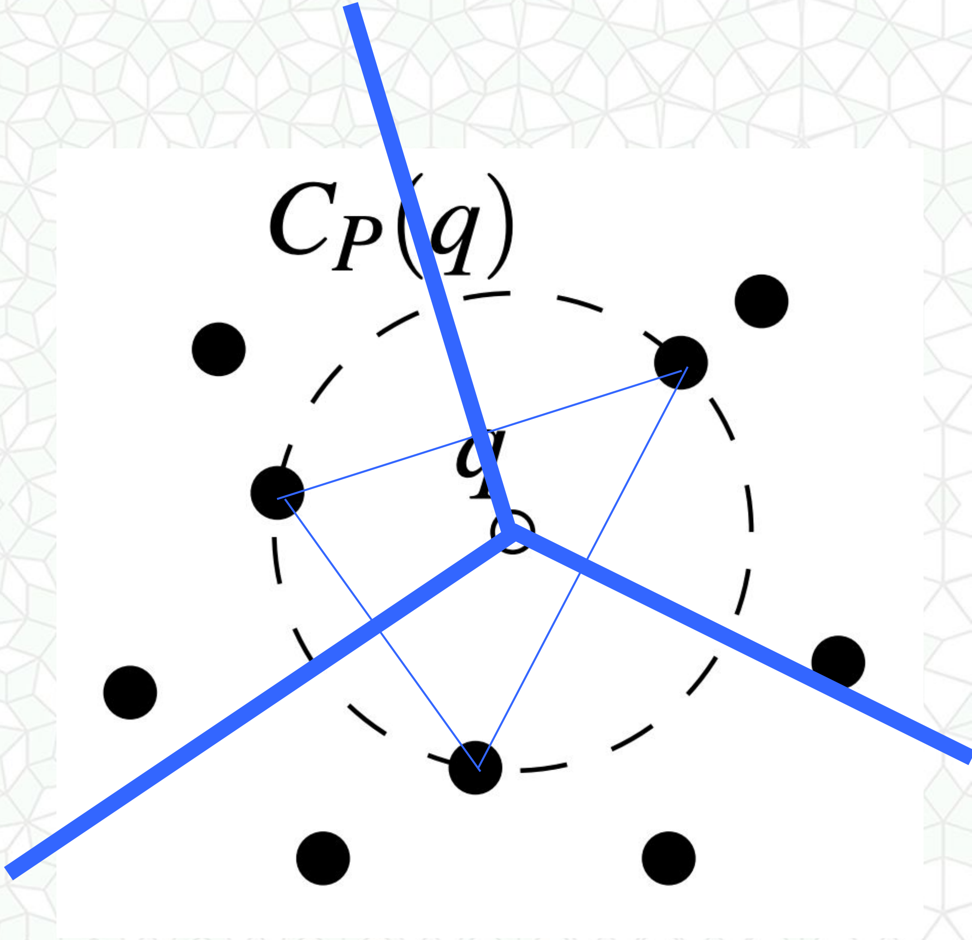
Voronoi Vertices

- Alternate Brute Force Construction algorithm:

- For all triples of Voronoi sites
  - If they are not collinear
  - Construct the circle
  - Check to see if any other Voronoi site lies within the circle
  - If not, keep the circle center as a Voronoi Vertex
- Overall:



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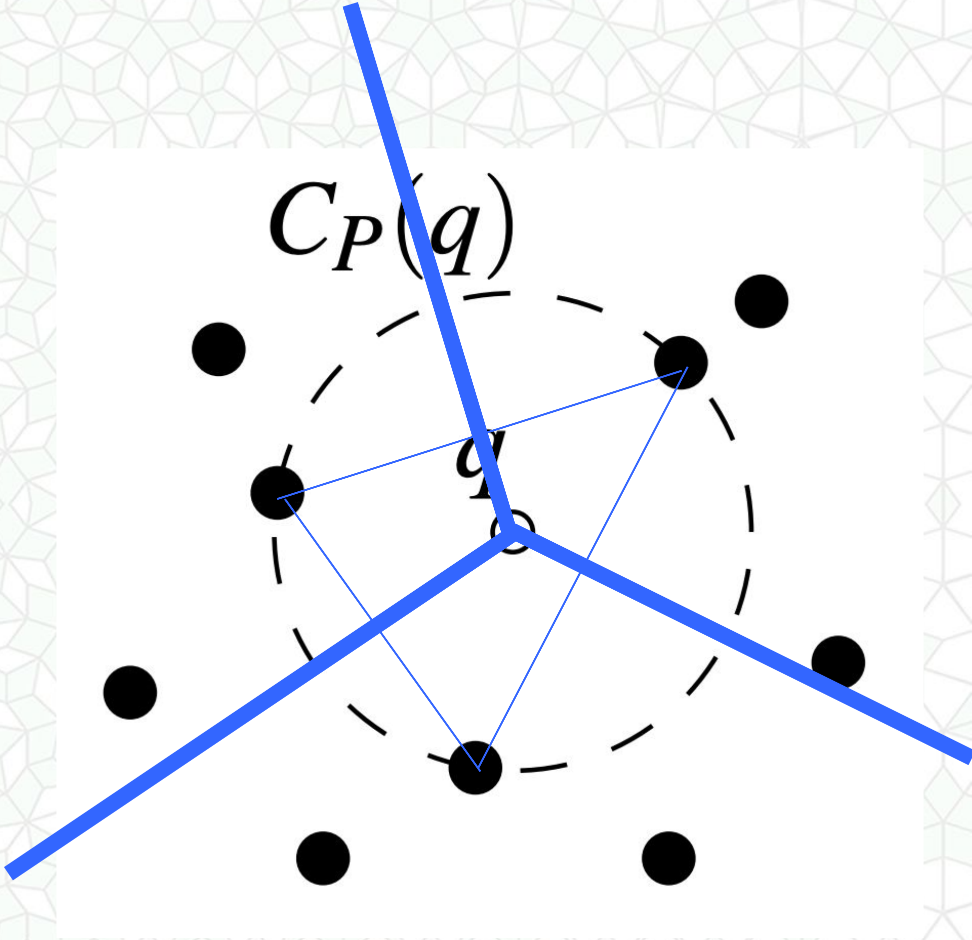
Voronoi Vertices

- Alternate Brute Force Construction algorithm:

- For all triples of Voronoi sites $O(n^3)$
  - If they are not collinear
  - Construct the circle
  - Check to see if any other Voronoi site lies within the circle $O(n)$
  - If not, keep the circle center as a Voronoi Vertex
- Overall: $\rightarrow O(n^4)$



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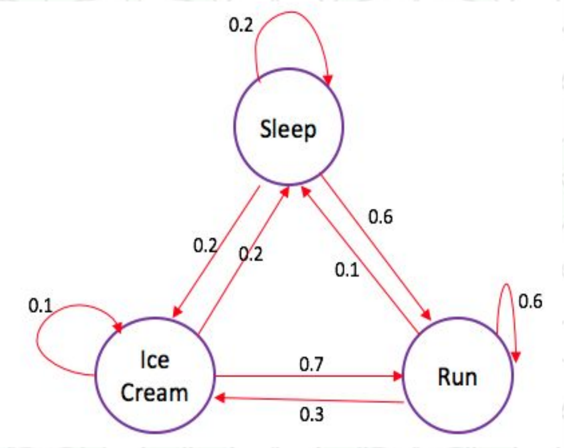
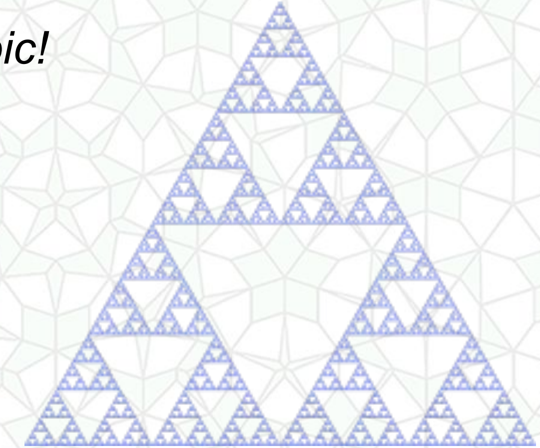
A Little History - Voronoi Diagram

- Ukrainian Mathematician Georgy Voronoy (1868-1908)
  - PhD advisor, Andrey Markov (1856-1922): Markov chains & Markov processes
  - PhD student, Wacław Sierpiński (1882-1969): Sierpiński Triangle (& other fractals)
  - PhD student, Boris Delaunay (1890-1980): Delaunay Triangulation – *foreshadowing a future topic!*
- a.k.a. “Dirichlet Tessellation” (only studied 2D & 3D?)
  - German Mathematician Peter Gustav Lejeune Dirichlet (1805-1859)
- a.k.a. “Thiessen Polygon”
  - Meteorologist Alfred H. Thiessen (1872-1956), *born in Troy, NY*! (PhD from Cornell)



https://www.datacamp.com/community/tutorials/markov-chains-python-tutorial



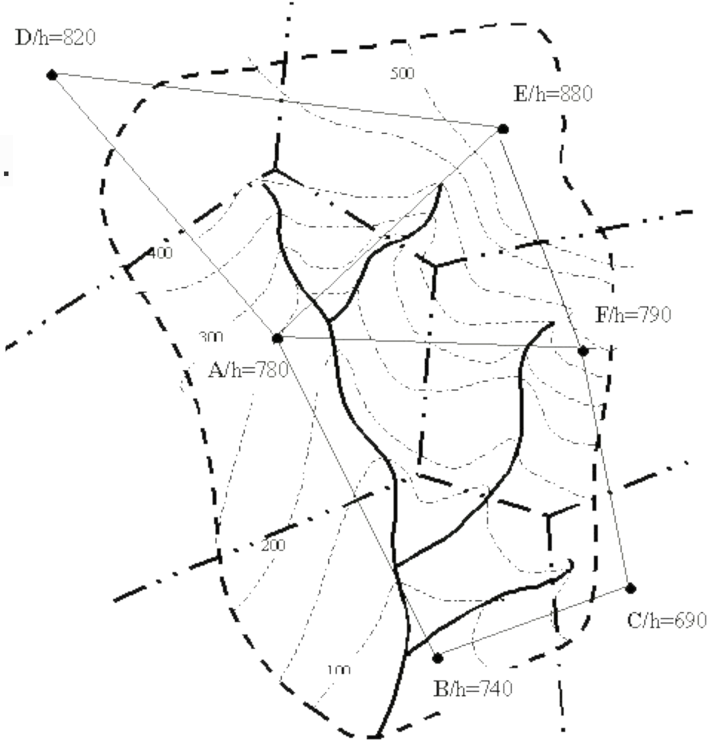
https://en.wikipedia.org/wiki/Sierpi%C5%84ski_triangle#/media/File:Sierpinski_triangle.svg

Thiessen Method - Computing Areal Precipitation

- Rainfall measurements at any station can be applied halfway to the next station in any direction.
- Rainfall is equal to observed rainfall at closest gauge.
- The weight of each rain gauge is computed by the area of the Thiessen Polygon.
- If the amount for any station is missing, the polygon must be changed.
- The Thiessen method is unable to consider orographic differences in rainfall distributions.

From: Encyclopedia of Earth Science, A. H. Schumann

Definition of **Orographic**: relating to mountains, e.g., resulting from the effects of mountains in forcing moist air to rise.



https://echo2.epfl.ch/VICAIRE/mod_1a/chapt_3/sol_exercices.htm

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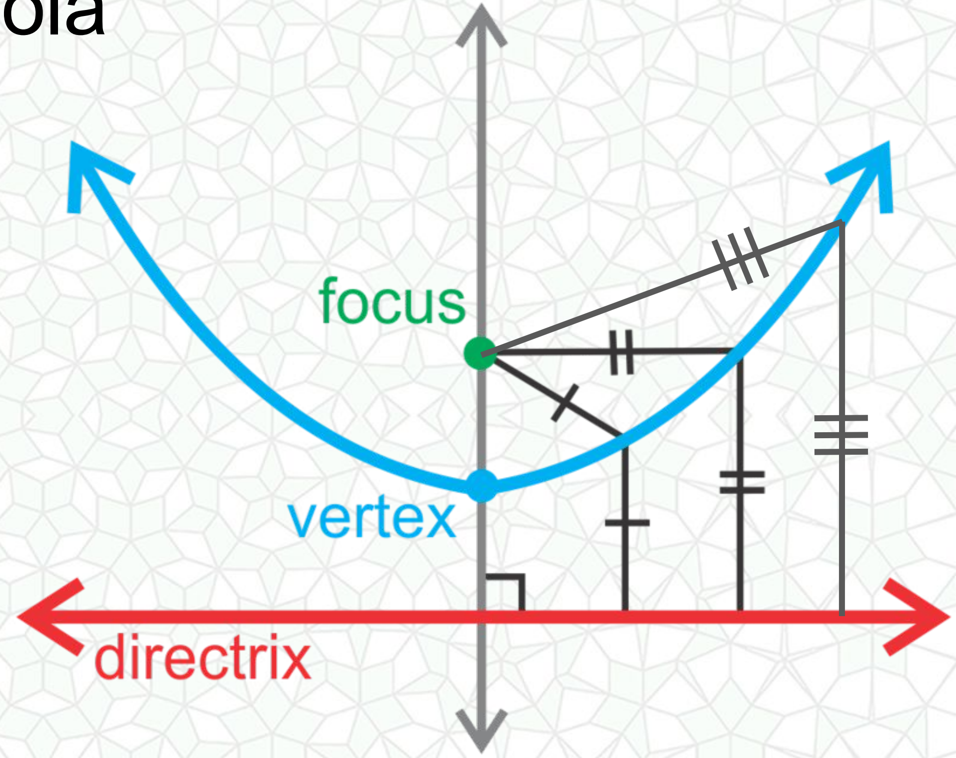
How to Graph a Parabola

All points on
a parabola
are equidistant
from
a point,
the focus
and
a line,
the directrix



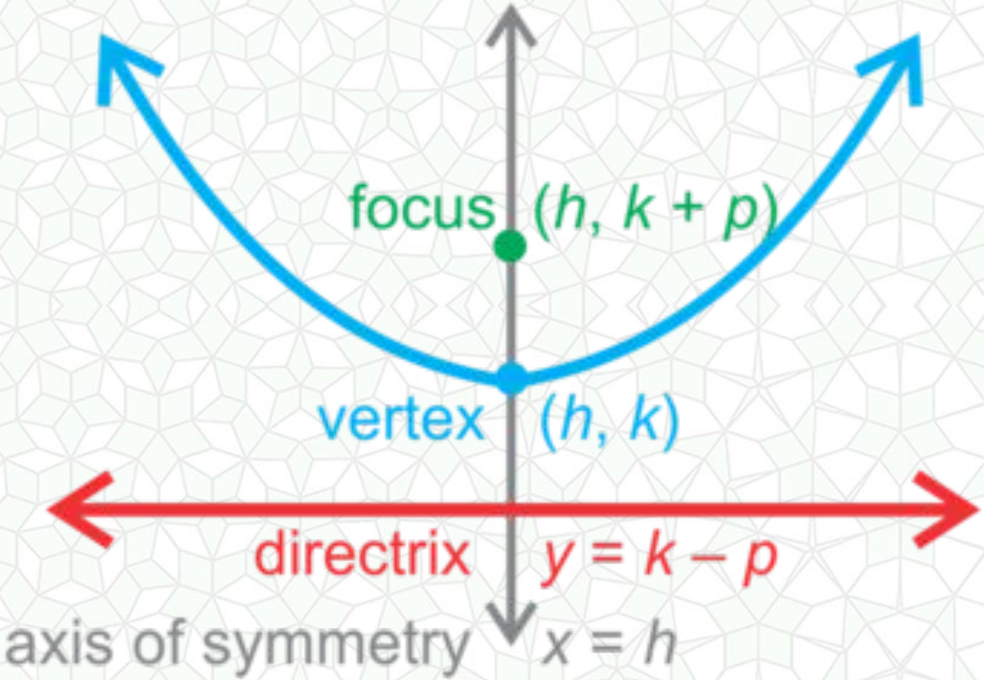
https://www.ck12.org/book/ck-12-algebra-ii-with-trigonometry-concepts/section/10.1/

How to Graph a Parabola

- Typical parabola equation:
  $y = ax^2 + bx + c$

- Rewrite as:
  $(x-h)^2 = 4p(y-k)$

- *h* gives you the vertical axis of symmetry
- *p* & *k* gives you the focus & directrix



https://www.ck12.org/book/ck-12-algebra-ii-with-trigonometry-concepts/section/10.2/

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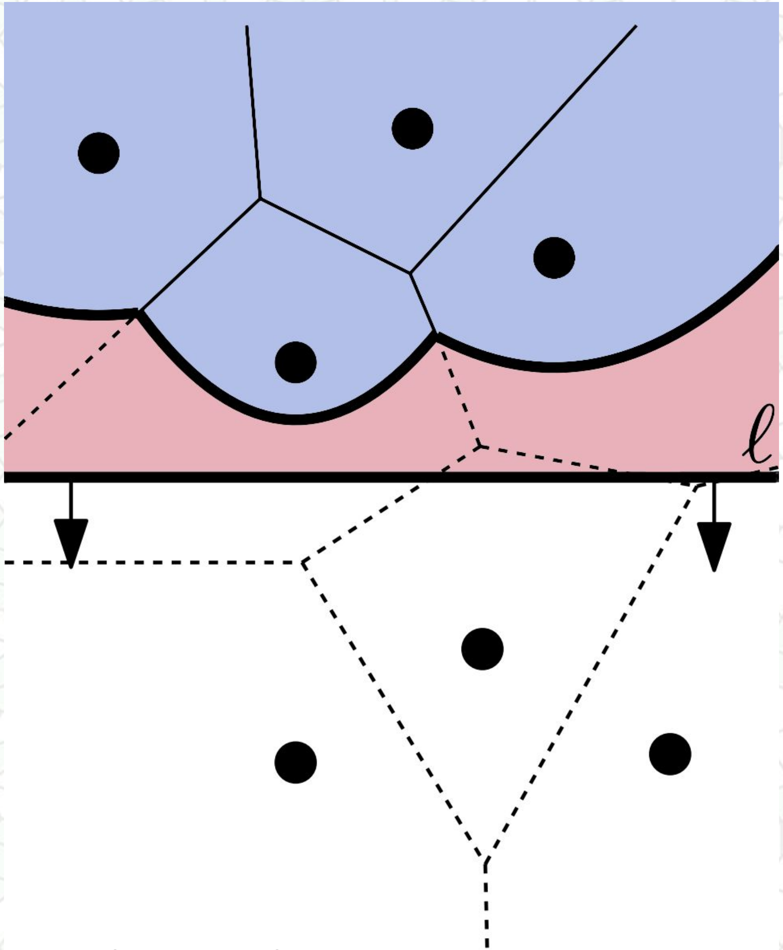
“Fortune’s Algorithm”

- Can we do better than $O(n^2 \log n)$ construction runtime? *Yes!*
- “A sweepline algorithm for Voronoi diagrams”, Steven Fortune, SoCG 1986

- Sort the Voronoi sites vertically
- Move sweep line from top to bottom

- Region above the “beach line” is known
- Region between the beach line and the sweep line may be impacted by Voronoi sites below the sweep line!



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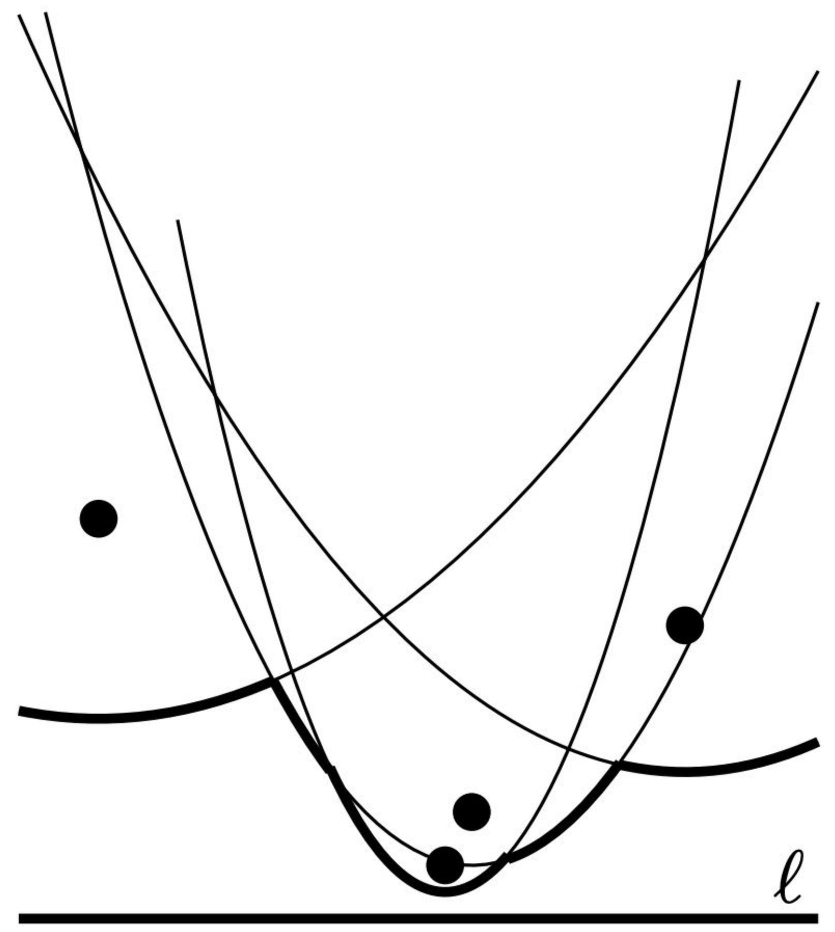
“Beach Line”

- Sequence of parabolic segments.
- Each Voronoi site above the sweep line creates a parabola. *site = focus, sweep line = directrix*
- Beach line is the lowest point of all of the parabolas for every x value.
- Beach line is x-monotone – every vertical line intersects it exactly once.
- Breakpoints – intersection of two parabolic ars – trace edges between two of Voronoi cells.



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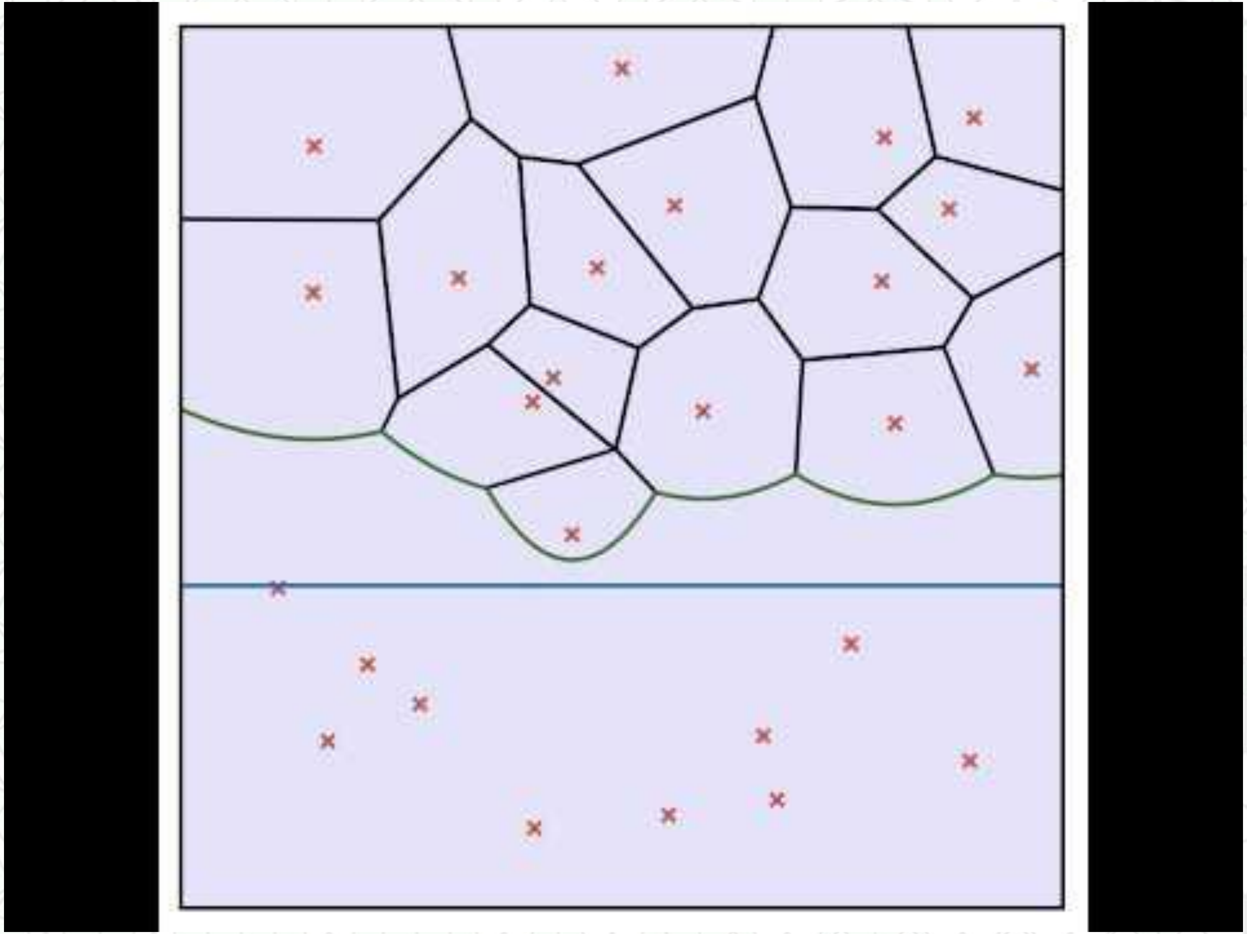
Sweep line algorithm - Voronoi tessellation

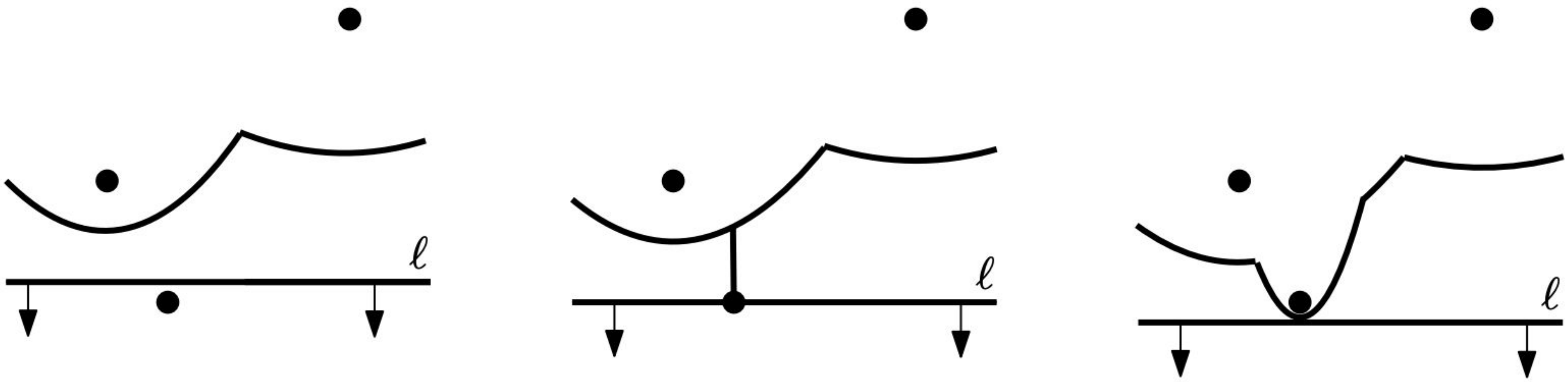
Kevin Schaal



https://www.youtube.com/watch?v=k2P9yWSMaXE

Event: New Parabola Appears

- Only happens when the sweep line reaches the next Voronoi Site
- Initially the parabola is degenerate, with width = 0



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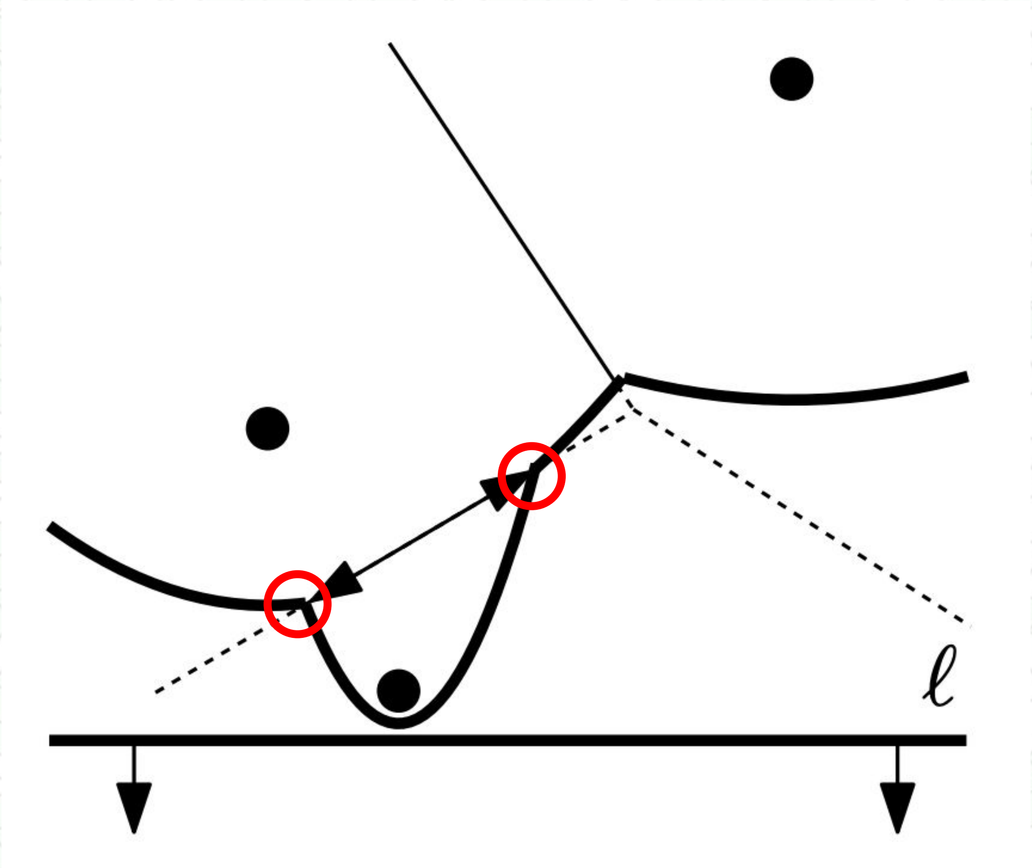
New Voronoi Edge

- New Parabolic Arc adds two new breakpoints
- New **Voronoi edge** is traced between those breakpoints



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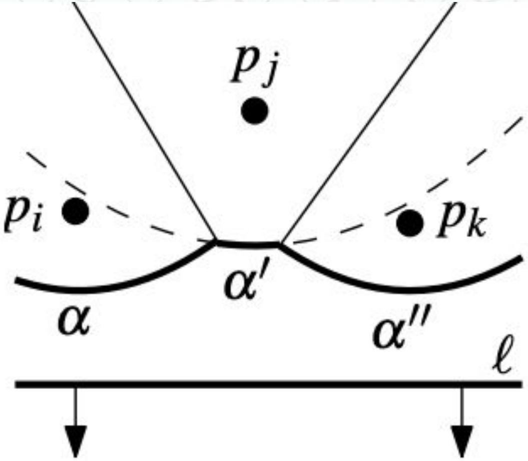
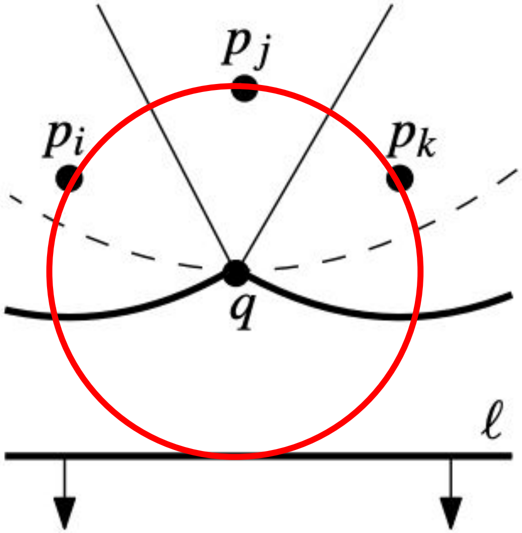
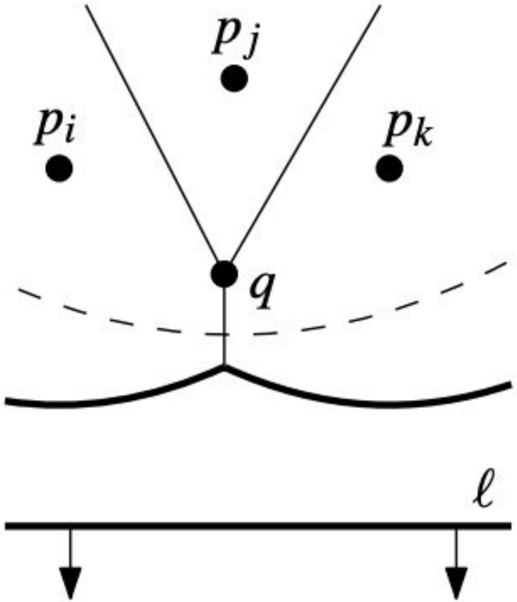
Event: Parabolic Arc is Absorbed

- Arc disappears when beach line reaches point q, a **Voronoi vertex,** center of circle that passes through 3 Voronoi sites / parabola foci







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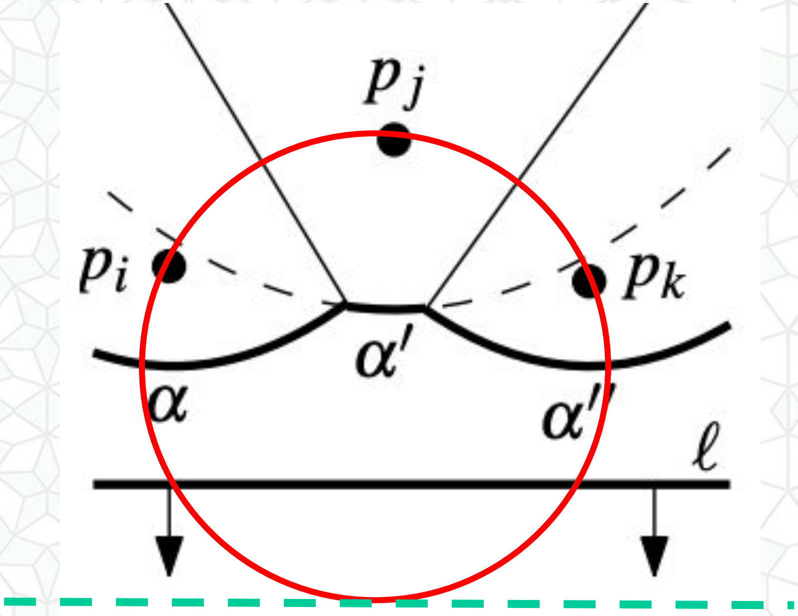
Outline for Today

- Homework 5 Questions?
- Last Time: Point Location & Trapezoidal Maps
- Motivating Application: Social Geography
- Observations about Voronoi Diagrams
- Brute Force Constructions & Analysis & Complexity
- A History of the Names Voronoi/Dirichlet/Thiessen
- How to Graph A Parabola
- Sweep Line Algorithm to Construct a Voronoi Diagram
- Analysis of Sweep Line Algorithm
- Next Time: More Voronoi Diagrams!

Sweep Line Algorithm

- For *n* Voronoi sites
- New Arc Events: Sort Voronoi sites vertically
- Keep a horizontal sorted ordering of the parabolic arcs on the current beachline.
- (Potential) Arc Absorption Events: For each triple of neighboring arcs α, α’, α’’ on the beachline, compute the **circle**, and **tangent sweep line**

- Move sweep line to the next event…
- Overall:



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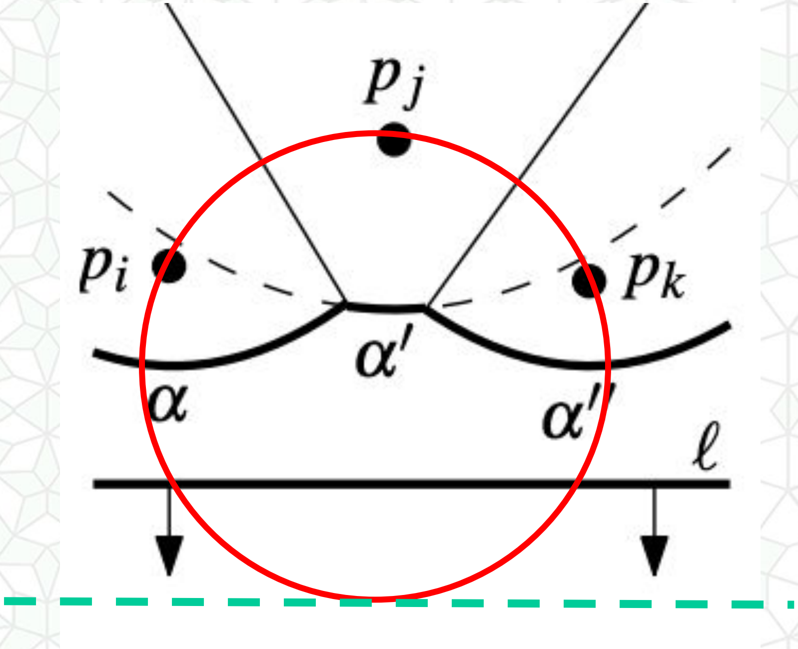
Sweep Line Algorithm

- For n Voronoi sites
- New Arc Events: Sort Voronoi sites vertically → $O(n \log n)$
- Keep a horizontal sorted ordering of the parabolic arcs on the current beachline. $2n$ arcs maximum
- (Potential) Arc Absorption Events: For each triple of neighboring arcs α, α', α'' on the beachline, compute the **circle**, and **tangent sweep line** → $O(n)$ Voronoi vertices
- Move sweep line to the next event…
- Overall: → $O(n \log n)$



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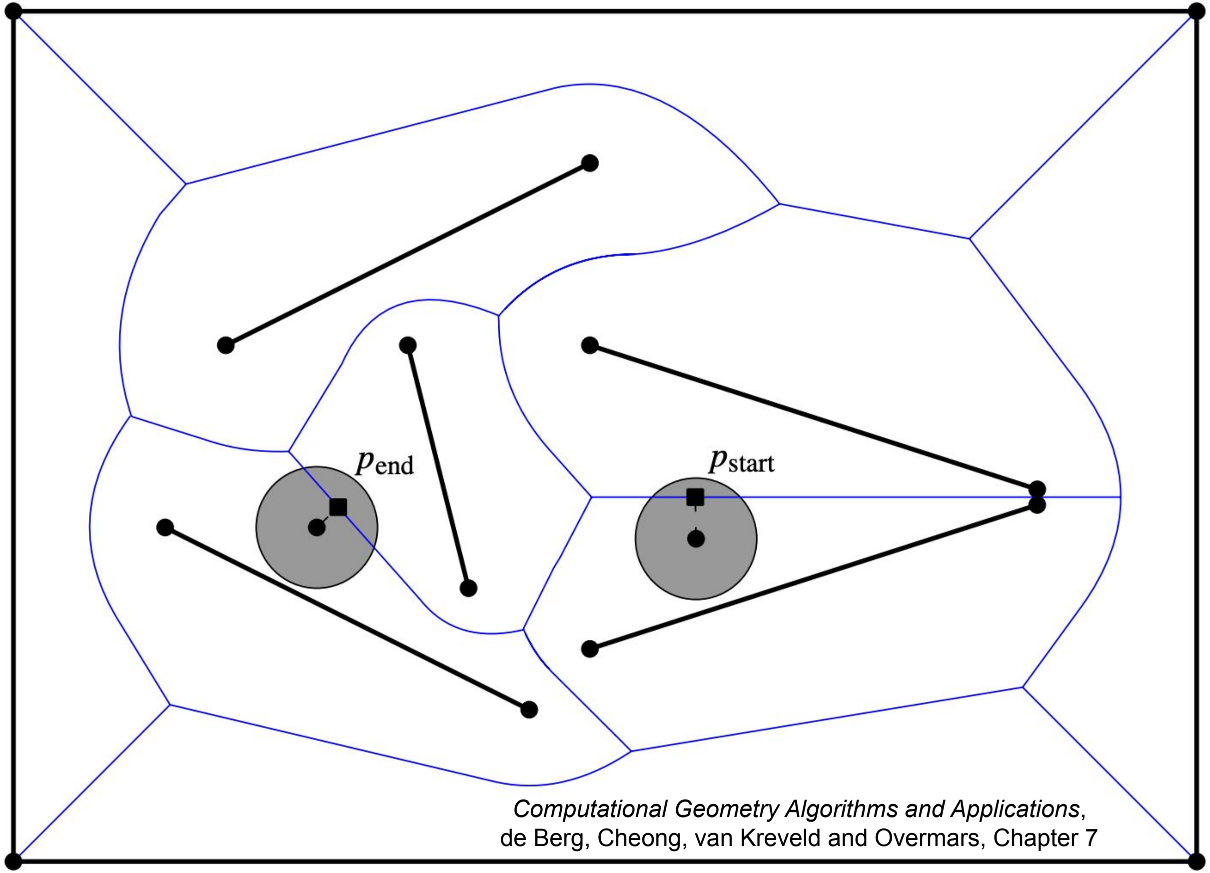
Next Time: Voronoi Diagram of Line Segments!



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