CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/F23/

Lecture 11: Voronoi Diagrams, Part 2

Outline for Today

- Quiz 1: Friday October 13th, 2-3:50pm, in class (no lecture)
- Last Time: Line Sweep construction of Voronoi Diagram
- Closest Point to a Line Segment
- Voronoi Diagram of Line Segments
- Motivation Application: Robotic Motion Planning
- Farthest Point Voronoi Diagram
- Motivating Application: Smallest Annulus
- Medial Axis & Higher-Order Voronoi Diagrams
- Next Time: Centroidal Voronoi Diagram & K-Means Clustering

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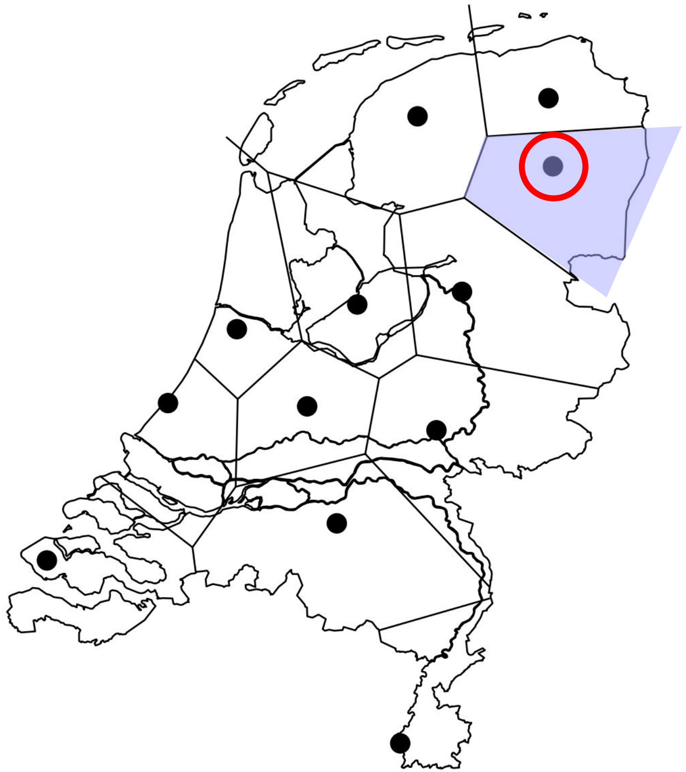
Voronoi Diagram - Social Geography

- There are a bunch of grocery stores spread across a large city.
- You're planning to open another grocery store at a specific location.
- How many customers can you expect at the new store location?
  Customers will choose the new store if it is closer to their home than their current store.

- a.k.a. The "Post Office Problem"



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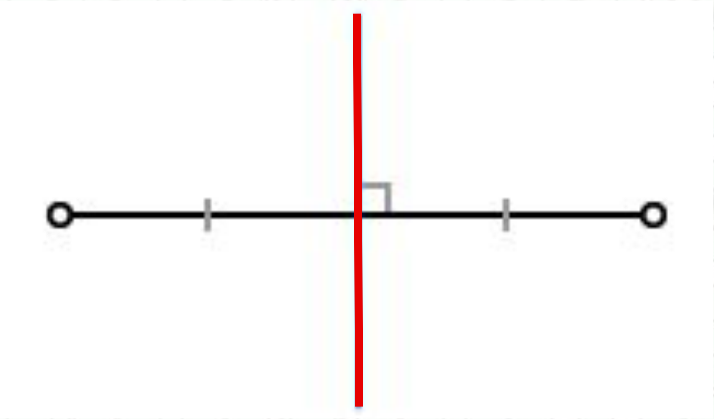
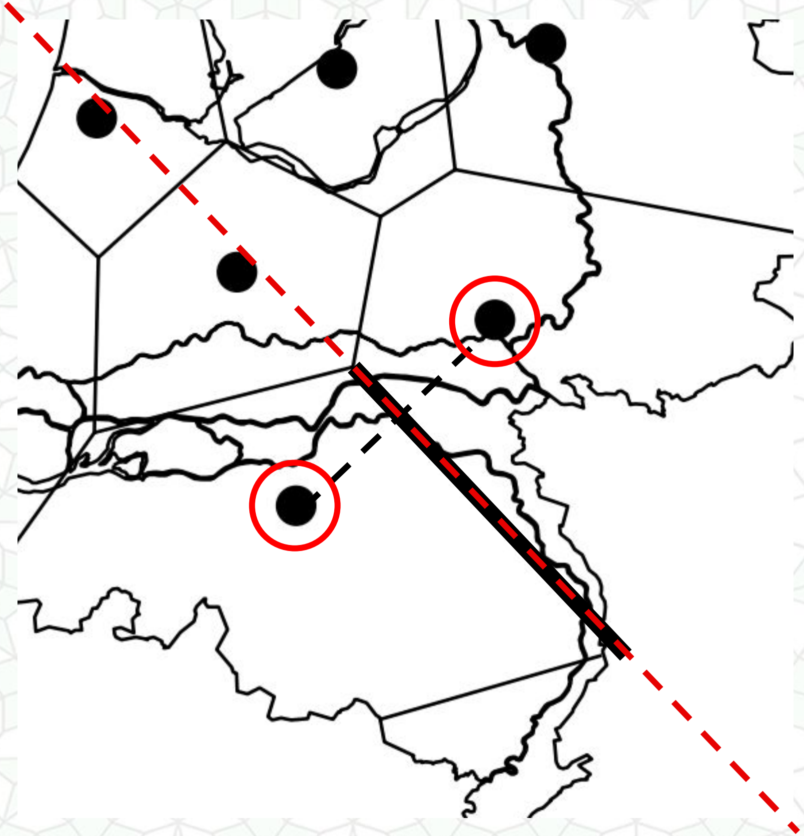
Observation: Perpendicular Bisector

- Points on the edge between two Voronoi cells are equidistant from two Voronoi sites.
- Edges of Voronoi cells are perpendicular bisectors of two Voronoi sites.



https://euclidea.fandom.com/wiki/Perpendicular_Bisector



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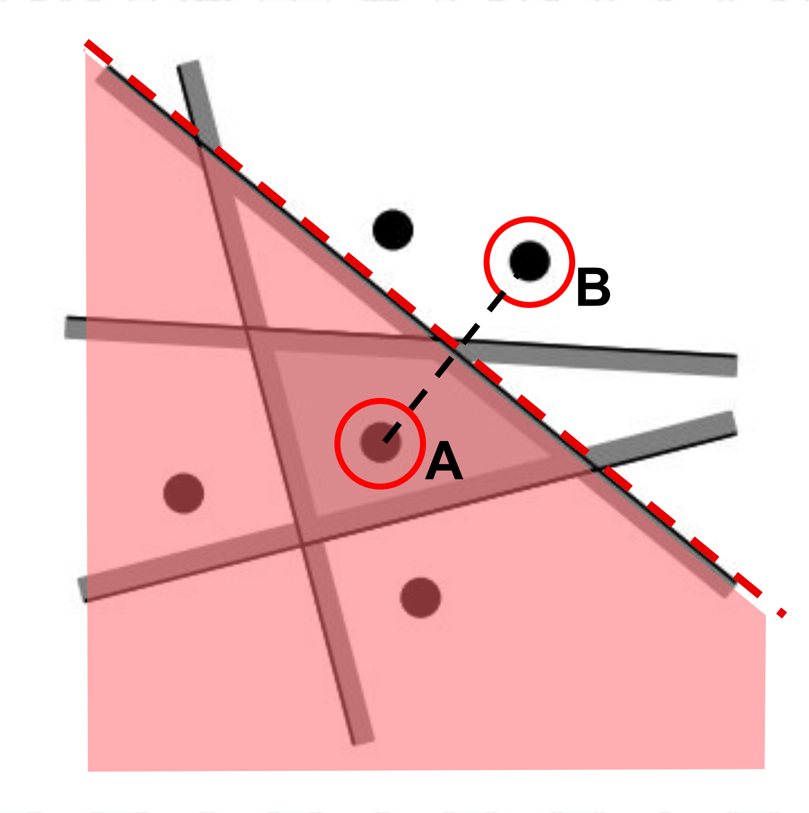
Observation: Intersection of Half Spaces

- All points that lie on one side of the perpendicular bisector,
- Are the half-space of points that will chose site A over site B because site A is closer than site B.

- This suggests a brute force construction algorithm…



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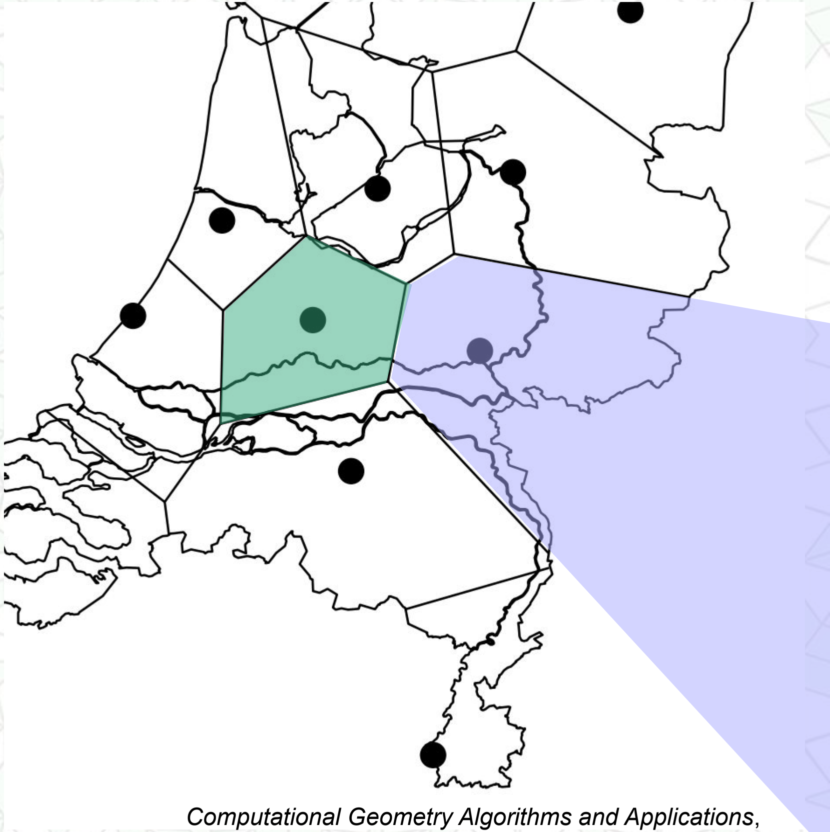
Observation: Voronoi Cells are Convex

- Because a Voronoi cell is the intersection of half-spaces…
- A Voronoi Cell must be convex
- *Note: Some Voronoi Cells are unbounded*



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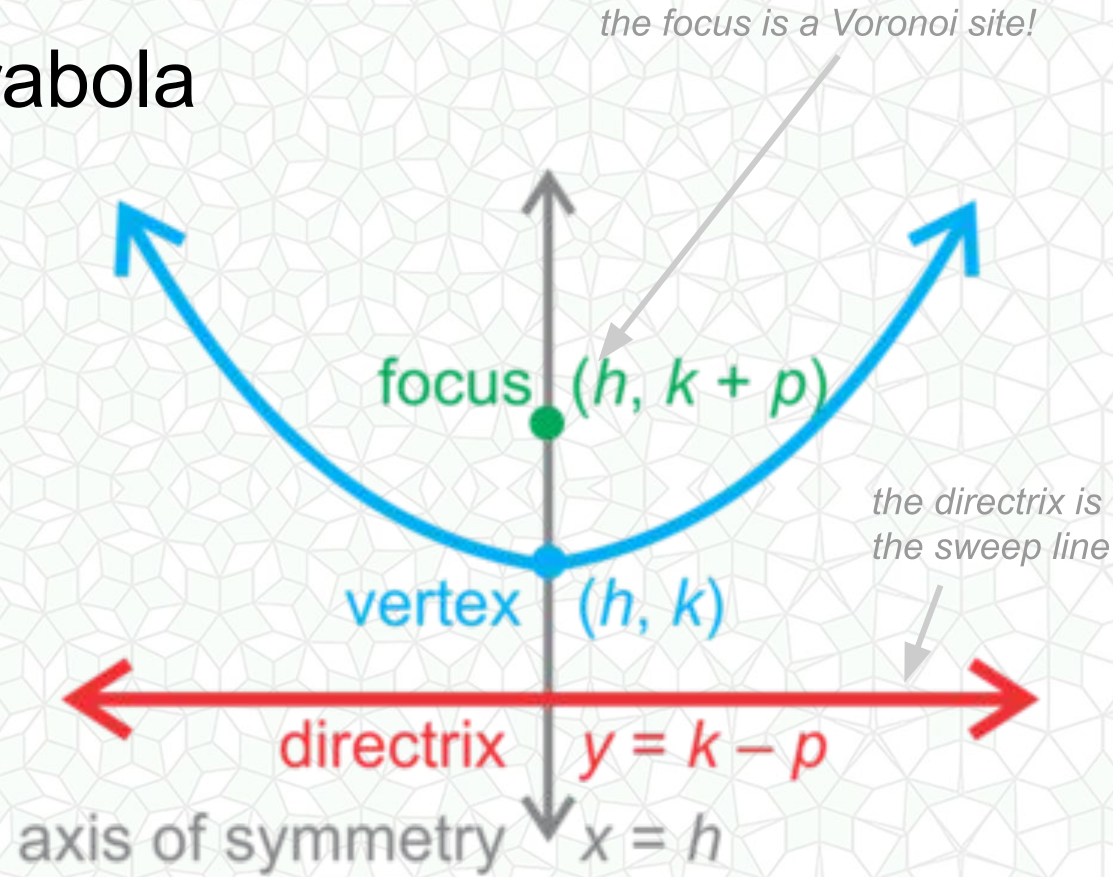
How to Graph a Parabola

- Typical parabola equation
  $y = ax^2 + bx + c$

- Rewrite as
  $(x-h)^2 = 4p(y-k)$

- *h* gives you the
  vertical axis of symmetry
- *p* & *k* gives you the
  focus & directrix

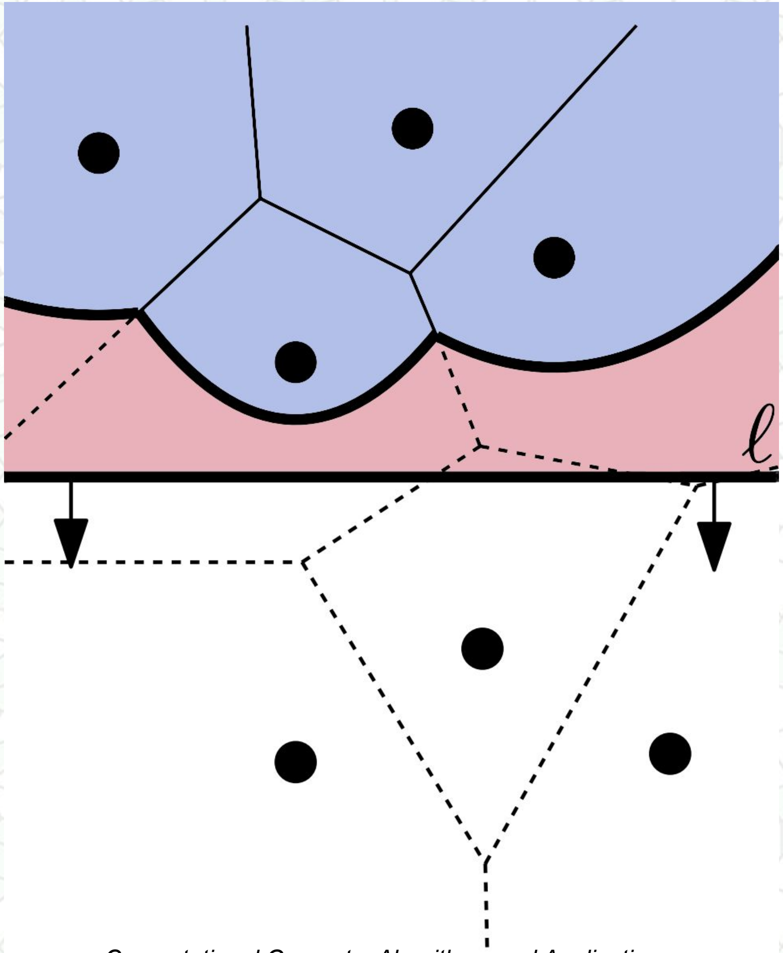


https://www.ck12.org/book/ck-12-algebra-ii-with-trigonometry-concepts/section/10.2/

“Fortune’s Algorithm”

- “A sweepline algorithm for Voronoi diagrams”, Steven Fortune, SoCG 1986
- Sort the Voronoi sites vertically
- Sort the events vertically
  - new parabola + new edge
  - remove arc + new vertex
- Sort the arcs horizontally
- Move sweep line from top to bottom
- Overall: *O(n log n)*

- Region above the “beach line” is known
- Region between the beach line and the sweep line may be impacted by Voronoi sites below the sweep line!



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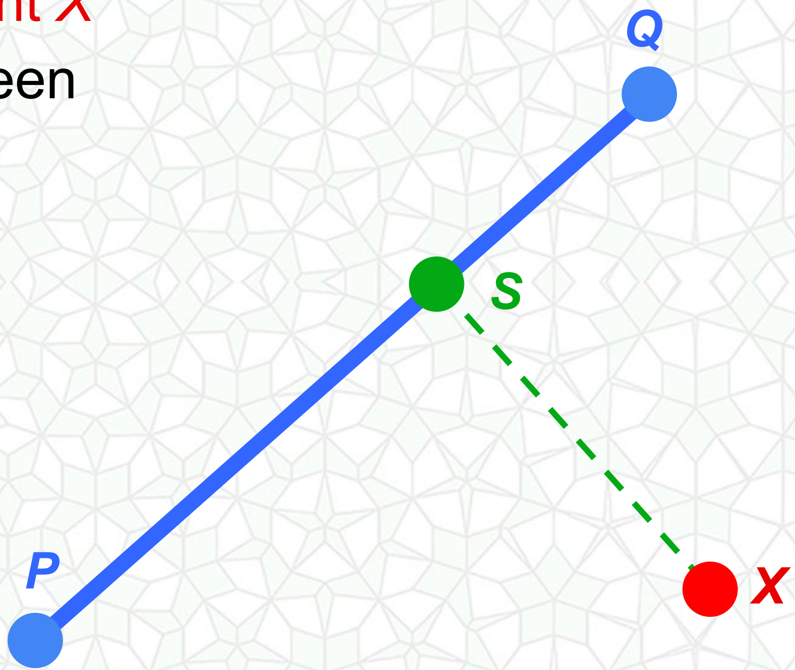
Closest Point on a Line Segment

- Given a line segment *PQ* and a point *X*
- What is the smallest distance between them?

- a.k.a. What is the closest point *S*, on the line segment?
- What is the distance between *S* and *X*?



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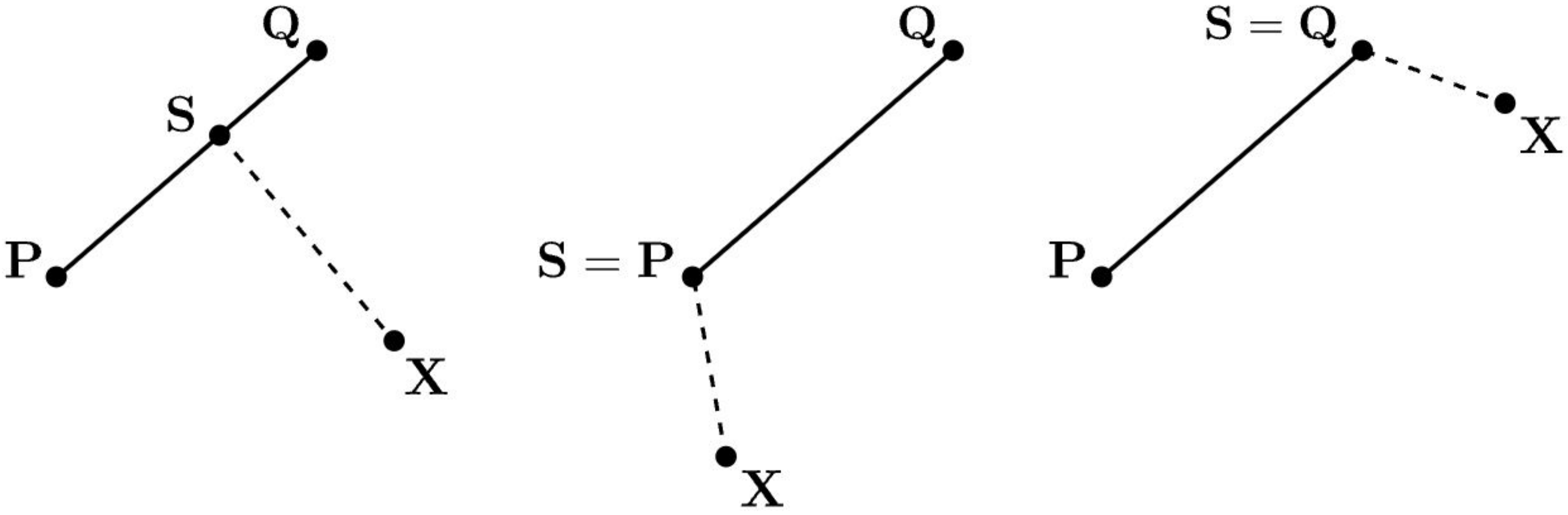
Closest Point on a Line Segment

Need to explicitly handle 3 cases:



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Closest Point on a Line Segment

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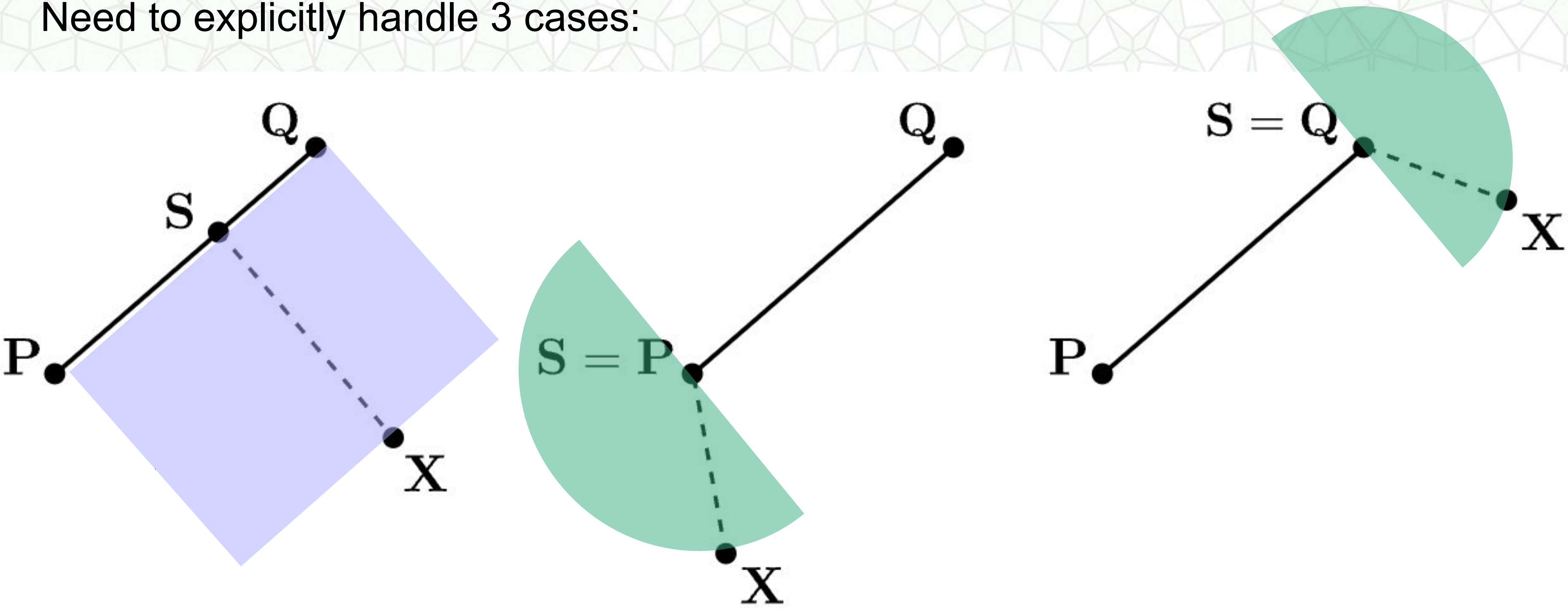

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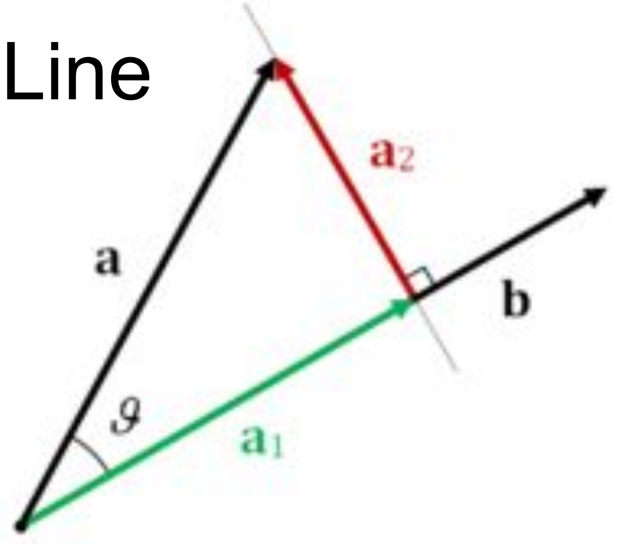
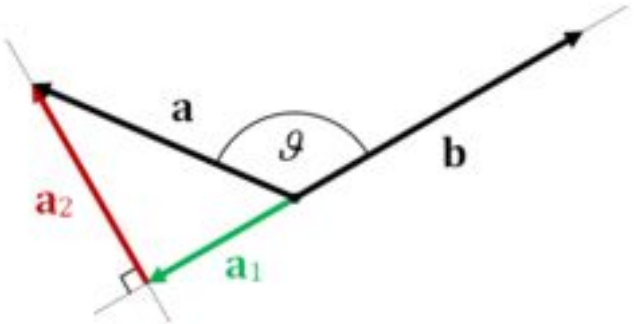
Orthogonal Projection of Point to Line





https://en.wikipedia.org/wiki/Vector_projection

Orthogonal Projection of Point to Line

- Break the vector **a** into two subvectors, one parallel to **b**, one perpendicular to **b**
- If ϴ < 90°, cos ϴ will be positive
- If ϴ > 90°, cos ϴ will be negative

length of $\boldsymbol{a_1}$

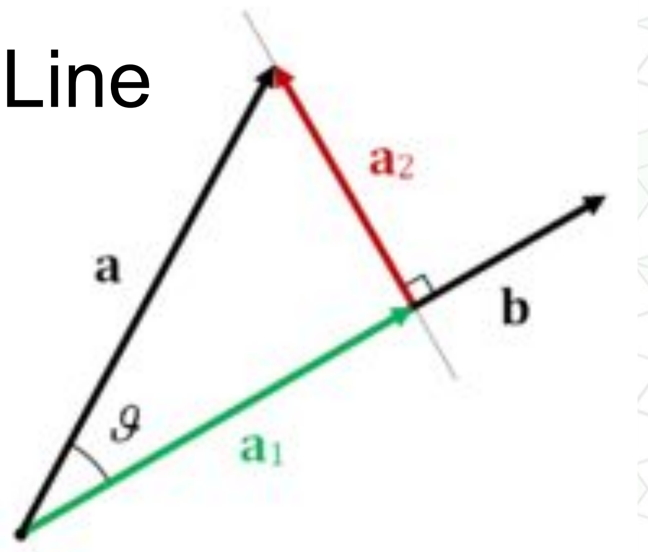
$\mathbf{b}$ normalized

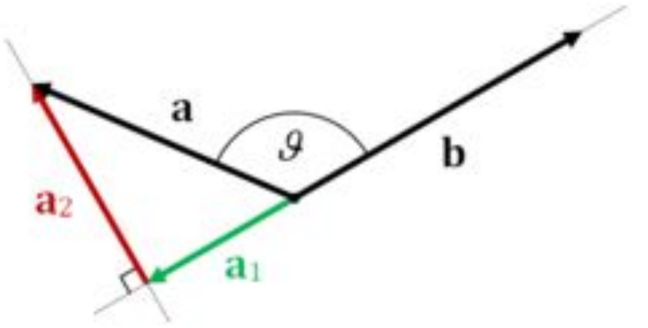
$$a_1 = \|\mathbf{a}\| \cos\theta = \mathbf{a} \cdot \hat{\mathbf{b}}$$

$$\mathbf{a}_1 = \left(\mathbf{a} \cdot \hat{\mathbf{b}}\right) \hat{\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}$$

vector $\boldsymbol{a_1}$

$$\mathbf{a}_2 = \mathbf{a} - \mathbf{a}_1$$





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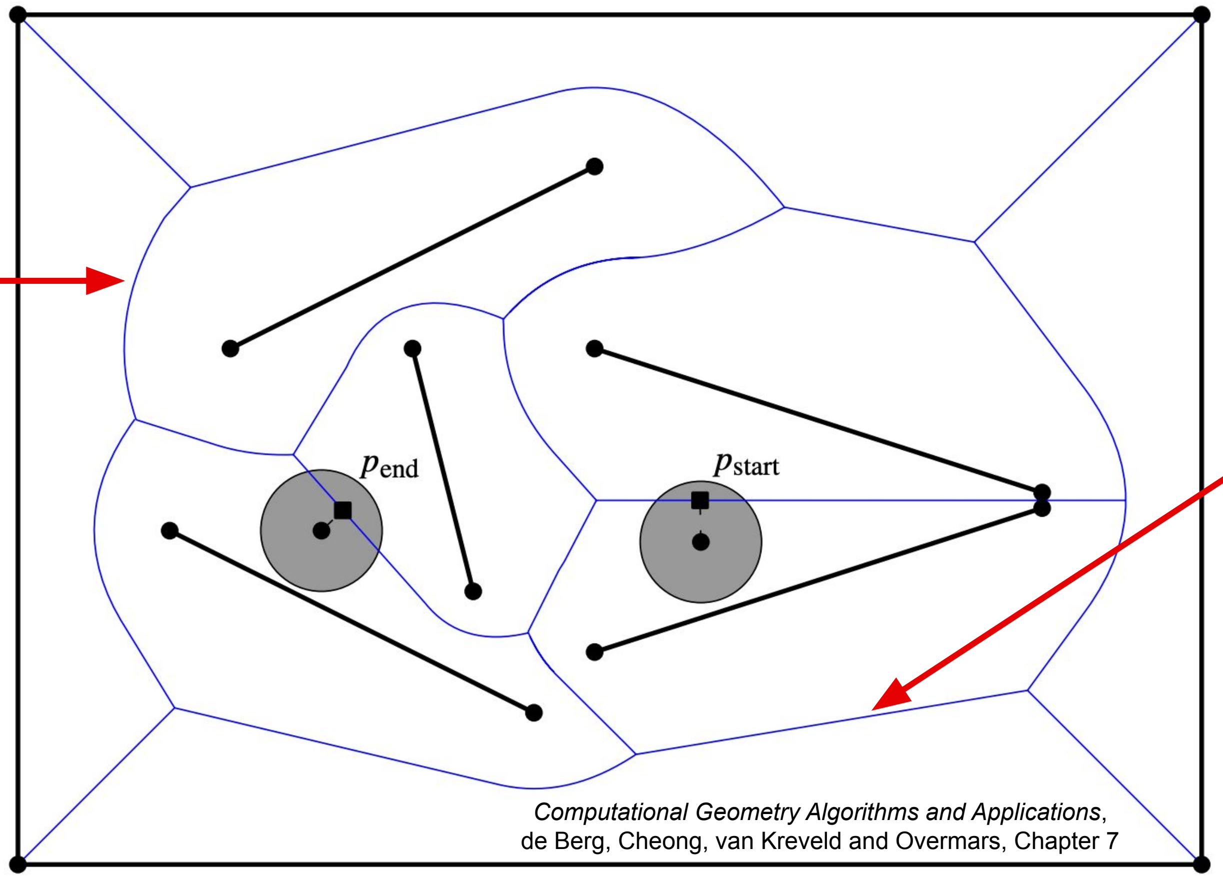
Voronoi Diagram of Line Segments

some edges are arcs



some edges are straight lines

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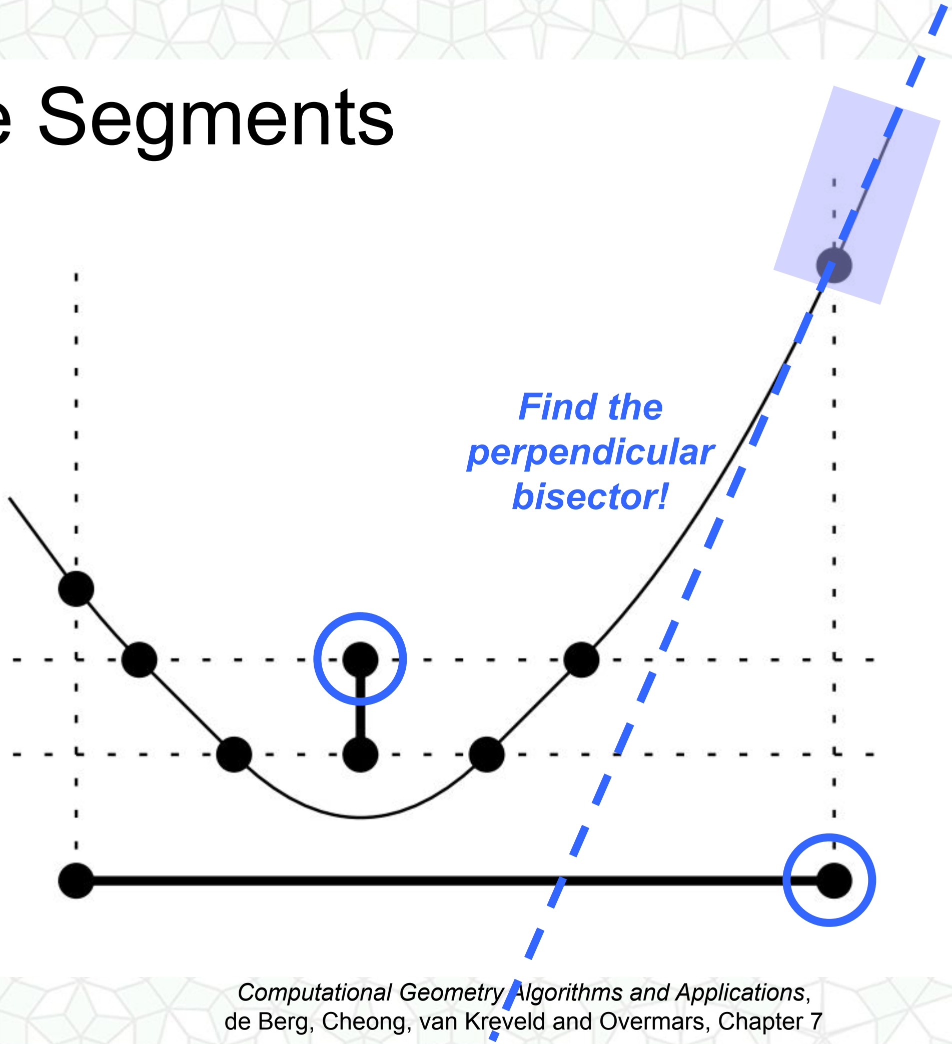
Voronoi Diagram of Line Segments

- Points equidistant between two points form a line.



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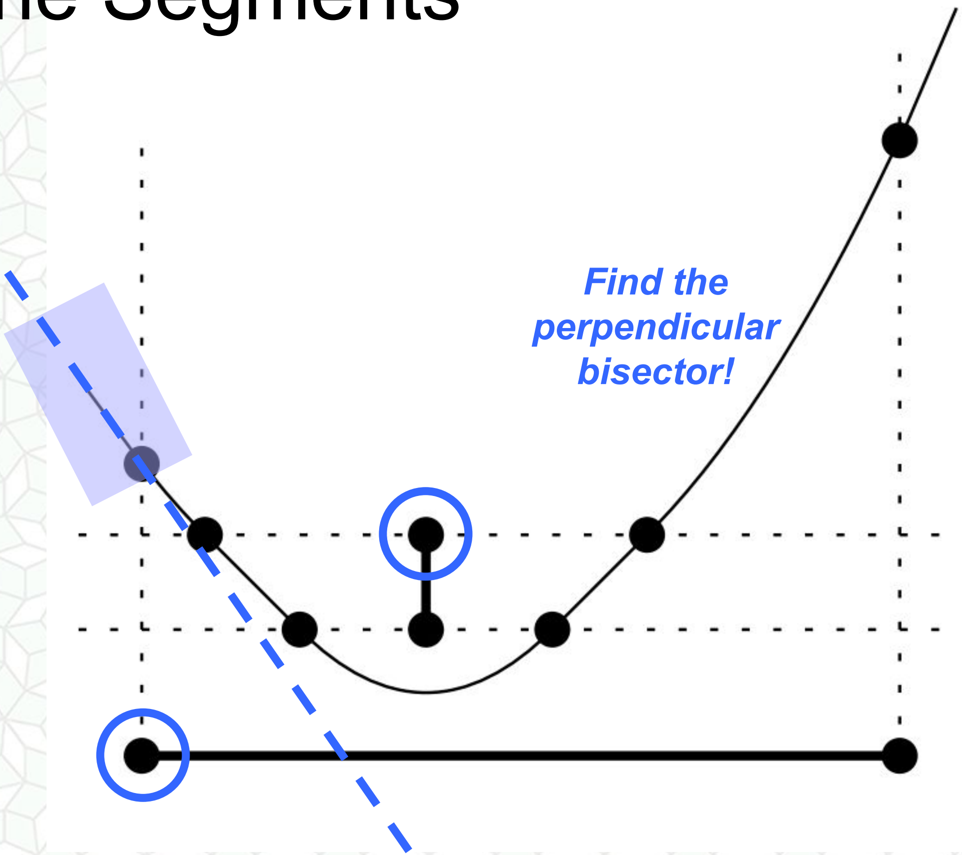

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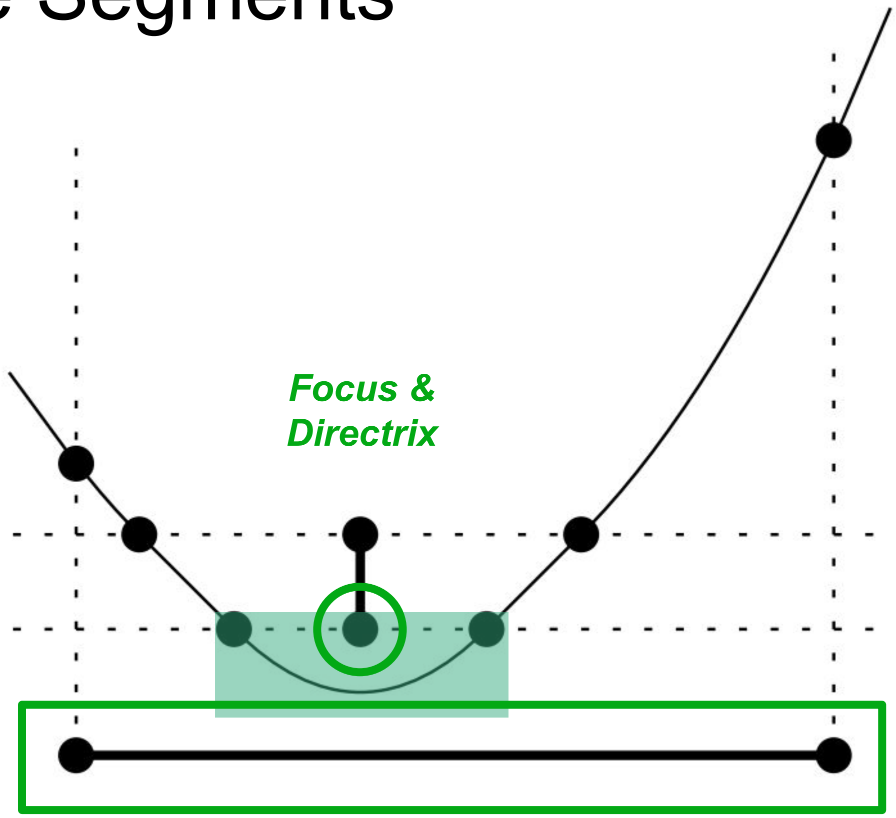
Voronoi Diagram of Line Segments

- Points equidistant between two points form a line.
- Points equidistant between a point and a line form a parabola.



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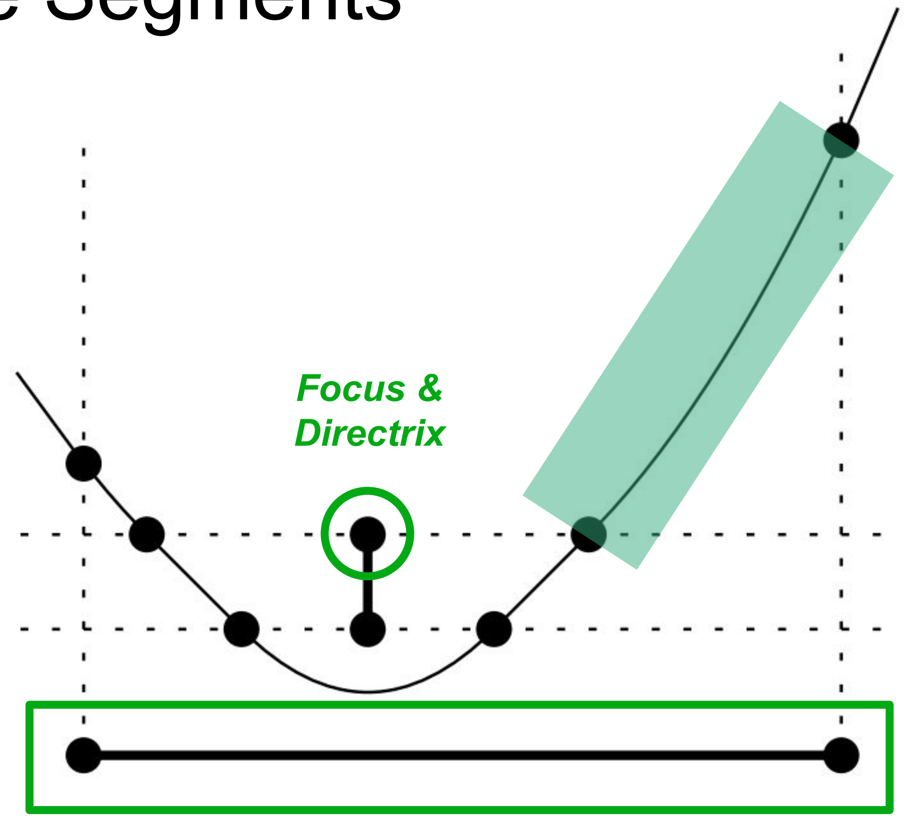

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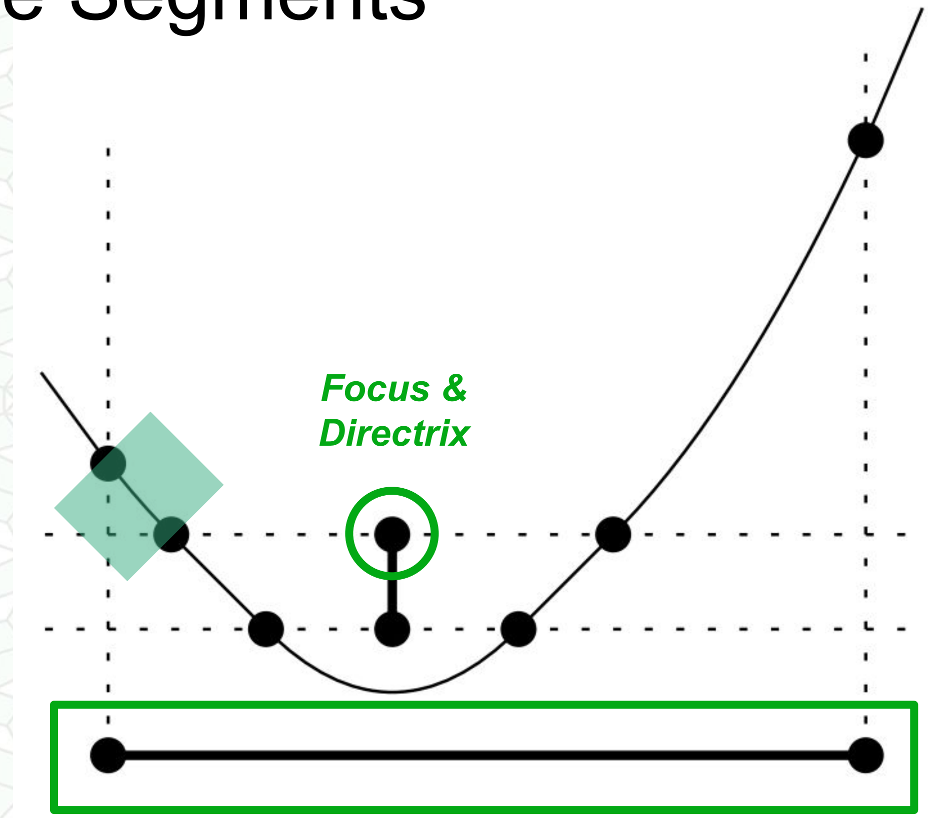

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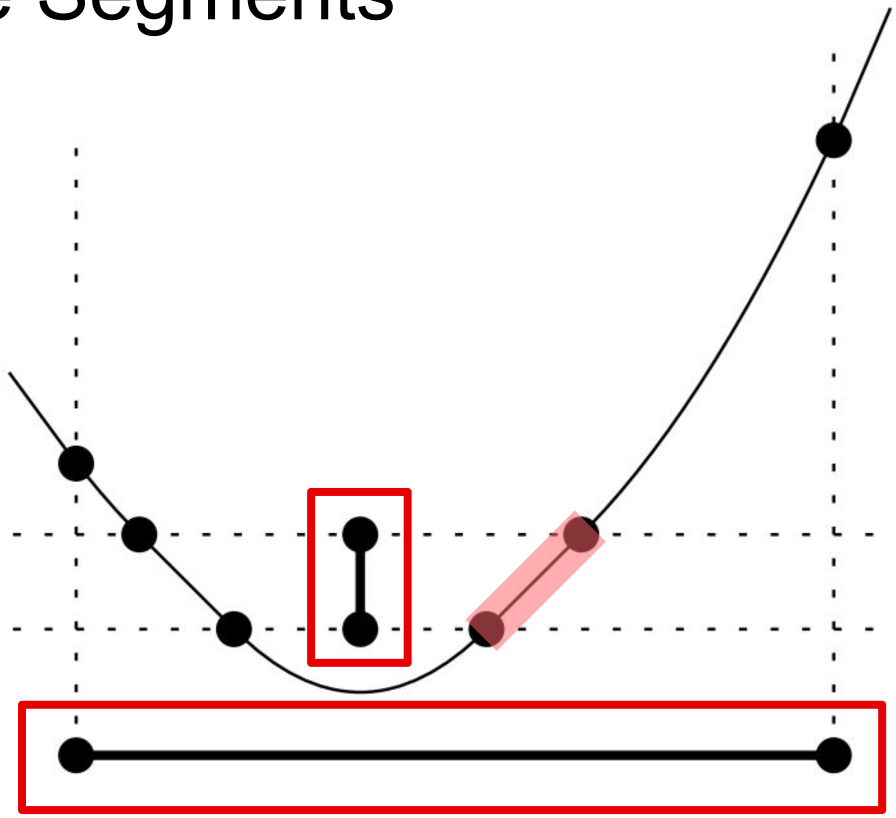
- Points equidistant between two lines form a line.



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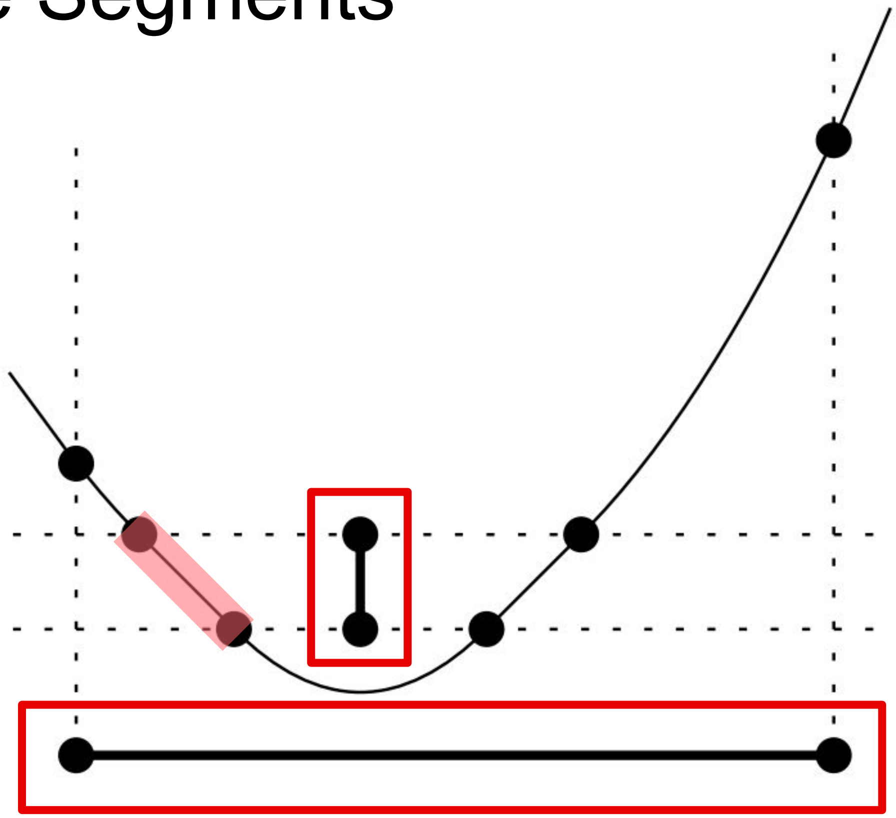

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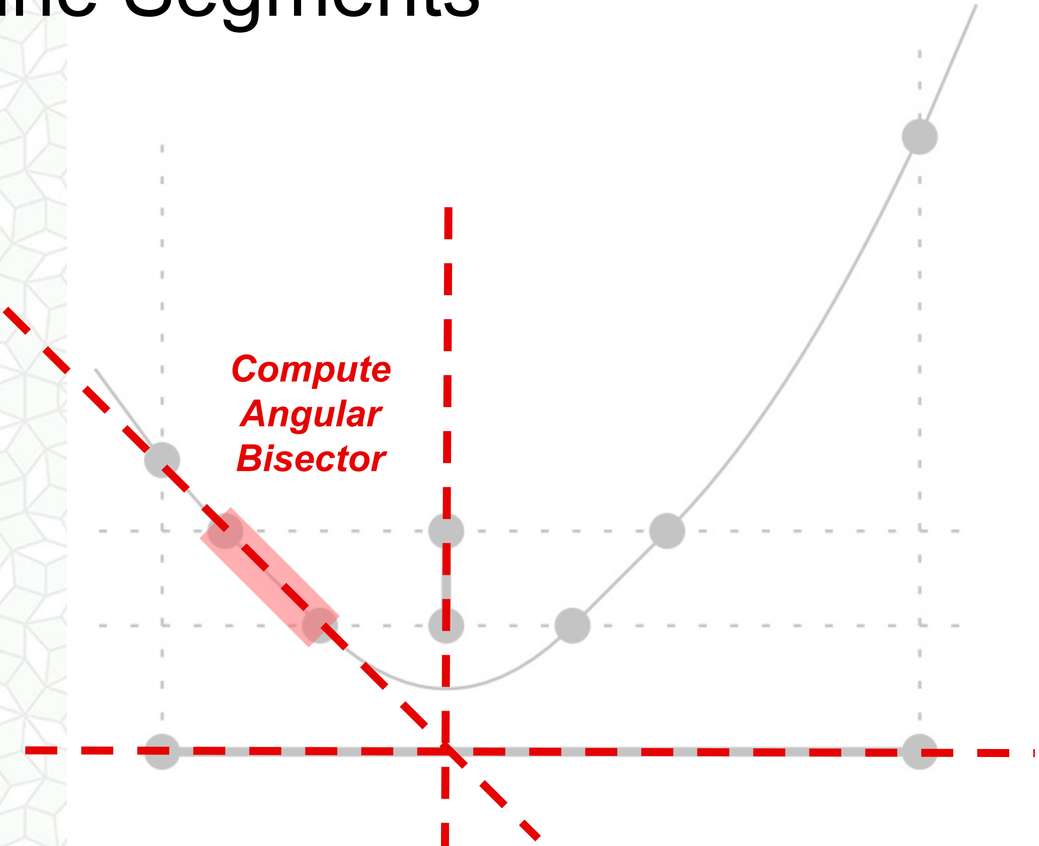

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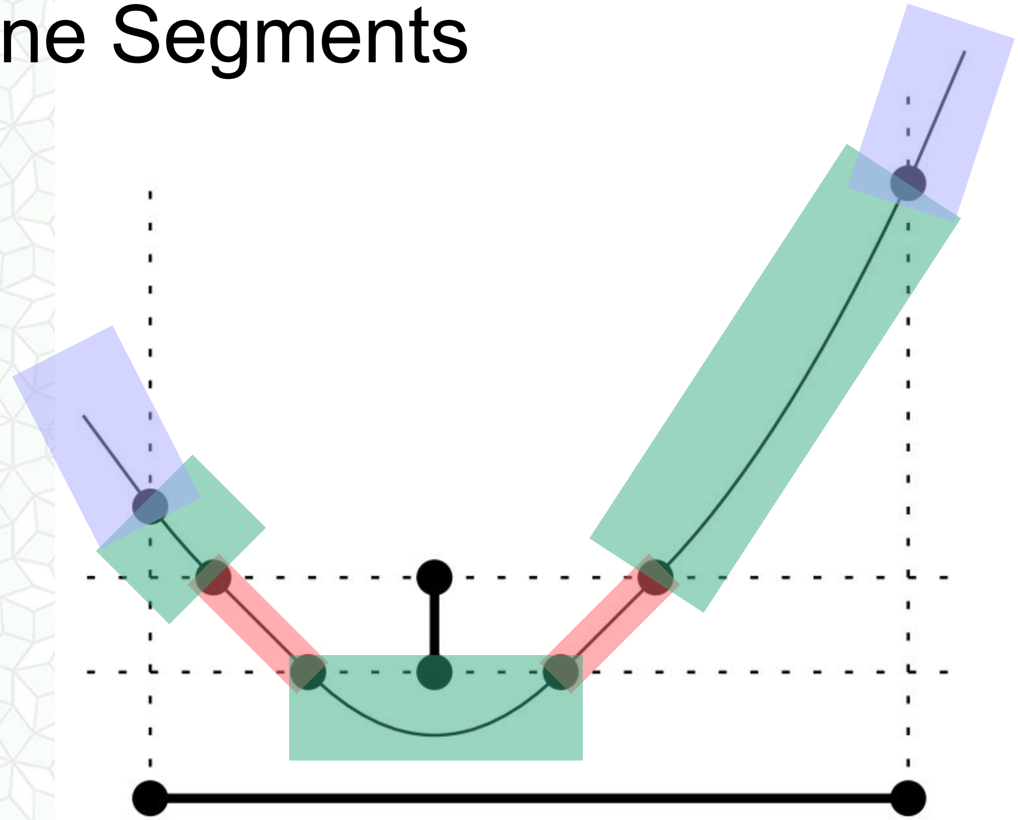

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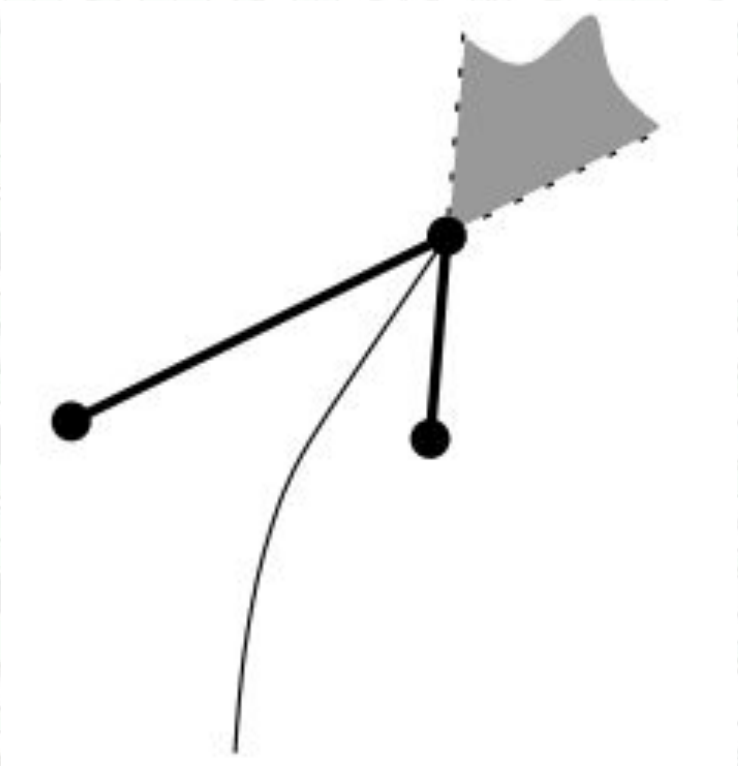

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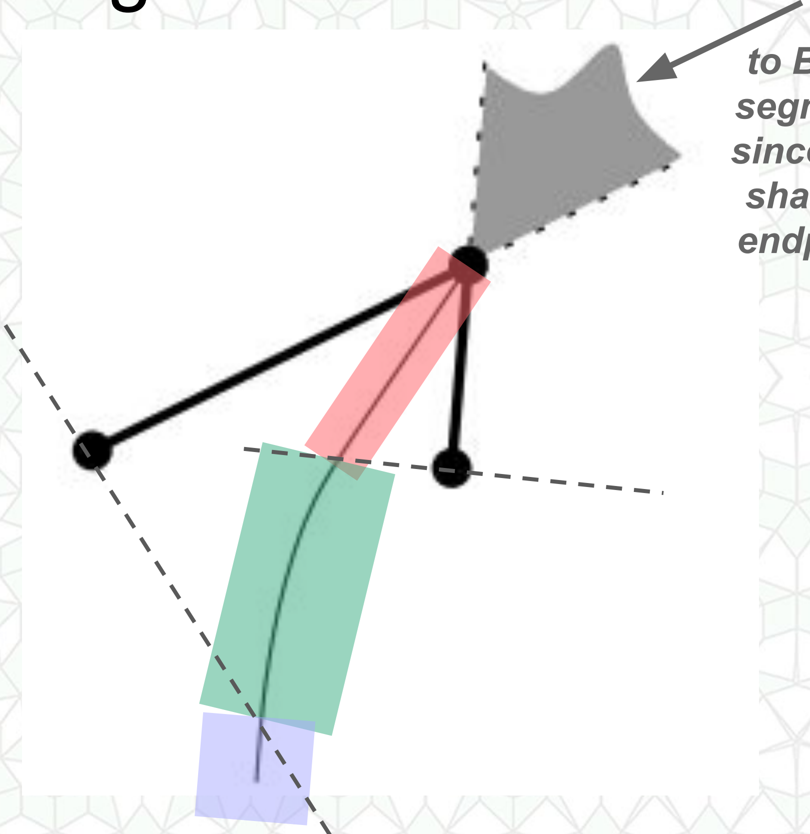

This region is equally close to BOTH segments since they share an endpoint!

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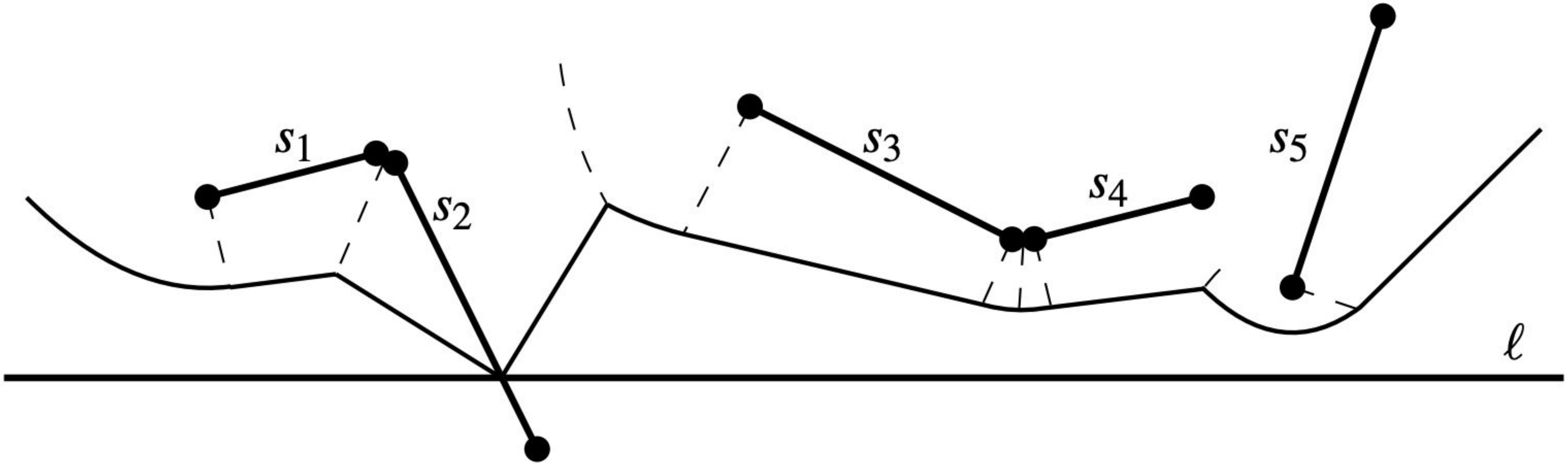
Sweep Line: More Complicated Beach Front

- Fortunately, the complexity (# of segments) is still *O(n)* in the size of the input – now line segments instead of just points!



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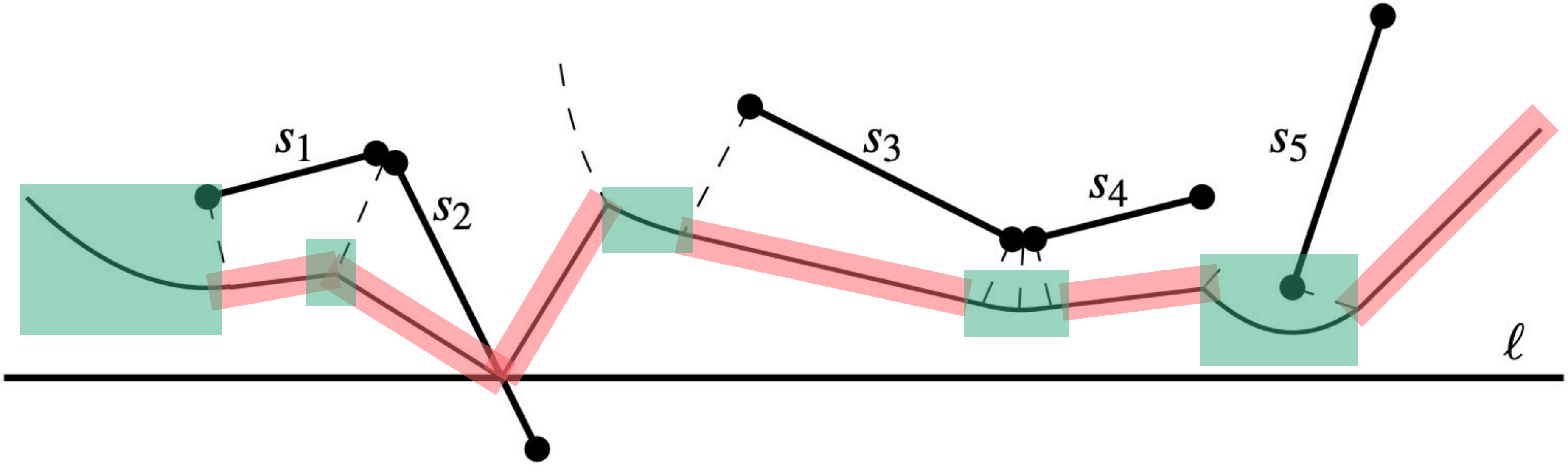

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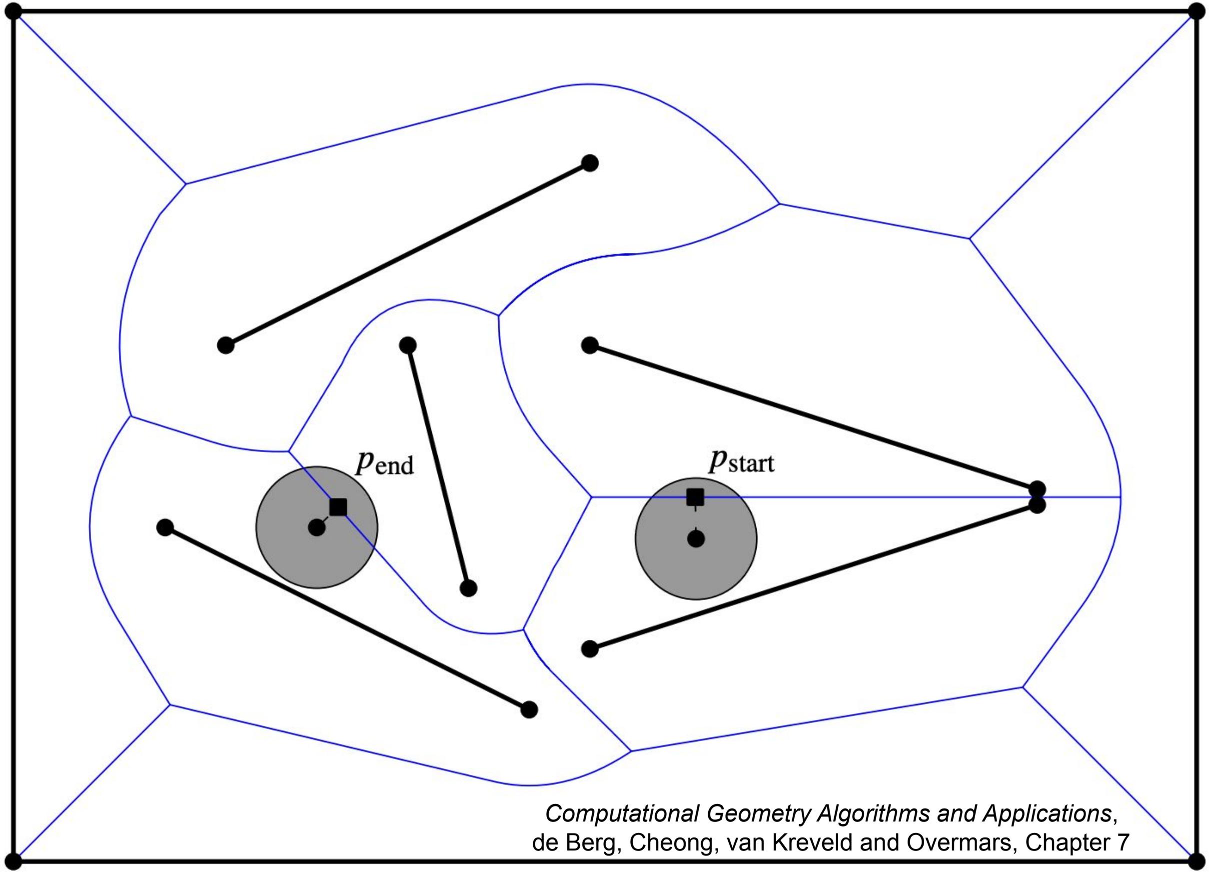
Finished Voronoi Diagram of Line Segments

- Finished diagram has lines and parabolic curved segments
- But is still *O(n)* in complexity - (# of segments)
- And can be computed in *O(n log n)*

- *But why is this useful?*



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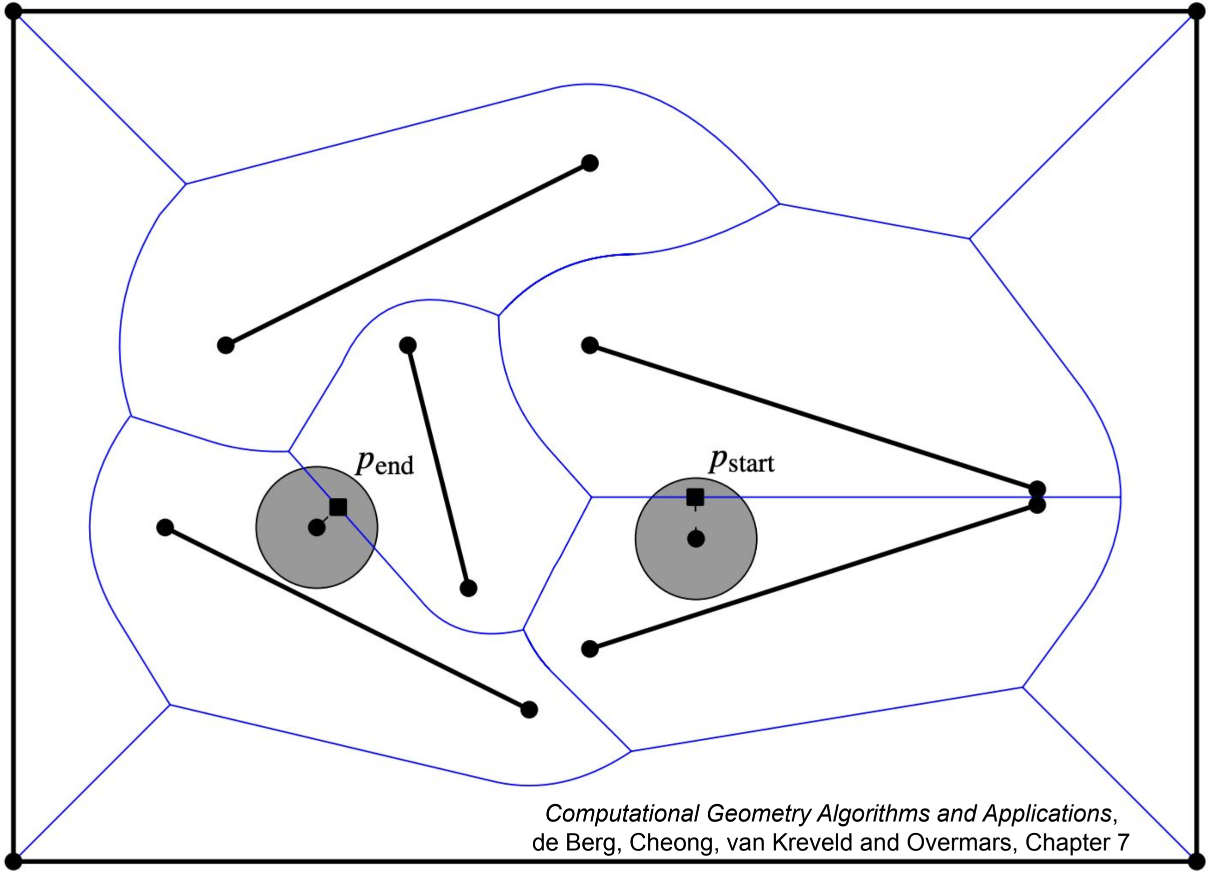
Application: Robotics & Motion Planning

- Let's move a circular/disk robot from the start position to the end position.

- Step 1: Project the robot center to the closest Voronoi edge (line segment or parabolic curve)



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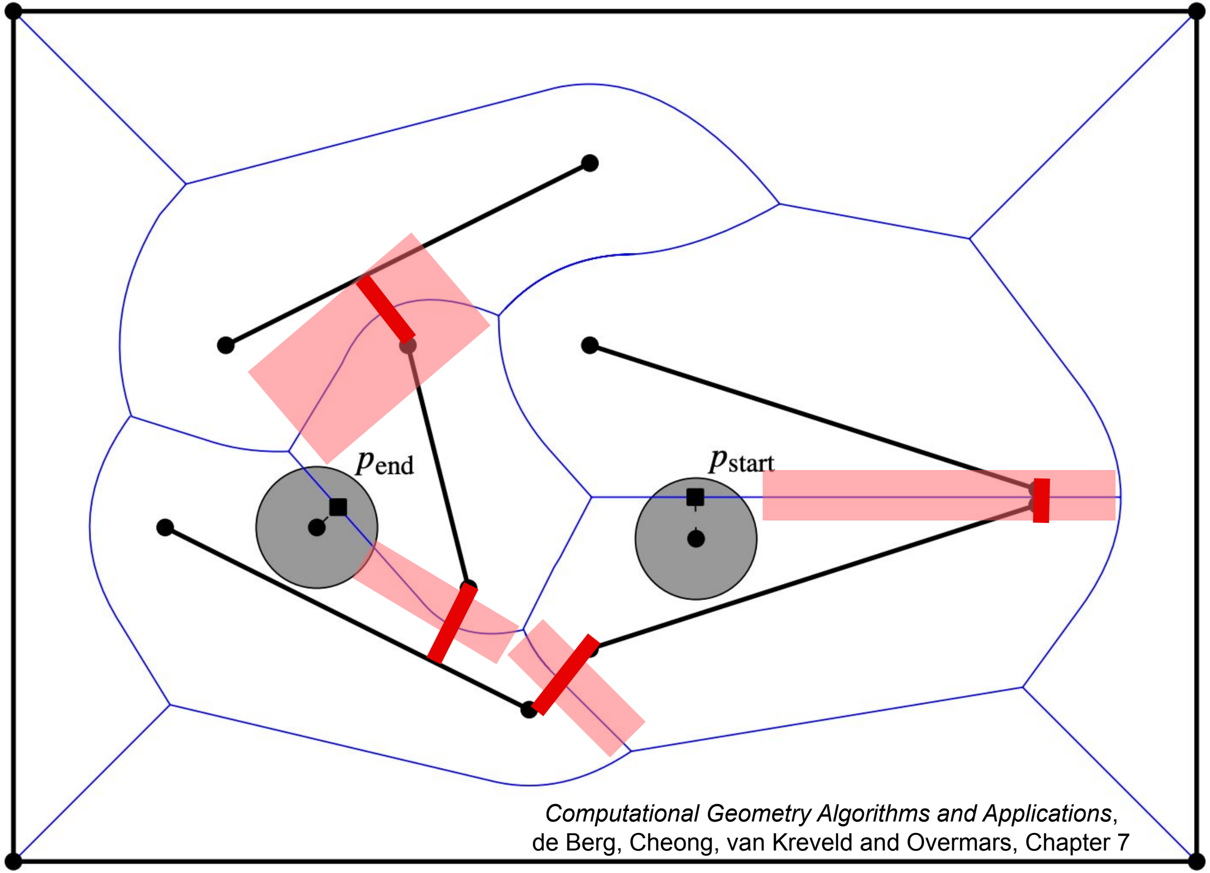
Application: Robotics & Motion Planning

- Step 1: Project the robot center to the closest Voronoi edge (line segment or parabolic curve)

- Step 2: Remove edges from the diagram graph with smallest distance to segment < radius.



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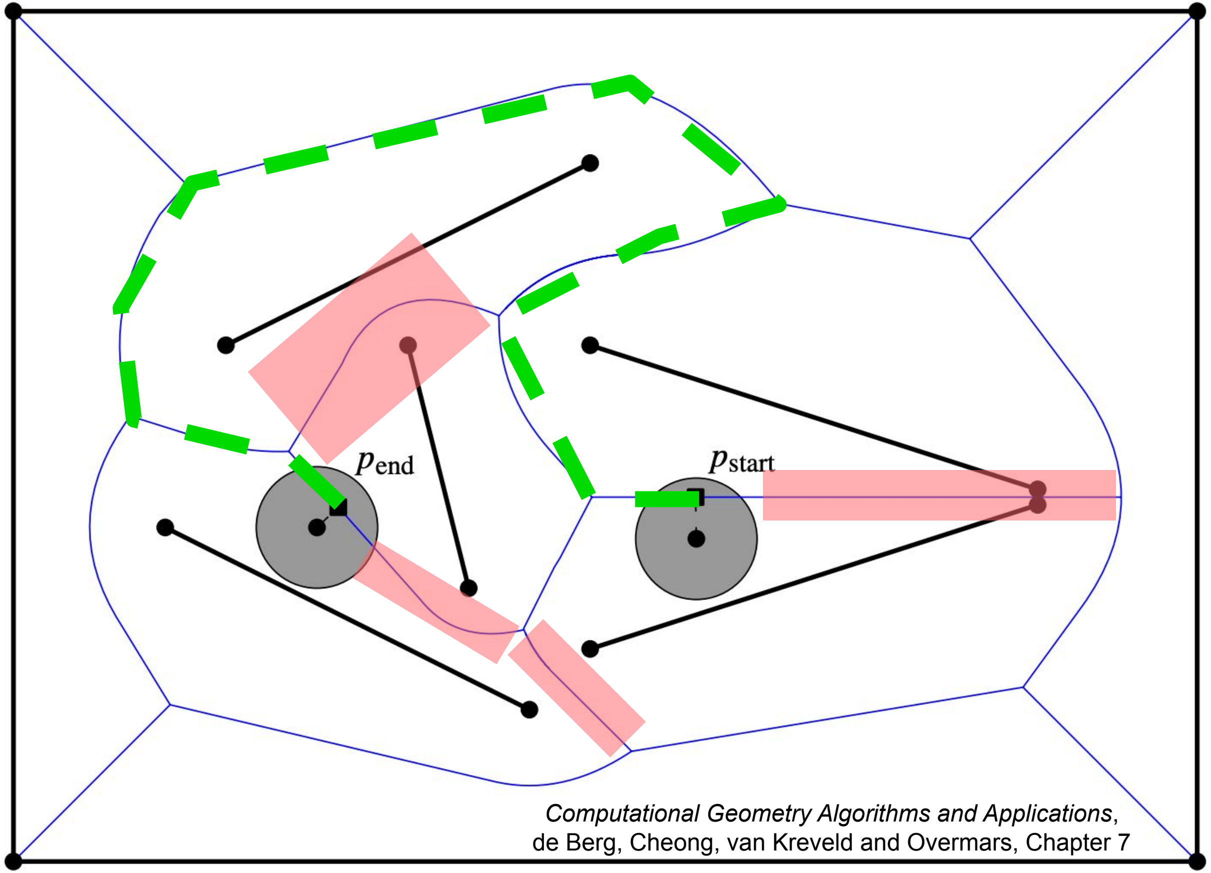
Application: Robotics & Motion Planning

- Step 2: Remove edges from the diagram graph with smallest distance to segment < radius.

- Step 3: Search the remaining graph for a connected path from start to end.



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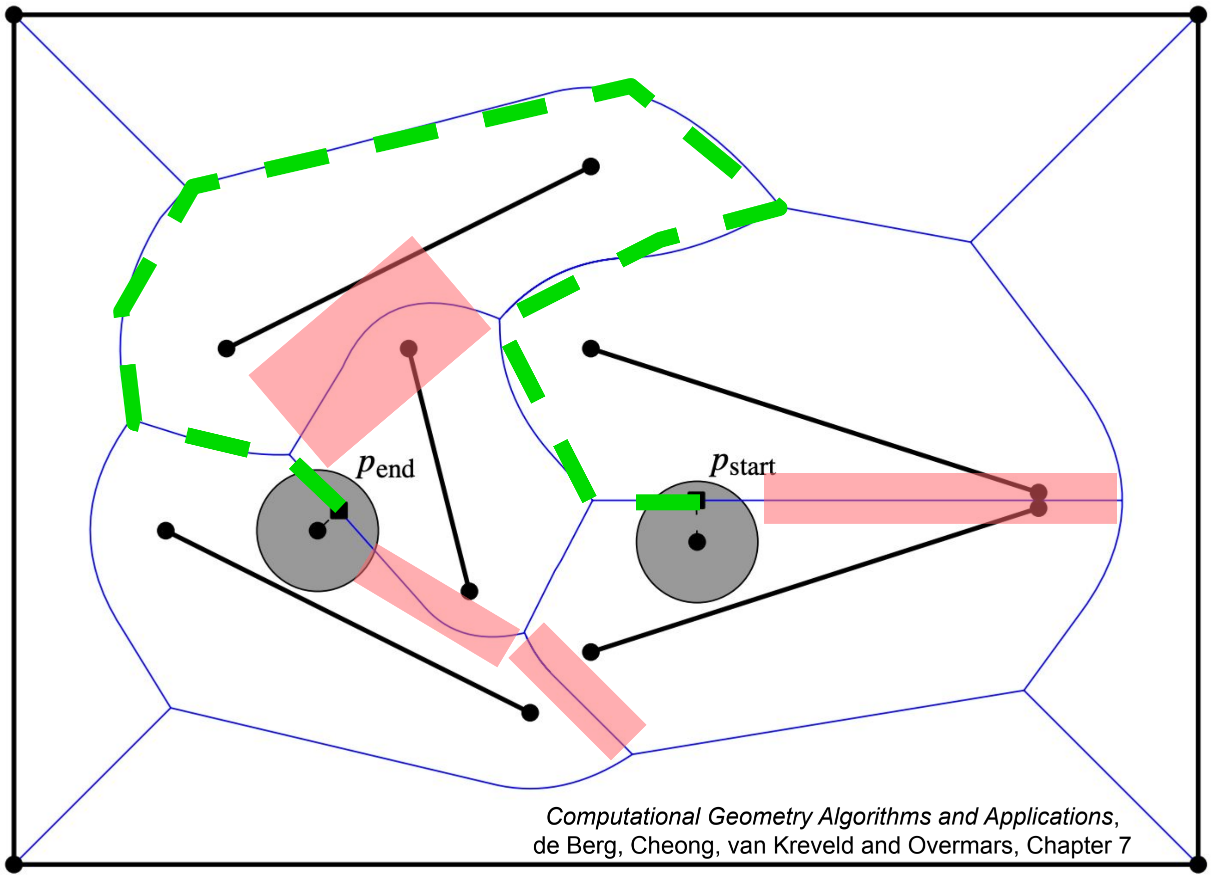
Application: Robotics & Motion Planning

- Step 3: Search the remaining graph for a connected path from start to end.

- *Is this guaranteed to be the SHORTEST path for the robot?*



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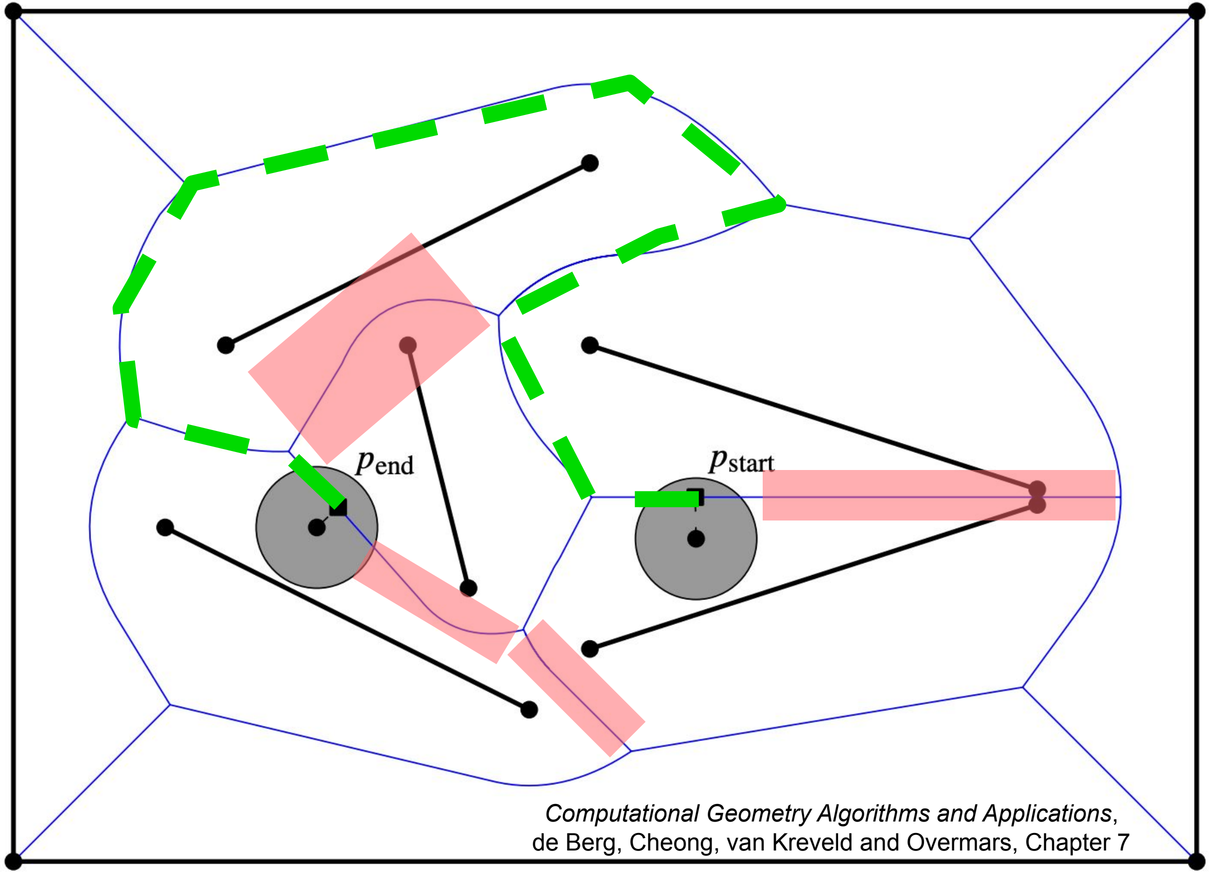
Application: Robotics & Motion Planning

- Step 3: Search the remaining graph for a connected path from start to end.

- *Is this guaranteed to be the SHORTEST path for the robot?*
- *No, but it will find a LEGAL path, if a legal path exists*



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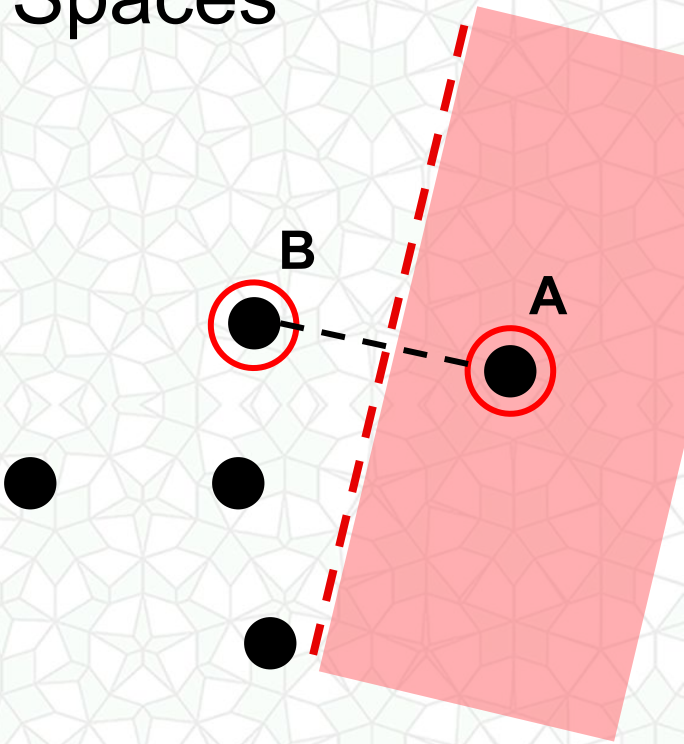
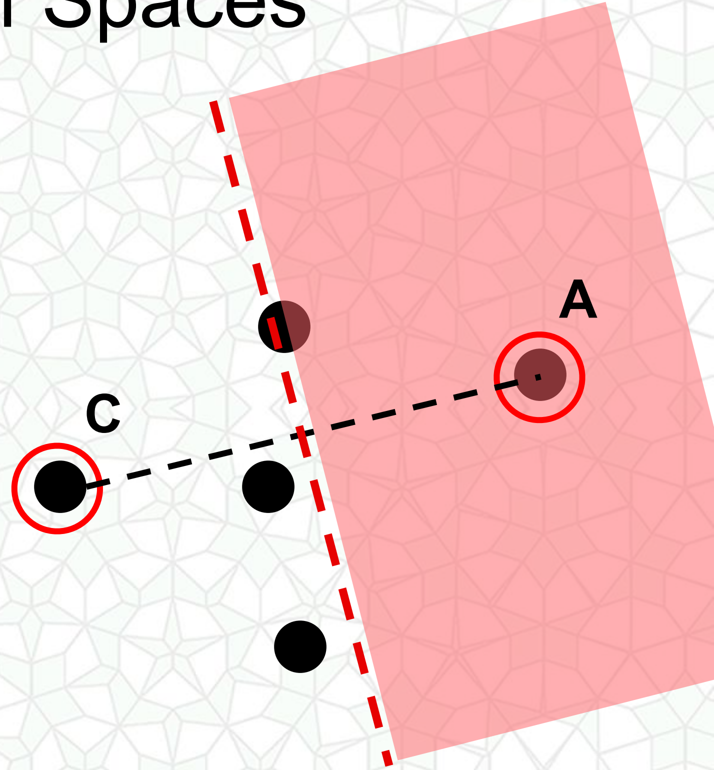
Voronoi Cell: Intersection of Half Spaces

- All points that lie on one side of the perpendicular bisector,
- Are the half-space of points that will chose site A over site B because site A is closer than site B.



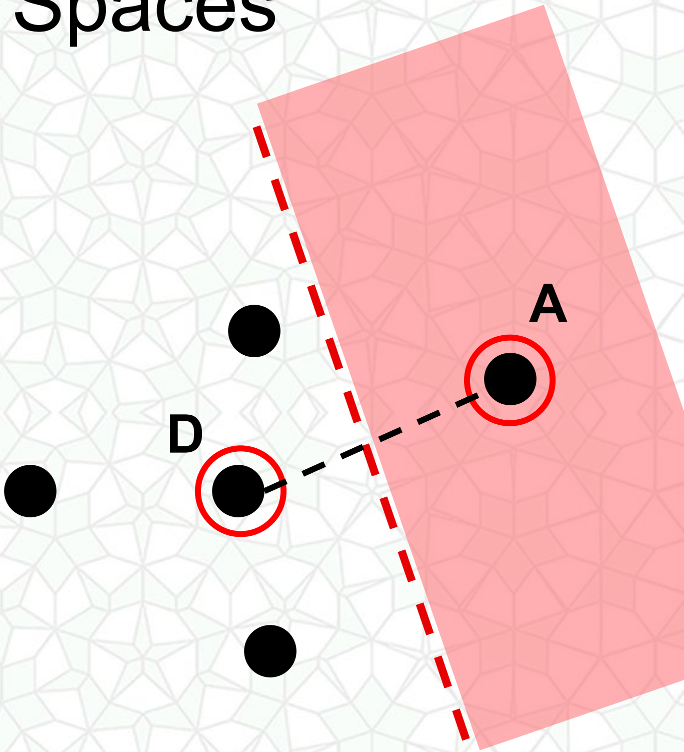
Voronoi Cell: Intersection of Half Spaces

- All points that lie on one side of the perpendicular bisector,
- Are the half-space of points that will chose site A over site C because site A is closer than site C.



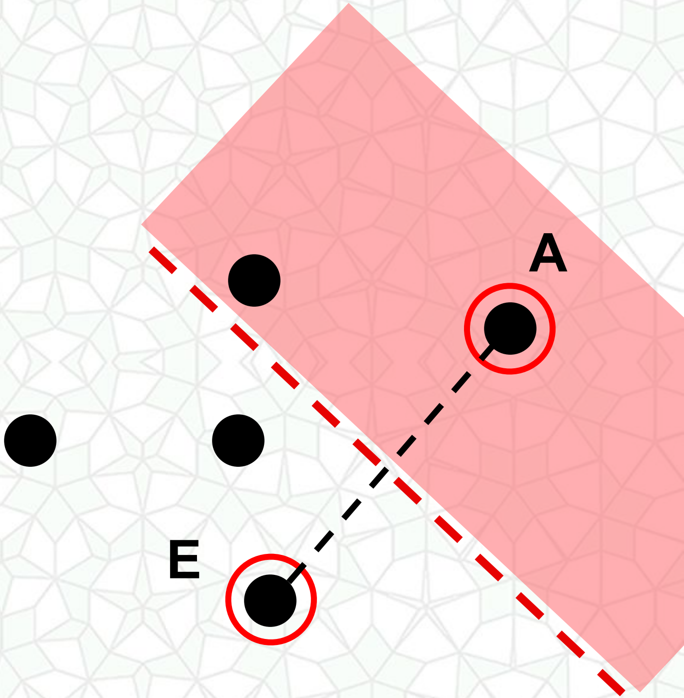
Observation: Intersection of Half Spaces

- All points that lie on one side of the perpendicular bisector,
- Are the half-space of points that will chose site A over site D because site A is closer than site D.



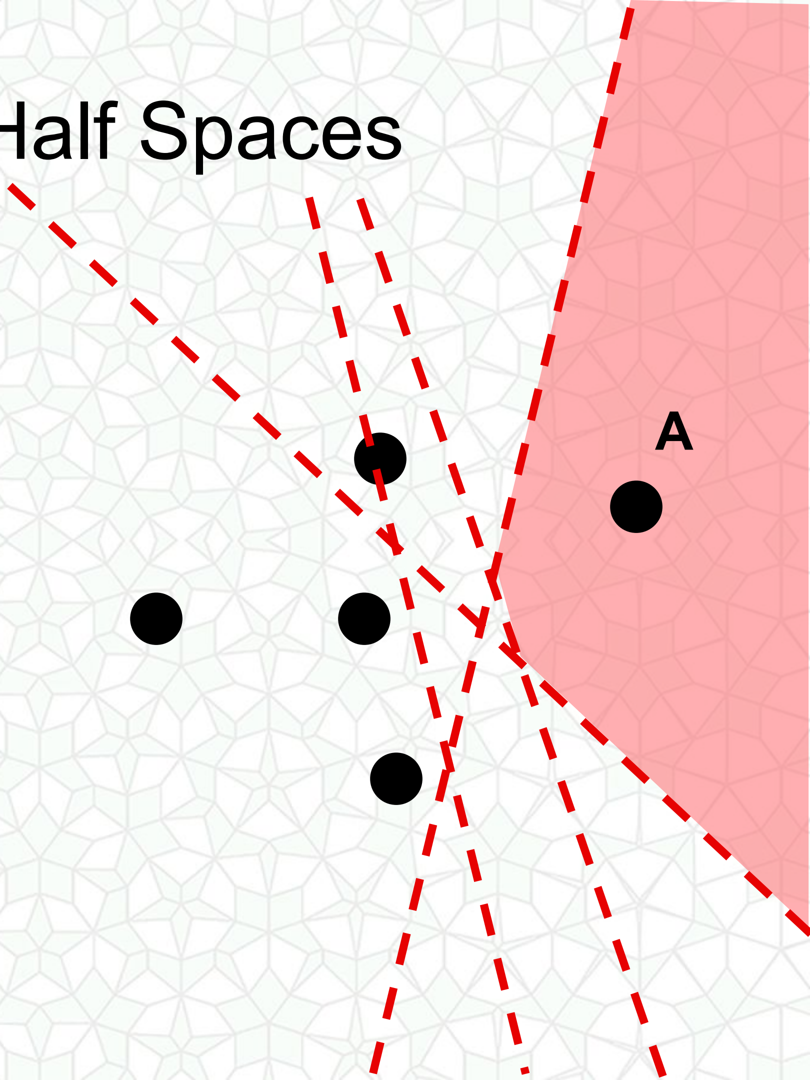
Observation: Intersection of Half Spaces

- All points that lie on one side of the perpendicular bisector,
- Are the half-space of points that will chose site A over site E because site A is closer than site E.



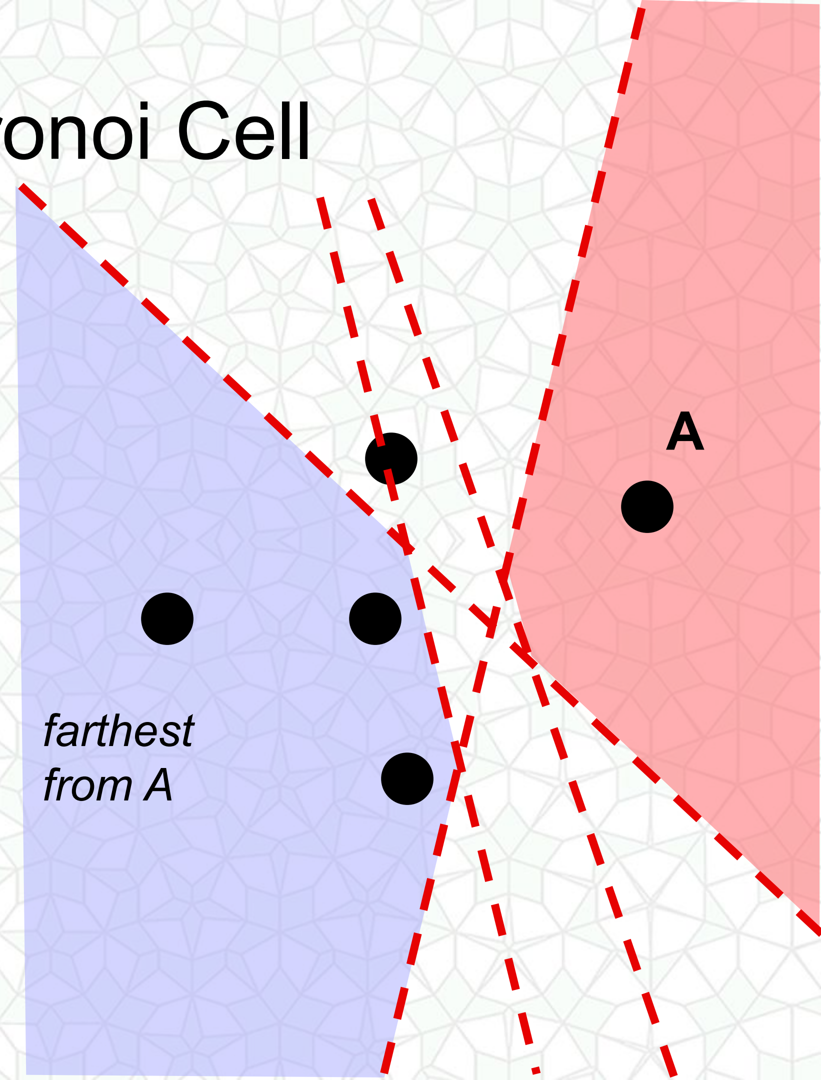
Voronoi Cell: Intersection of Half Spaces

- The intersection of these half-spaces is the Voronoi Cell for A – *all points that choose A as their closest Voronoi site*.



Definition: Farthest Point Voronoi Cell

- The intersection of these half-spaces is the Voronoi Cell for A – *all points that choose A as their closest Voronoi site*.
- The intersection of the opposite half-space is the Farthest Point Voronoi Cell - all points that indicate that A is their furthest Voronoi site.

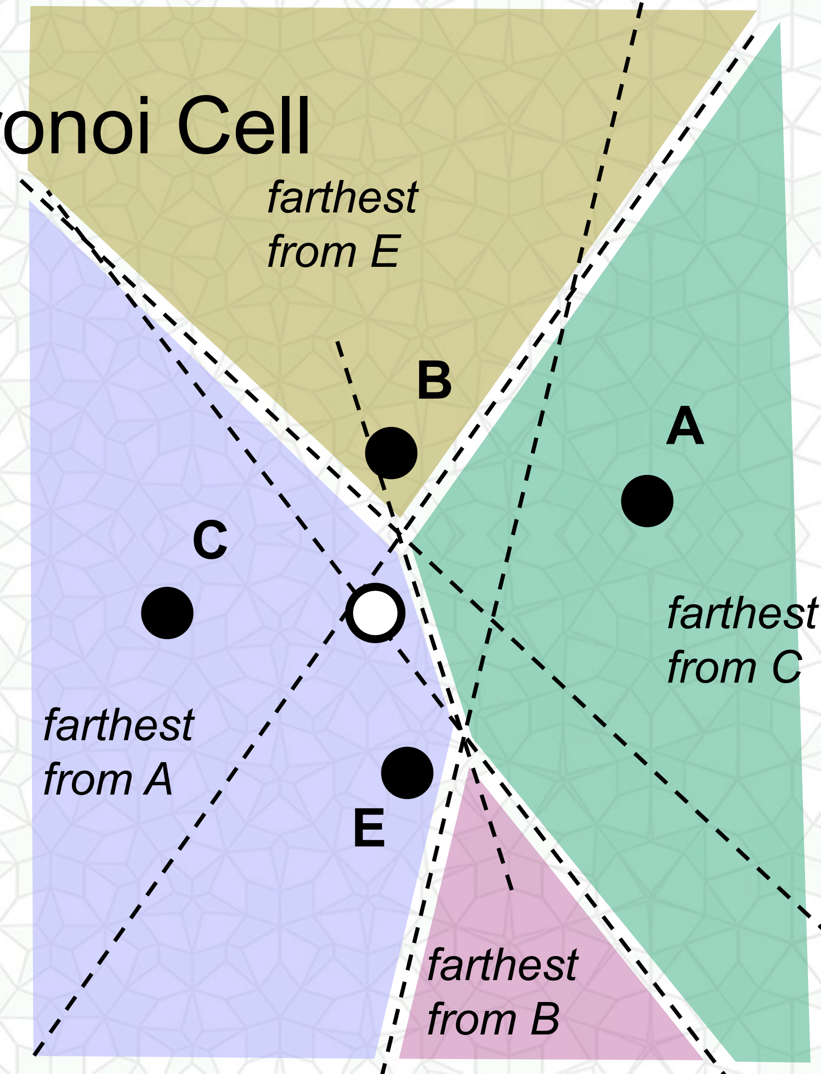


Definition: Farthest Point Voronoi Cell

- The intersection of these half-spaces is the Voronoi Cell for A – *all points that choose A as their closest Voronoi site*.

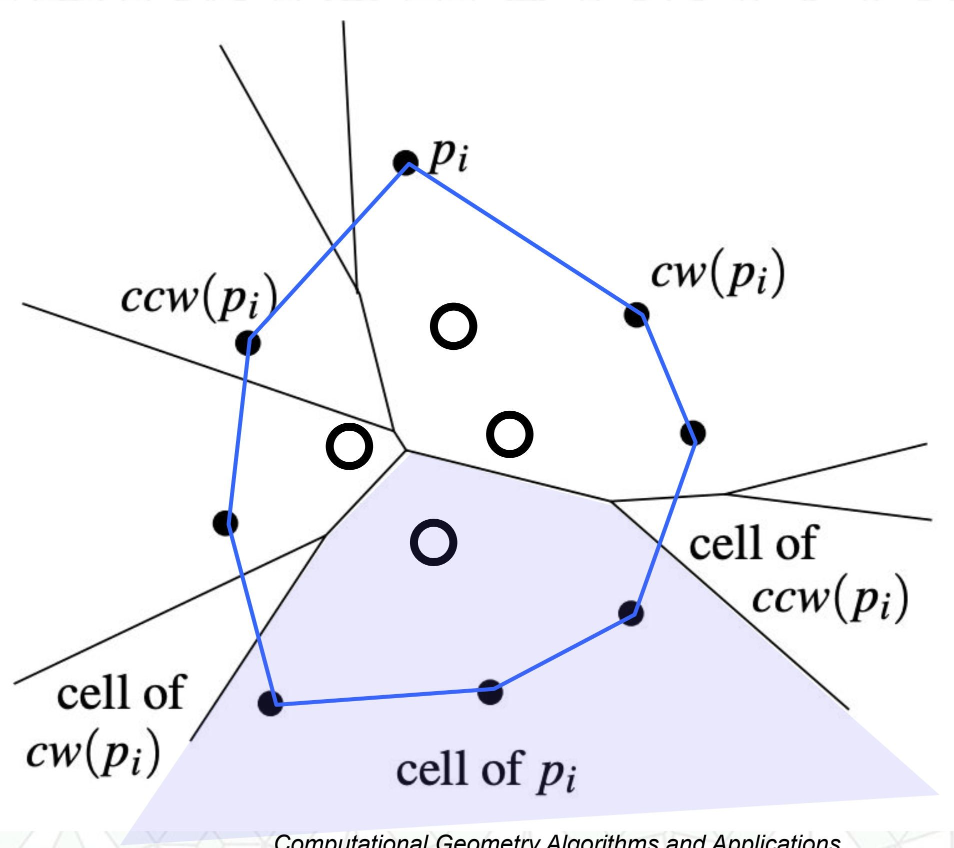
- The intersection of the opposite half-space is the Farthest Point Voronoi Cell - all points that indicate that A is their furthest Voronoi site.

NOTE: In this example, no points are farthest from D!



Farthest-Point Voronoi Diagram

- Observation: Only sites on the convex hull will have a cell in the farthest point diagram.
- Observation: All farthest-point cells are *unbounded.*
- Observation: The diagram is a tree – no cycles!
  If there were a cycle, that would mean we had a bounded cell.



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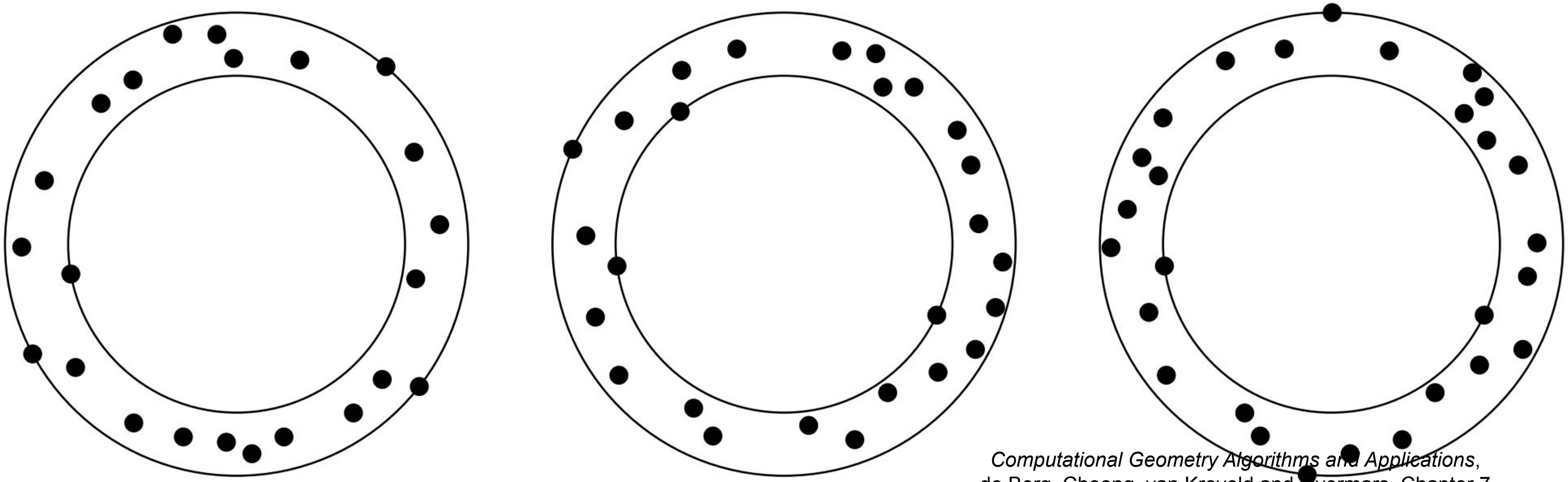
Finding the Smallest-Width Annulus

- Manufacturing Application:
  This object is supposed to be perfectly round (spherical), given precise measurements of the actual object, what are the error bounds?



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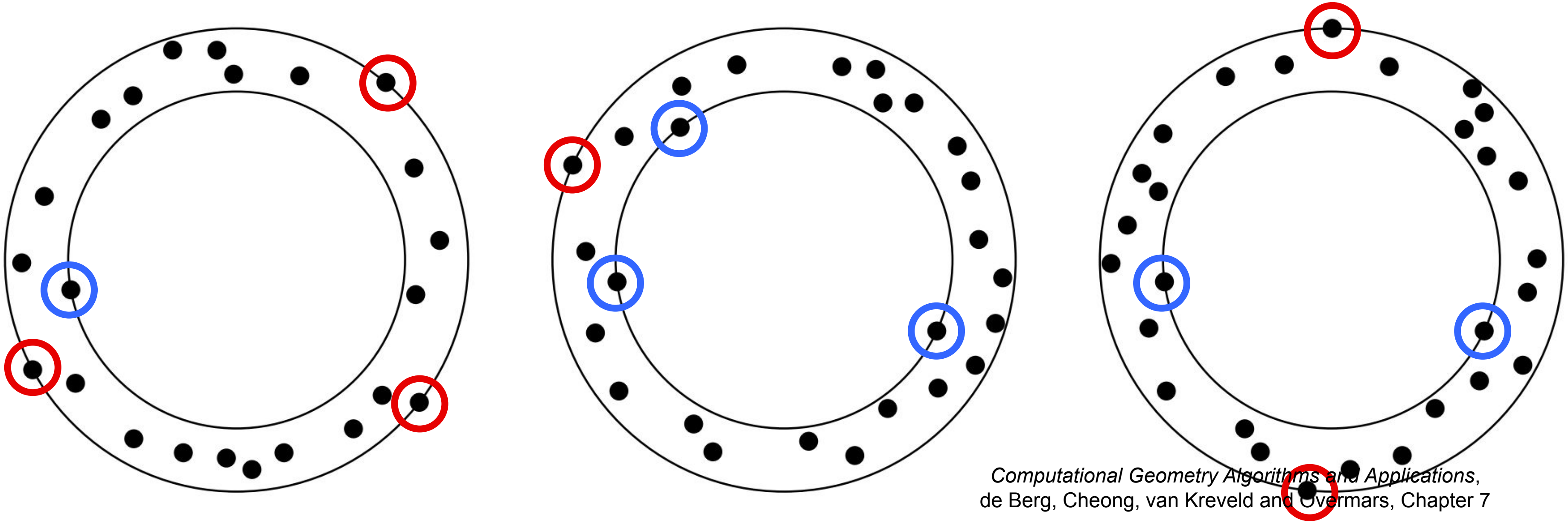
Finding the Smallest-Width Annulus

- 3 points on the outer circle, 1 point on the inner circle
- 1 point on the outer circle, 3 points on the inner circle
- 2 points on the outer circle, 2 points on the inner circle

3 Possible Cases



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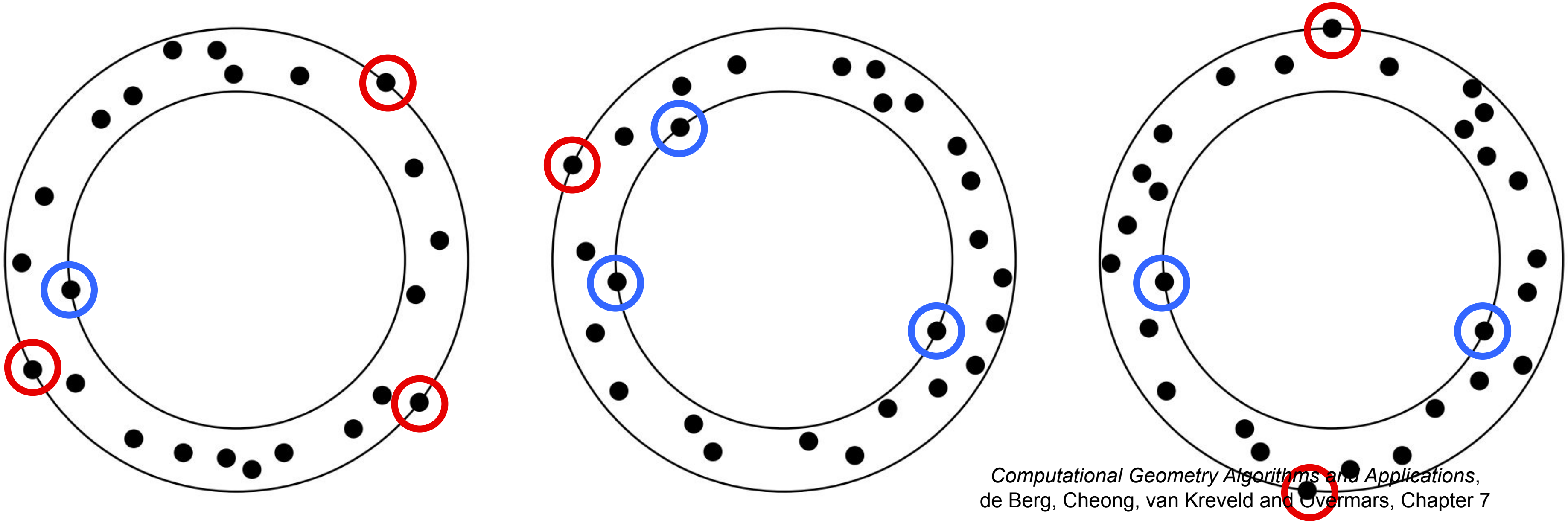
Finding the Smallest-Width Annulus

- Easy to compute once we know the center
  (it is the center of both the inner & outer circle)
- *What points might be the center? Any point on the plane?*



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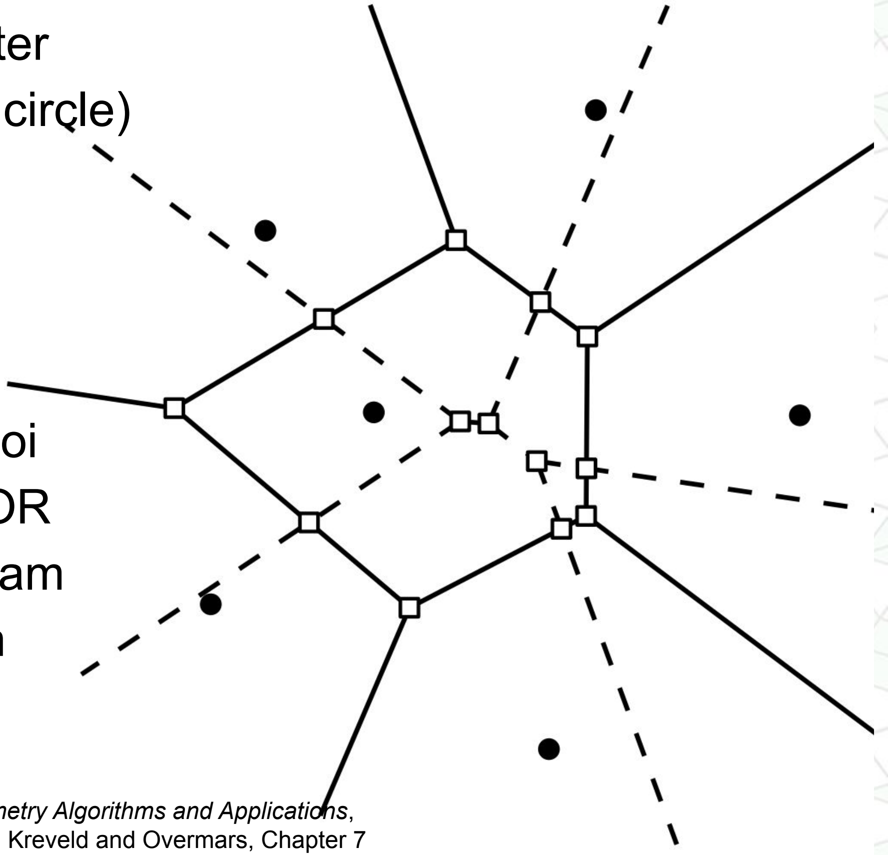
Finding the Smallest-Width Annulus

- Easy to compute once we know the center (it is the center of both the inner & outer circle)
- *What points might be the center?*
  It must be:
  - A vertex of the Voronoi Diagram (equally close to 3 sites) OR
  - A vertex of the Farthest Point Voronoi Diagram (equally far from 3 sites) OR
  - An intersection of the Voronoi Diagram and Farthest Point Voronoi Diagram (equally close to 2 sites AND equally far from 2 sites)



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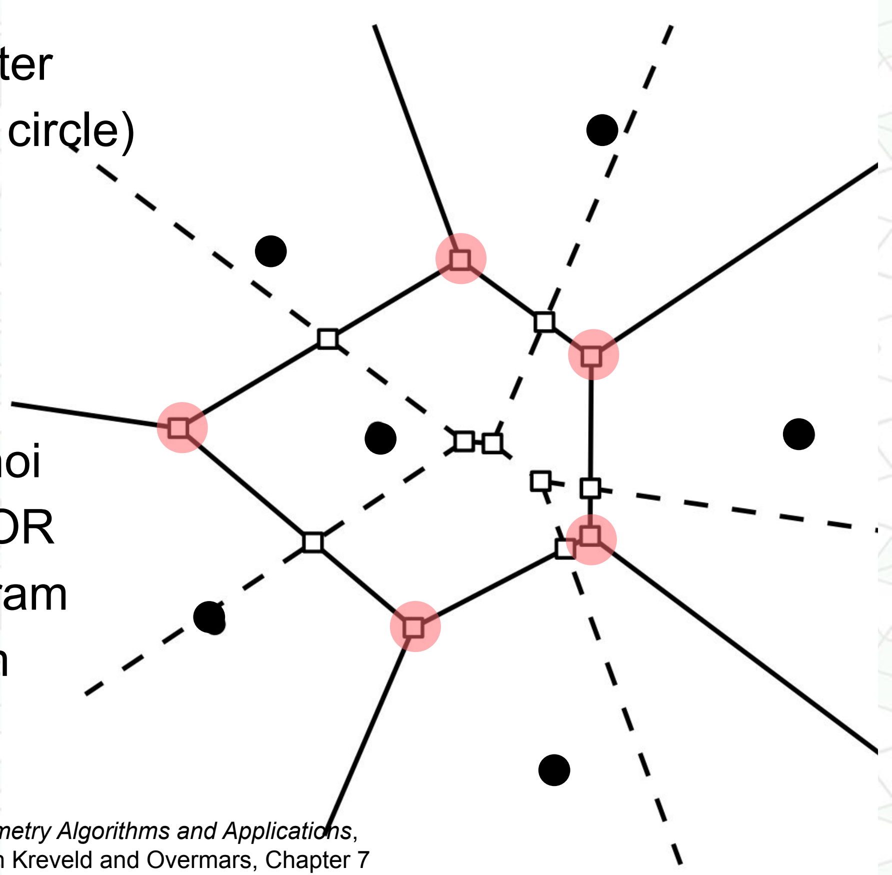

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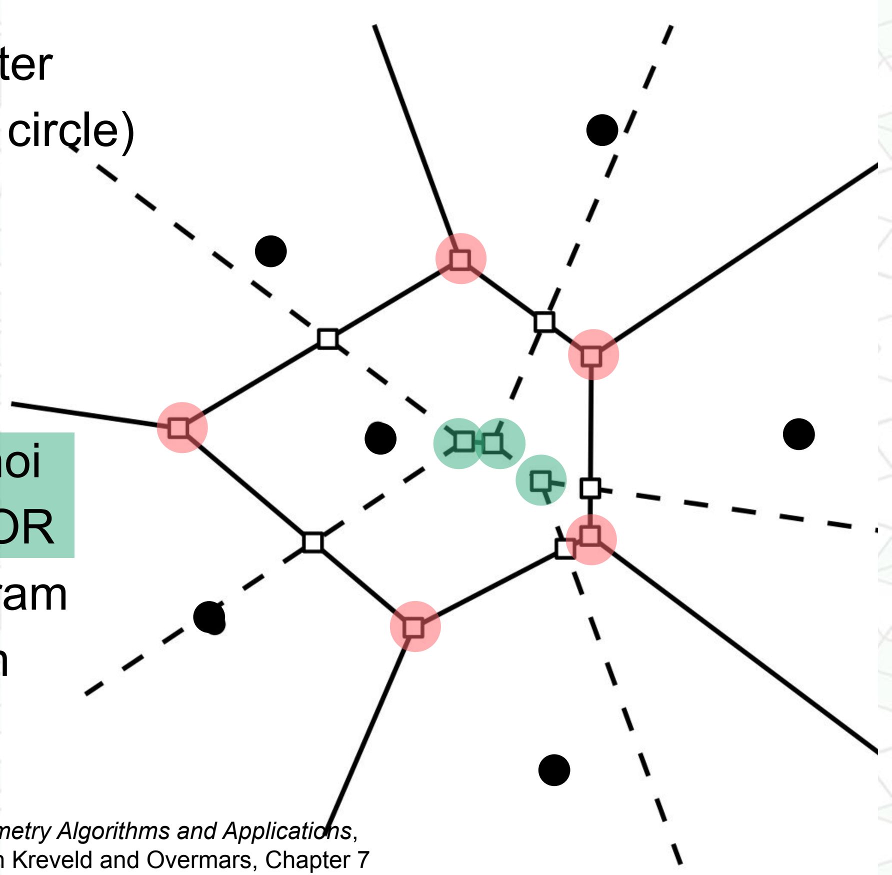

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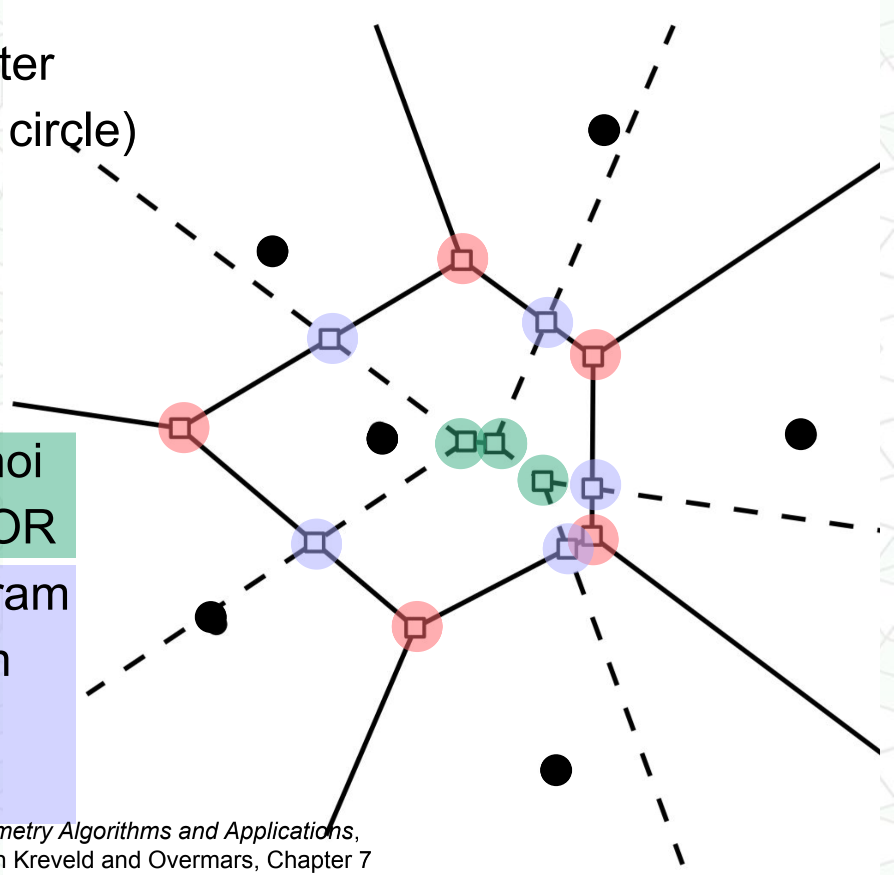

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Finding the Smallest-Width Annulus

Brute force check of *O(n)* = a FINITE number of possible center positions

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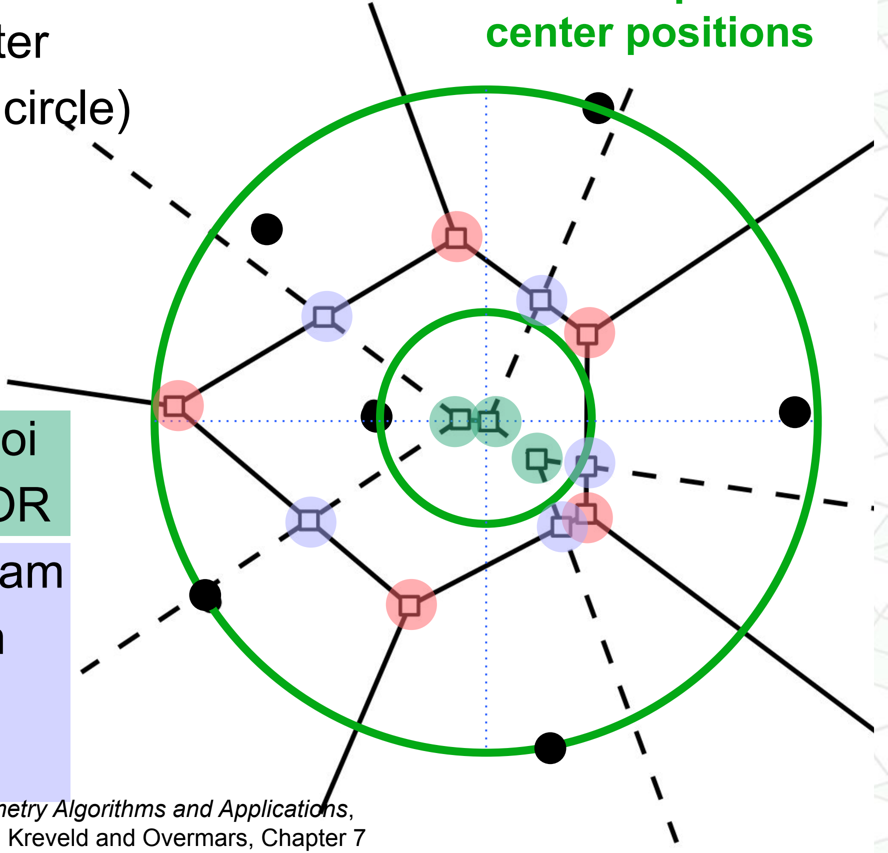

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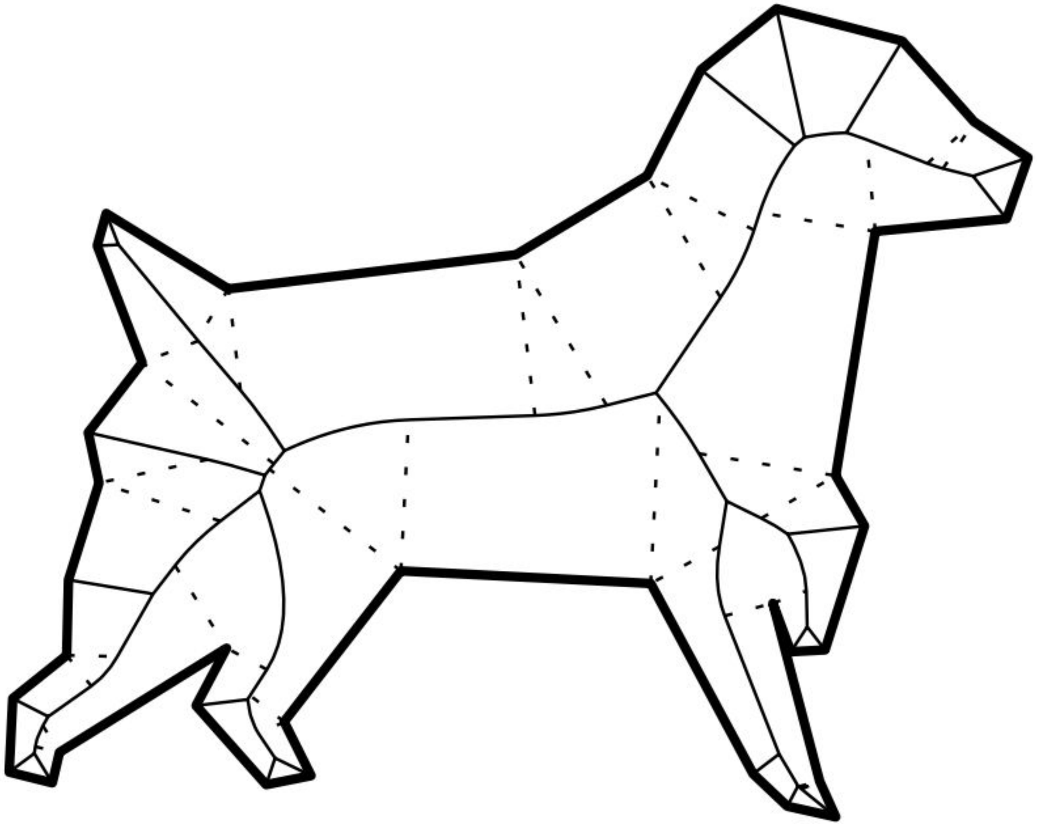
Medial Axis - Voronoi Diagram of Simple Polygon

- a.k.a. Skeleton

- Applications to:
  - Shape Analysis
  - Deformation



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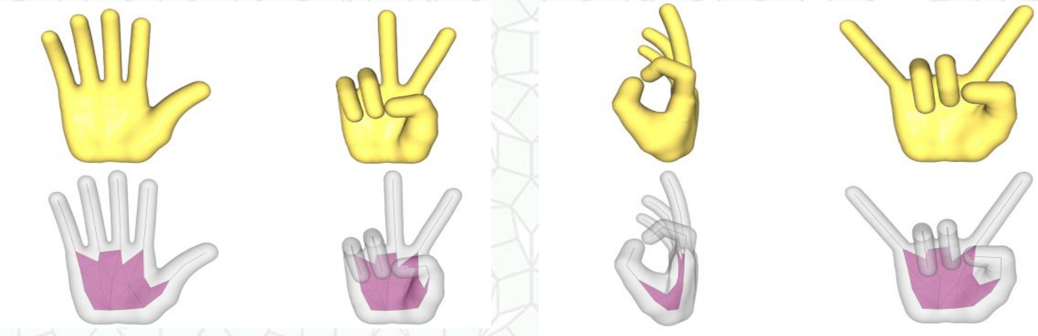
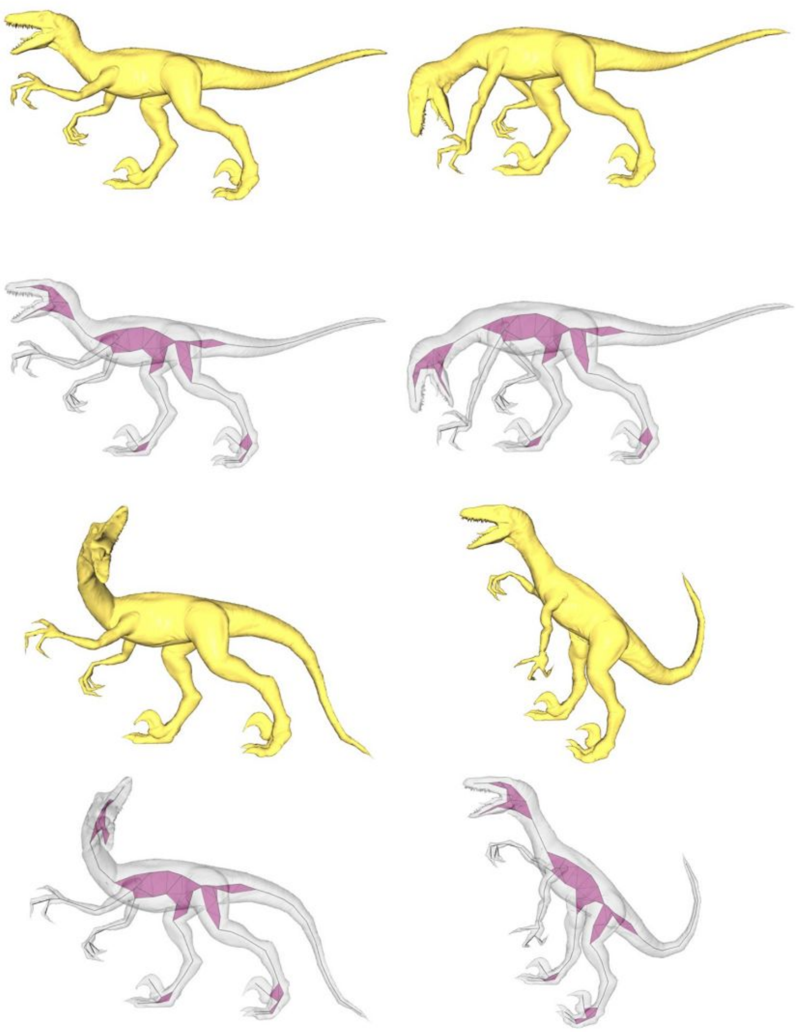
Application of Medial Axis

- Placing the skeleton inside of a 3D object
- Automated Rigging

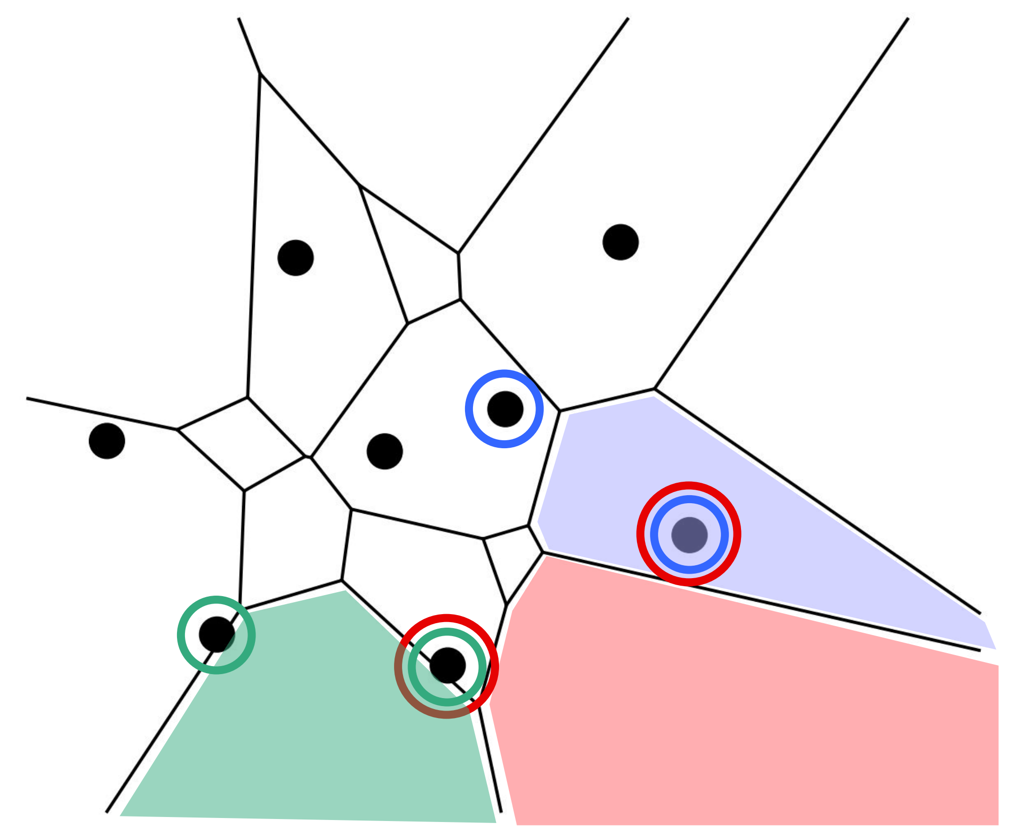




"Medial-Axis-Driven Shape Deformation with Volume Preservation" Lan, Yao, Huang, Guo, Visual Computer 2017

Higher-Order Voronoi / k-Closest Sites

- For example, k = 2…
- Subdivide the plane into regions that have the same closest *and* second closest sites



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