CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/F23/

# Lecture 12: Voronoi Diagrams, Part 3

# Outline for Today

- Friday Oct 13th: Quiz 1, Logistics
- Homework 6 will be posted tomorrow
- Last Time: More Voronoi Diagrams
- Higher-Order vs Higher Dimension Voronoi Diagrams
- Centroidal Voronoi Diagram
- K-Means Clustering
- Application: Architectural Geometry
- More Spatial Query & Search Problems / Applications
- Reducing Other Problems to the Voronoi Diagram
- Future Topic (after Quiz 1): Delaunay Triangulation

# Quiz 1

- In class, Friday Oct 13th, 2-3:50pm
- Will involve simple sketching, you are welcome (but not required) to bring colored pencils/markers/crayons/etc.
- 1 double-sided page of notes allowed.  It is NOT open book.

- You may complete the quiz entirely on paper – *OR* –
- Do the sketching problem(s) on paper and use your laptop and type the written answers in a simple text file and upload to Submitty.

- **We need a volunteer** to pick up the quizzes from Shannon (Lally 207) Friday morning/early afternoon & slide under my office door (Lally 302) after the quiz?

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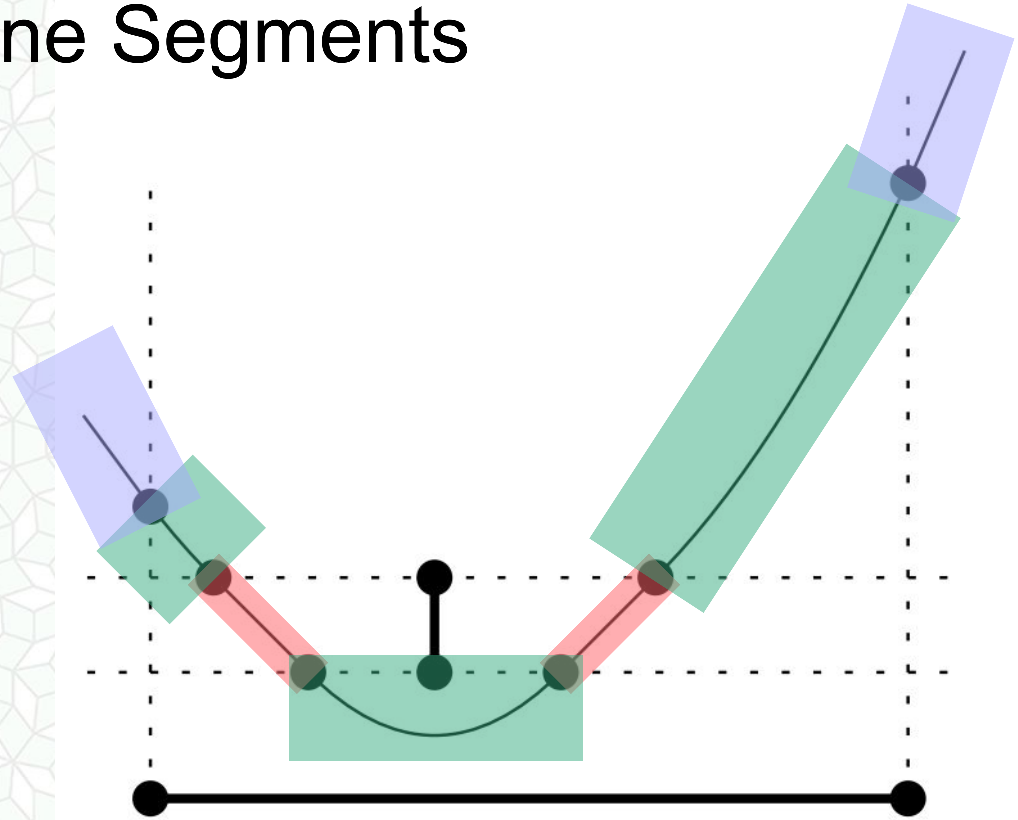
# Voronoi Diagram of Line Segments

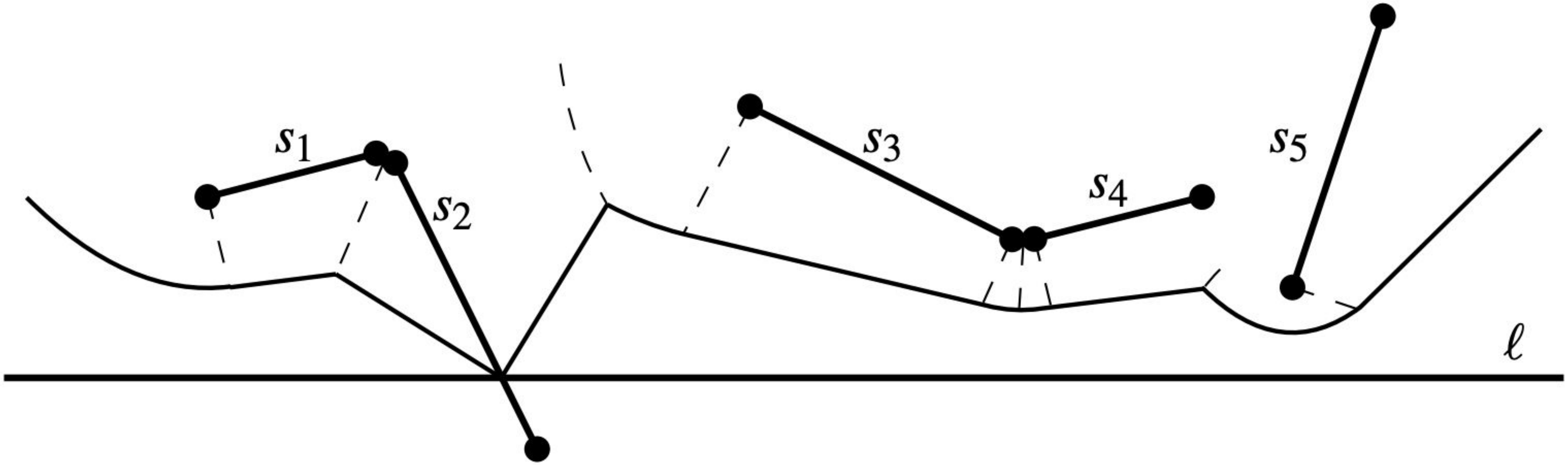
- Points equidistant between two points form a line.
- Points equidistant between a point and a line form a parabola.
- Points equidistant between two lines form a line.



*Computational Geometry Algorithms and Applications*,
de Berg, Cheong, van Kreveld and Overmars, Chapter 7

# Sweep Line: More Complicated Beach Front

- Fortunately, the complexity (# of segments) is still *O(n)* in the size of the input – now line segments instead of just points!

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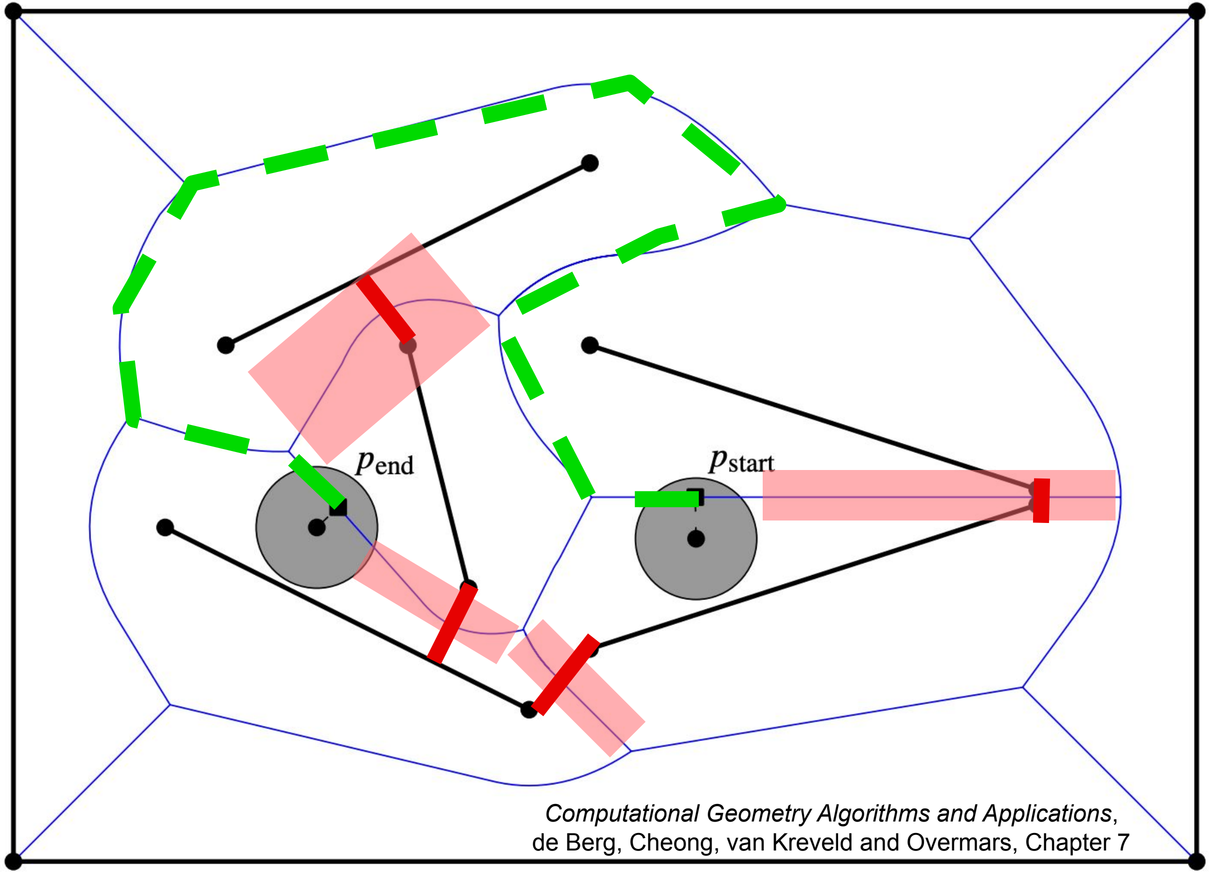
# Application: Robotics & Motion Planning

- Step 1: Project robot center to closest Voronoi edge.
- Step 2: Remove Voronoi edges from diagram graph where smallest distance to segment < radius.
- Step 3: Search the remaining graph for a connected path from start to end.

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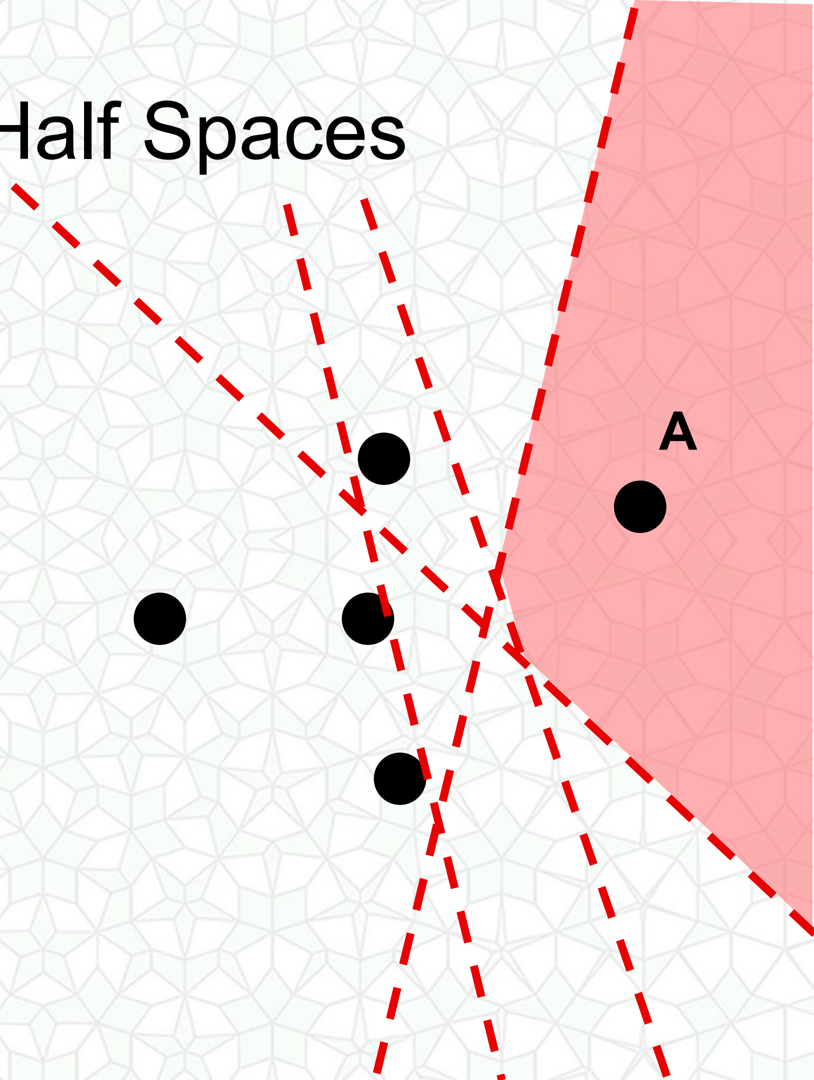
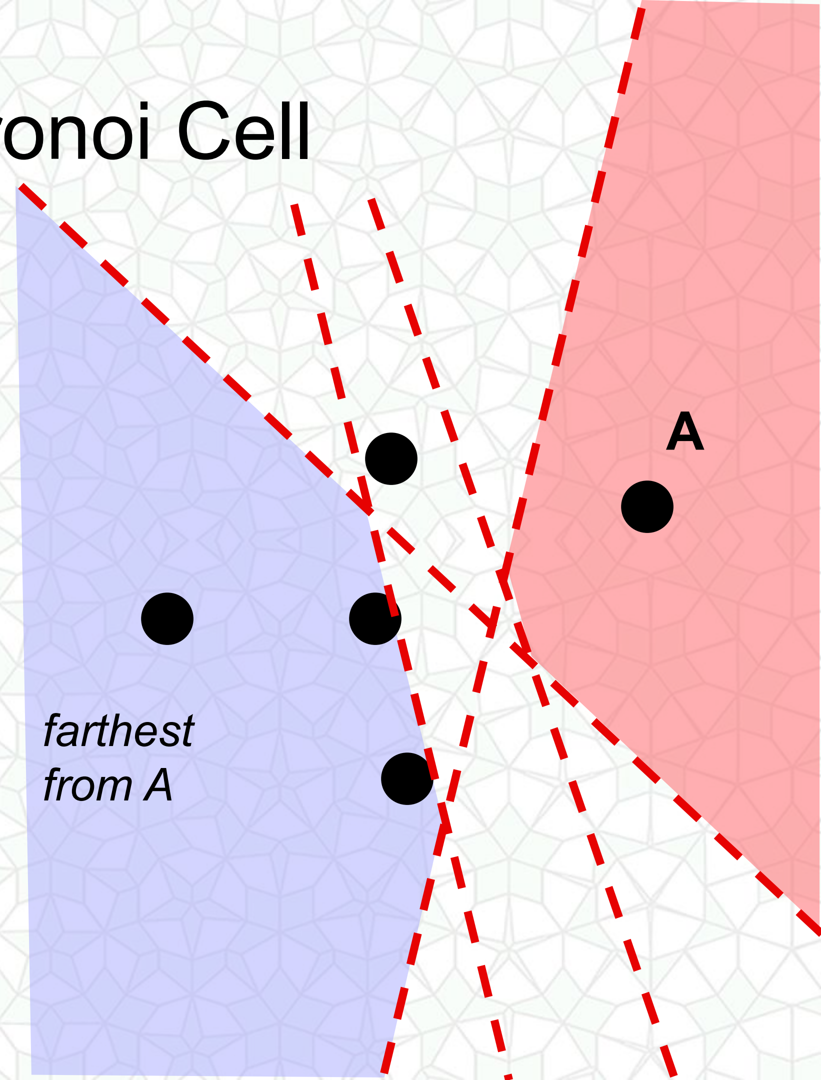
# Voronoi Cell: Intersection of Half Spaces

- The intersection of these half-spaces is the Voronoi Cell for A – *all points that choose A as their closest Voronoi site*.



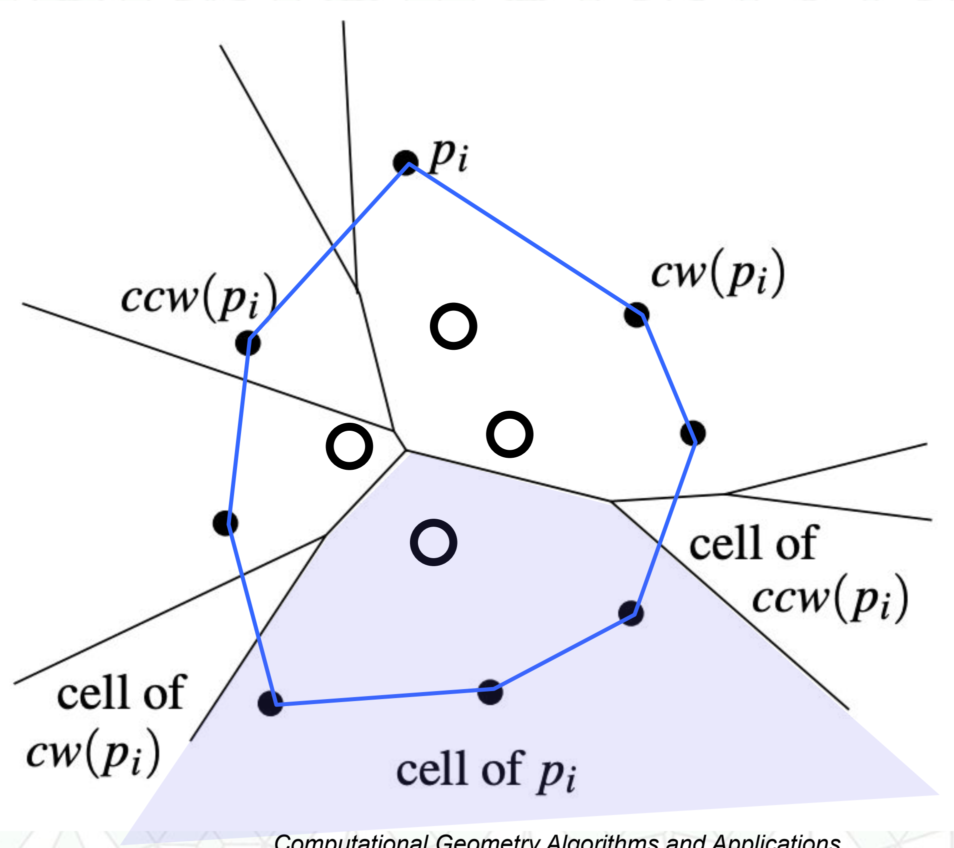
# Definition: Farthest Point Voronoi Cell

- The intersection of these half-spaces is the Voronoi Cell for A – *all points that choose A as their closest Voronoi site*.
- The intersection of the opposite half-space is the Farthest Point Voronoi Cell - all points that indicate that A is their furthest Voronoi site.



# Farthest-Point Voronoi Diagram

- Observation: Only sites on the convex hull will have a cell in the farthest point diagram.
- Observation: All farthest-point cells are *unbounded.*
- Observation: The diagram is a tree – no cycles!
  *If there were a cycle, that would mean we had a bounded cell.*

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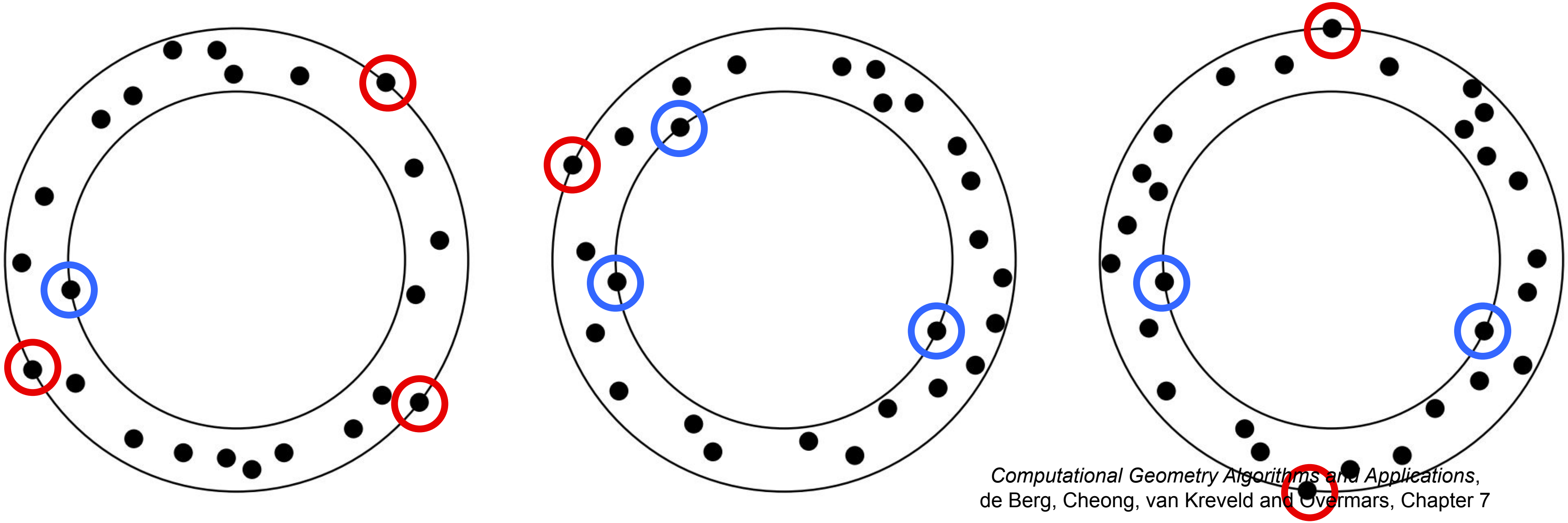
# Finding the Smallest-Width Annulus

- Easy to compute once we know the center
  (it is the center of both the inner & outer circle)
- *What points might be the center? Any point on the plane?*



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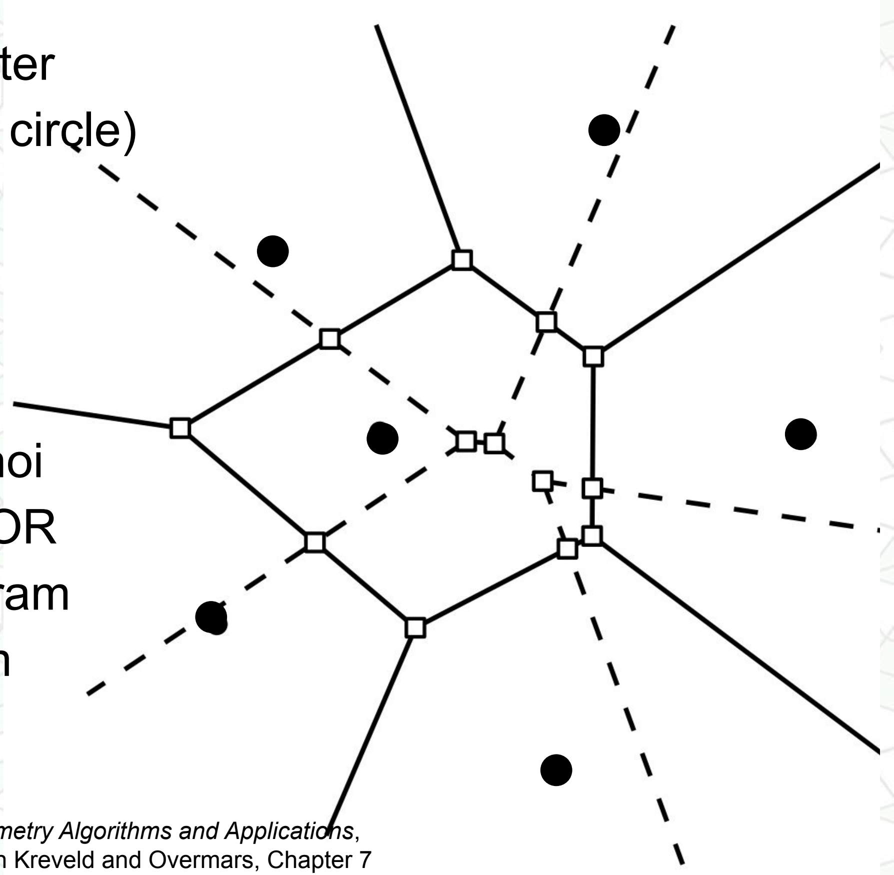
de Berg, Cheong, van Kreveld and Overmars, Chapter 7

# Finding the Smallest-Width Annulus

- Easy to compute once we know the center (it is the center of both the inner & outer circle)
- *What points might be the center?*
  It must be:
  - A vertex of the Voronoi Diagram (equally close to 3 sites) OR
  - A vertex of the Farthest Point Voronoi Diagram (equally far from 3 sites) OR
  - An intersection of the Voronoi Diagram and Farthest Point Voronoi Diagram (equally close to 2 sites AND equally far from 2 sites)



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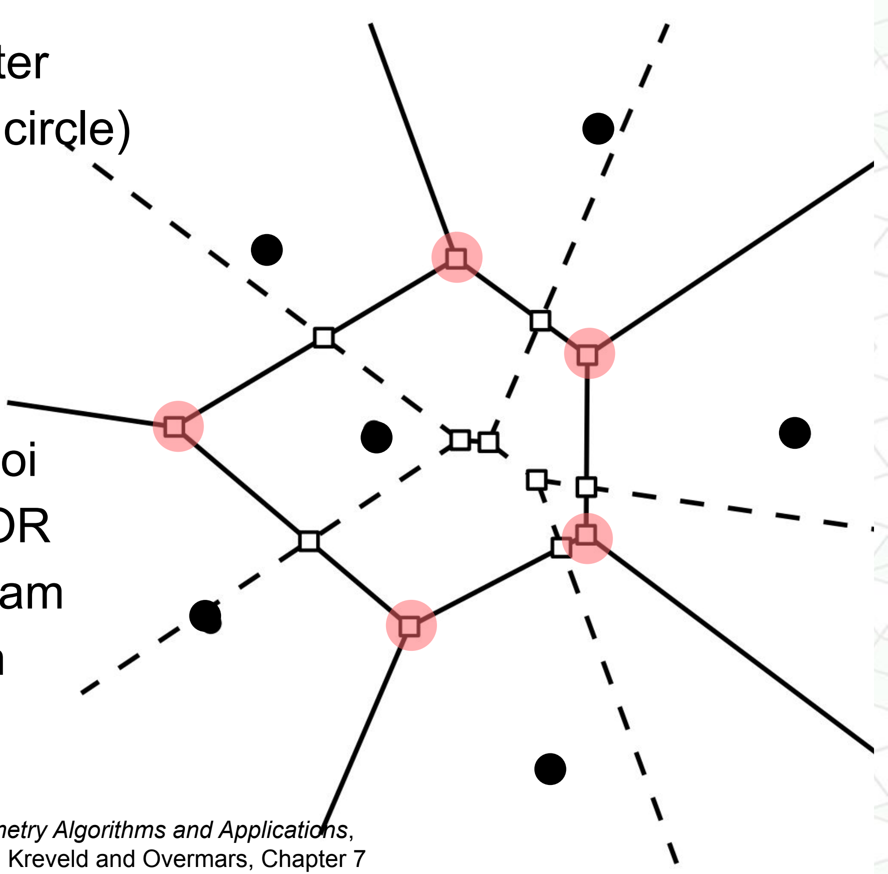

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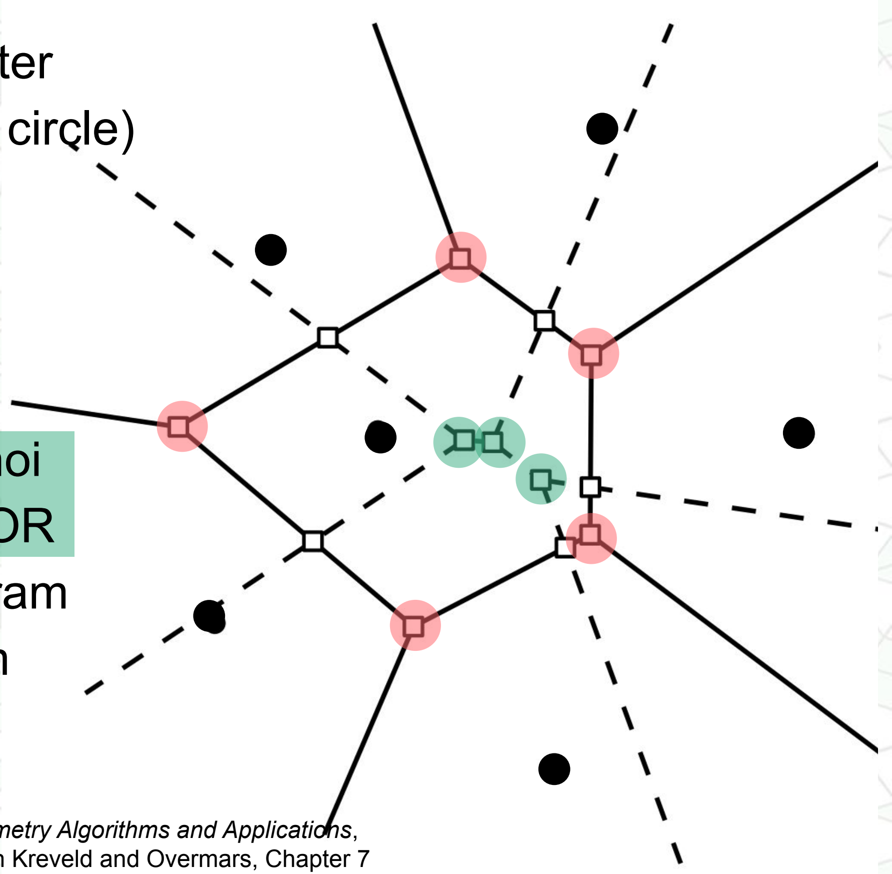

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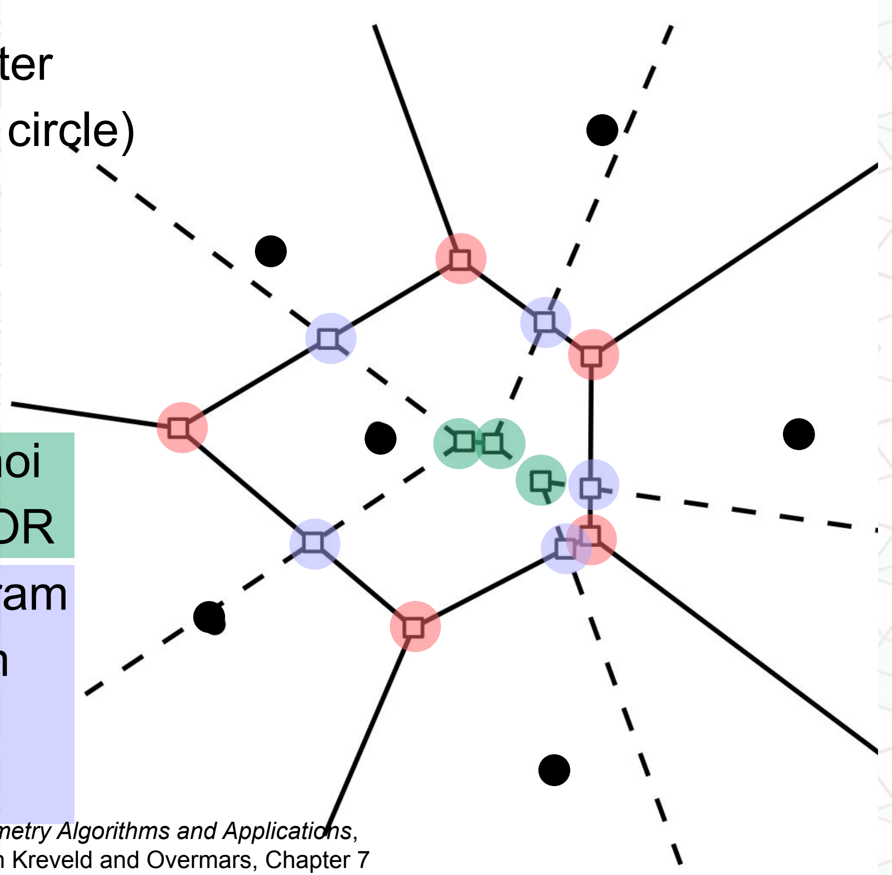

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# Finding the Smallest-Width Annulus

**Brute force check of *O(n)* = a FINITE number of possible center positions**

- Easy to compute once we know the center (it is the center of both the inner & outer circle)
- *What points might be the center?*

  It must be:
  - A vertex of the Voronoi Diagram (equally close to 3 sites) OR
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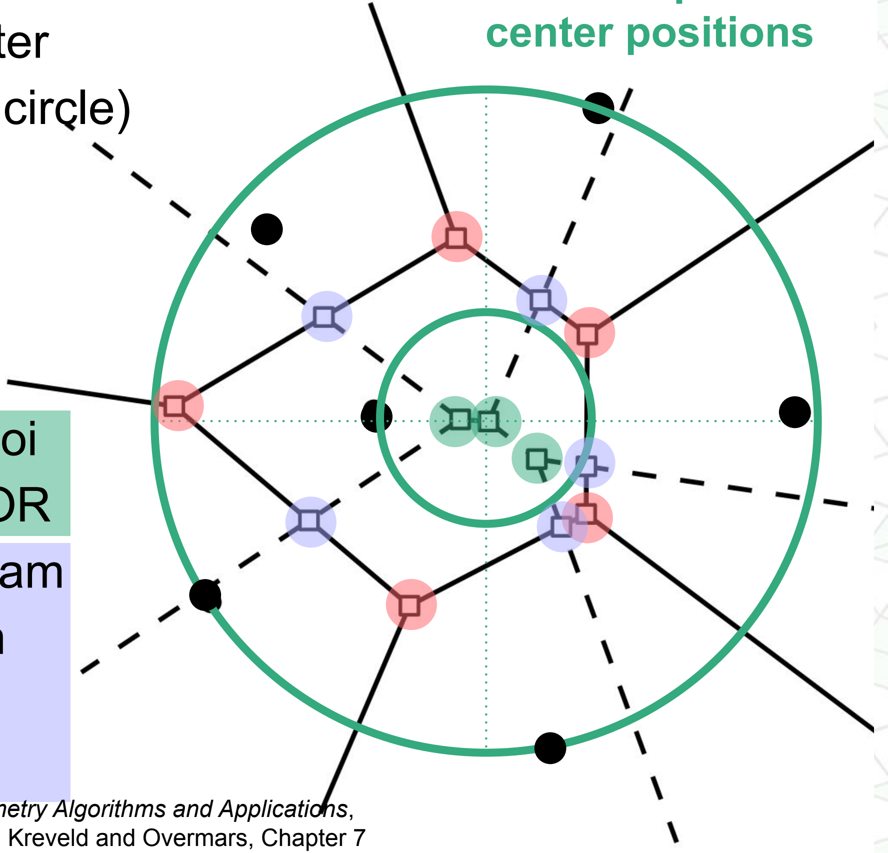

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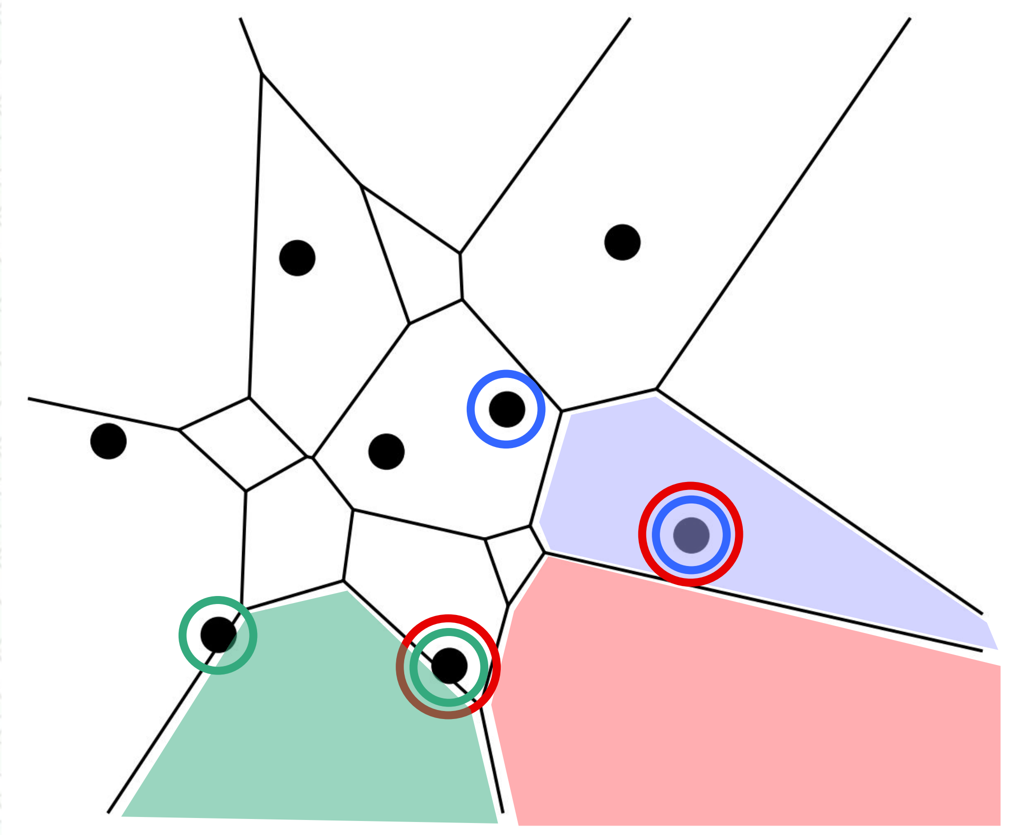
# Higher-Order Voronoi / k-Closest Sites

- For example, k = 2…

- Subdivide the plane into regions that have the same closest *and* second closest sites



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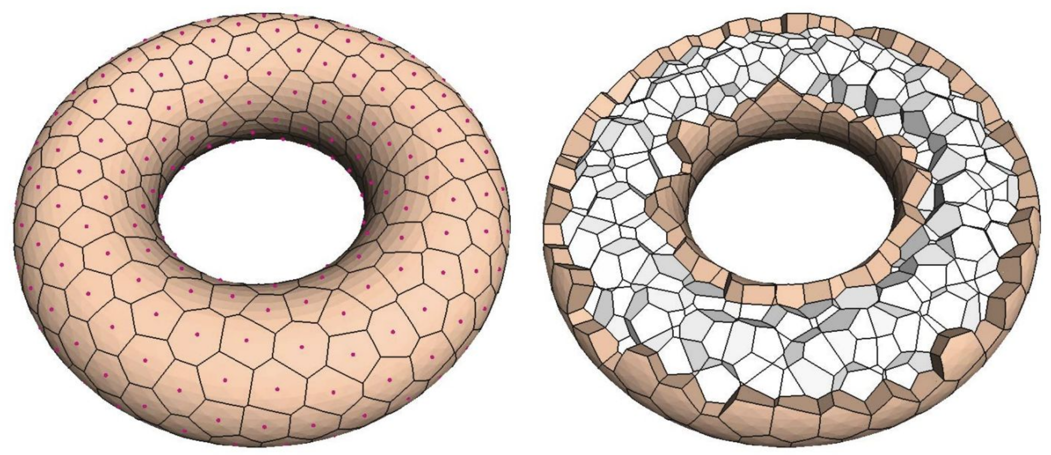
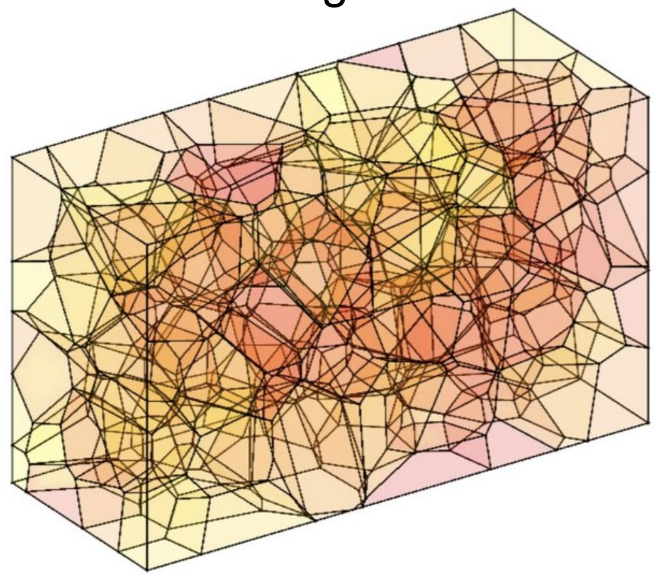
de Berg, Cheong, van Kreveld and Overmars, Chapter 7

# Voronoi Diagram in 3D or Higher Dimension

- *Not the same as "Higher-Order Voronoi Diagram"*
- Well defined in higher dimensions, but hard to visualize & debug!
- Each Voronoi cell is convex!



"Efficient Computation of Clipped Voronoi Diagrams for Mesh Generation", Yan, Wang, & Liu, 2011



"Simulation and Optimization of Porous Bone-Like Microstructures with Specific Mechanical Properties", Wit, Wronski, & Tarasiuk, 2019

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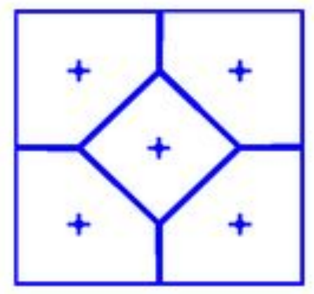
# Centroidal Voronoi Diagram



- *What if we could place all of the grocery stores?*
- Where should we place the grocery stores so that they are centrally located for all of their customers?

https://en.wikipedia.org/wiki/Centroidal_Voronoi_tessellation

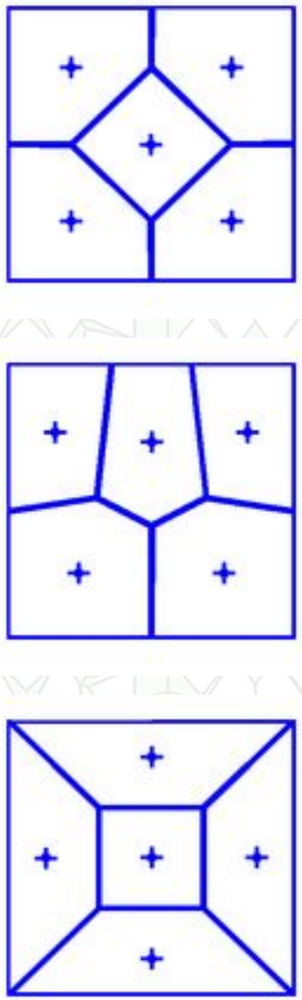
# Centroidal Voronoi Diagram

- *What if we could place all of the grocery stores?*
- Where should we place the grocery stores so that they are centrally located for all of their customers?
- But if you change the position of the store, the closest store will change for some customers…

- Points are at the center of mass of their cell
- Constructed using k-means clustering / Lloyd's algorithm - an iterative relaxation algorithm

- *Note: May be multiple solutions!*

https://en.wikipedia.org/wiki/Centroidal_Voronoi_tessellation



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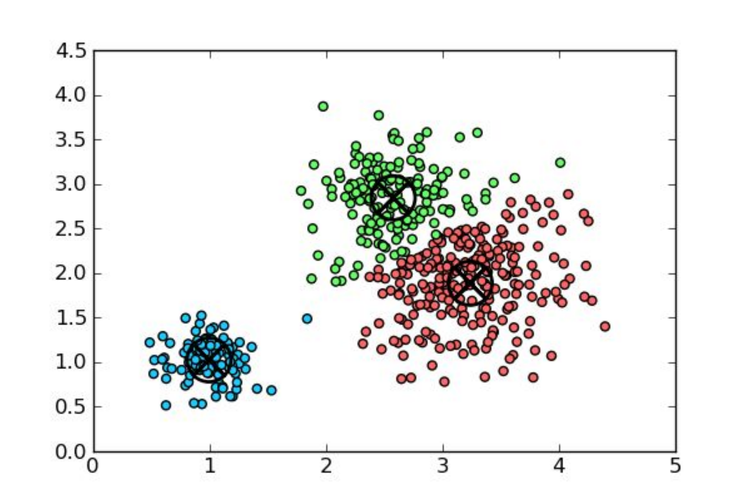
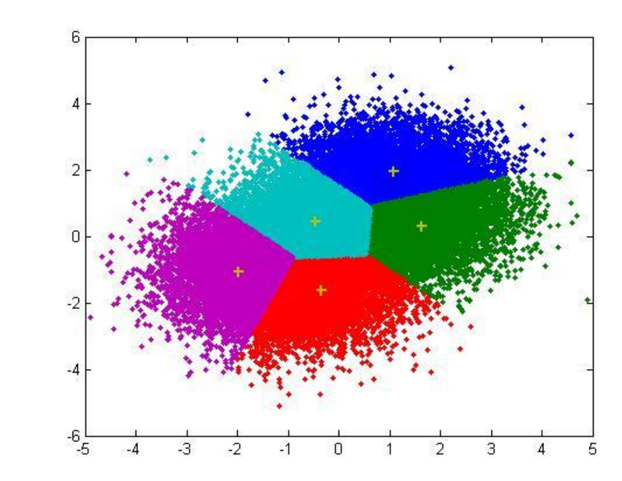
# K-Means Clustering

- Works quite well, when the data can be meaningfully classified (and we know how many clusters to use).
- With dense data, output is visually similar to Voronoi diagram (k-Means chooses the data points that define the cells)




http://blog.mpacula.com/2011/04/27/k-means-clustering-example-python/



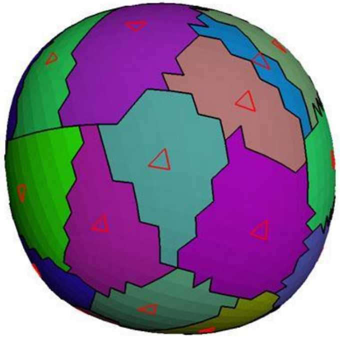

"Efficient K-Means Clustering using JIT" Yi Cao

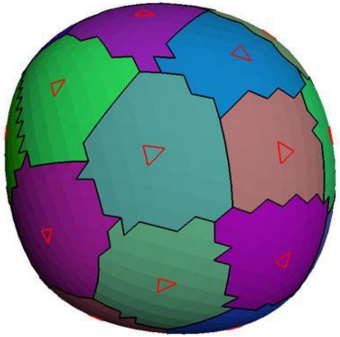
# K-Means Clustering

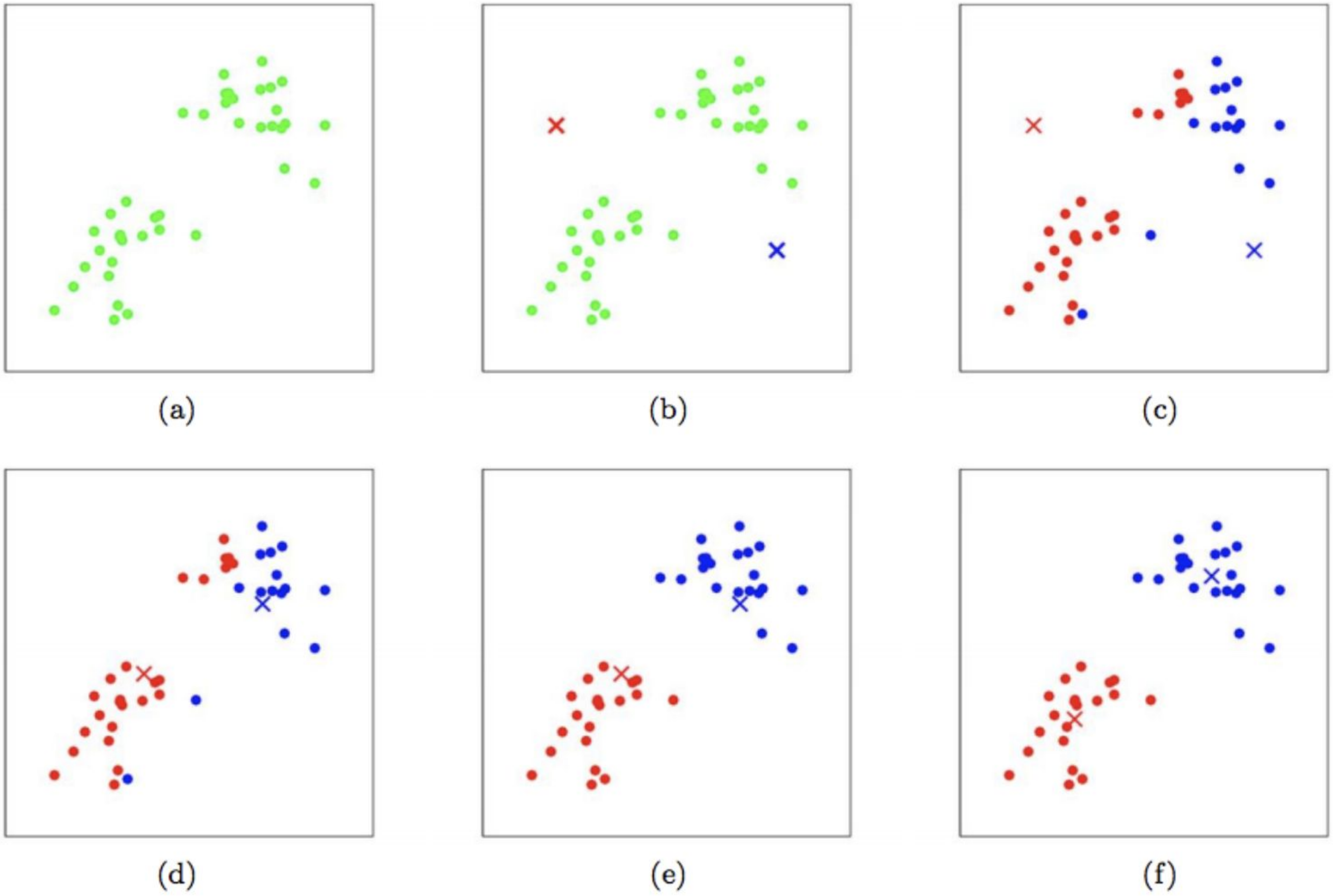
For a set of 2D/3D/*n*D points:

- Choose *k*, # of clusters (maybe an “oracle” tells us...)
- Select *k* points from your data at random as initial team representatives
- Every other point determines which team representative it is closest to and joins that team
- The team averages the positions of all members, this is the team’s new representative
- Repeat *x* times or until change < threshold

*Same/Similar to: Lloyd’s Algorithm*








Wei Zhang
https://wei2624.github.io/MachineLearning/usv_kmeans/

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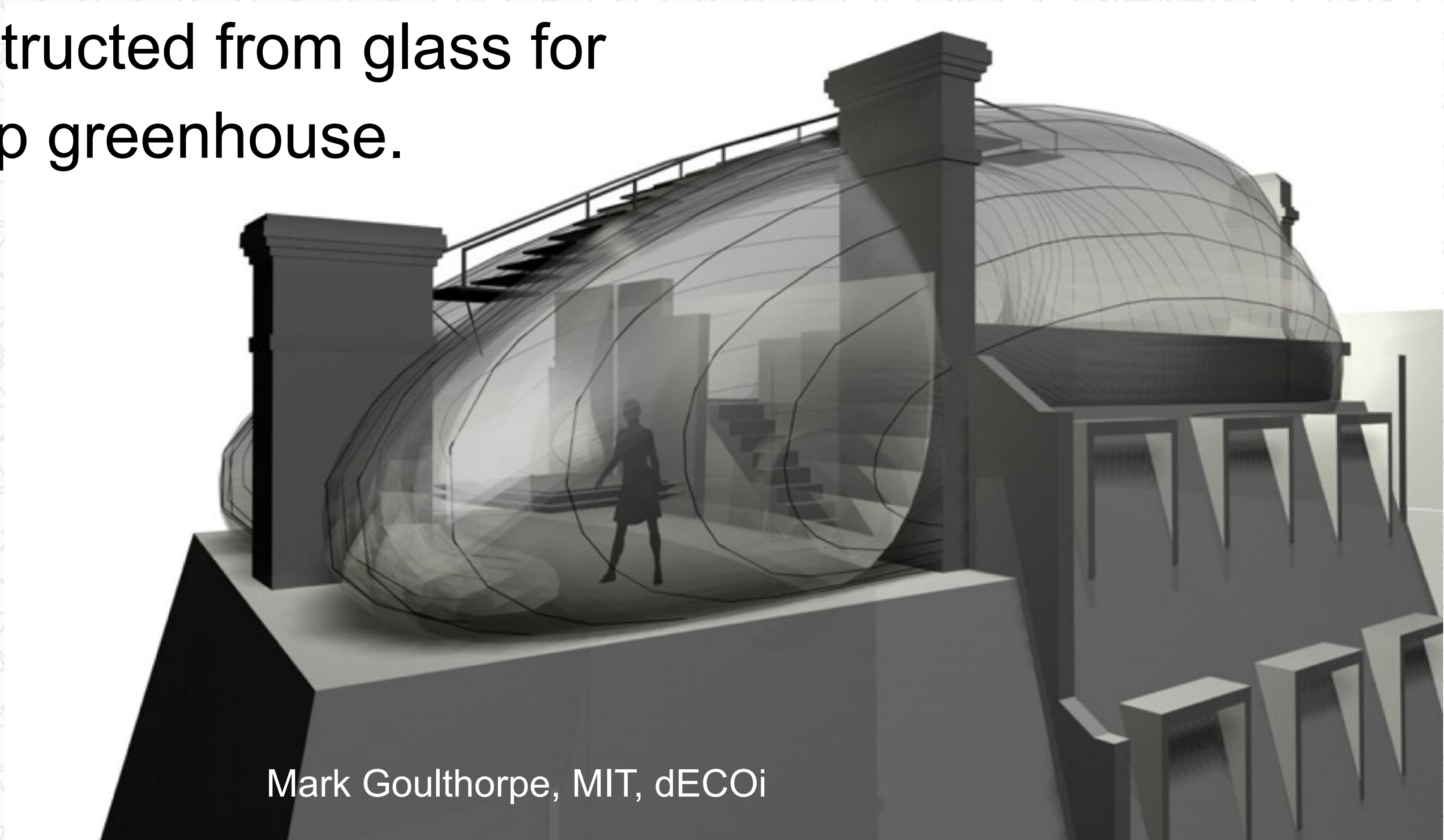
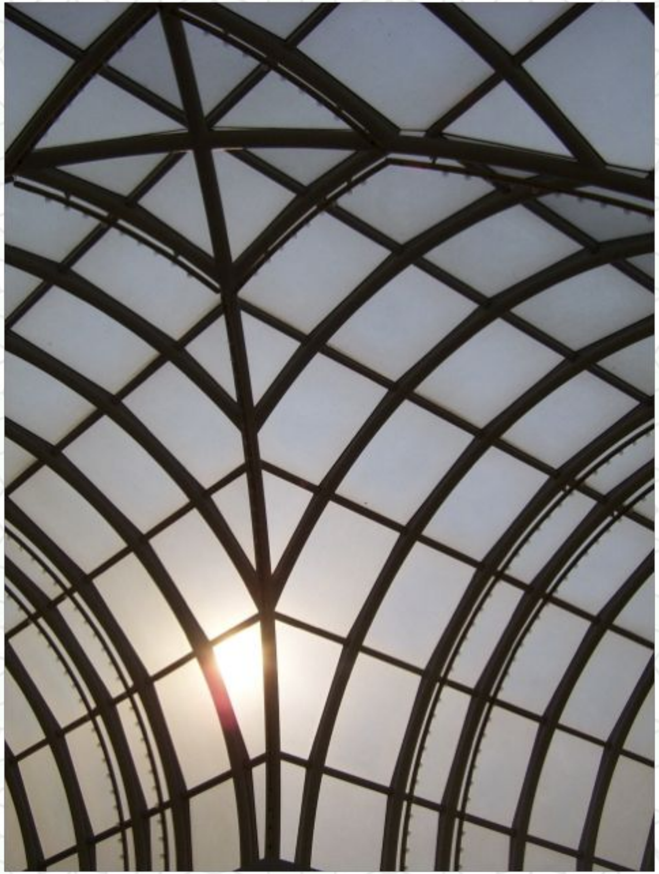
The Problem: Mesh this curved surface so that it can be constructed from glass for a rooftop greenhouse.



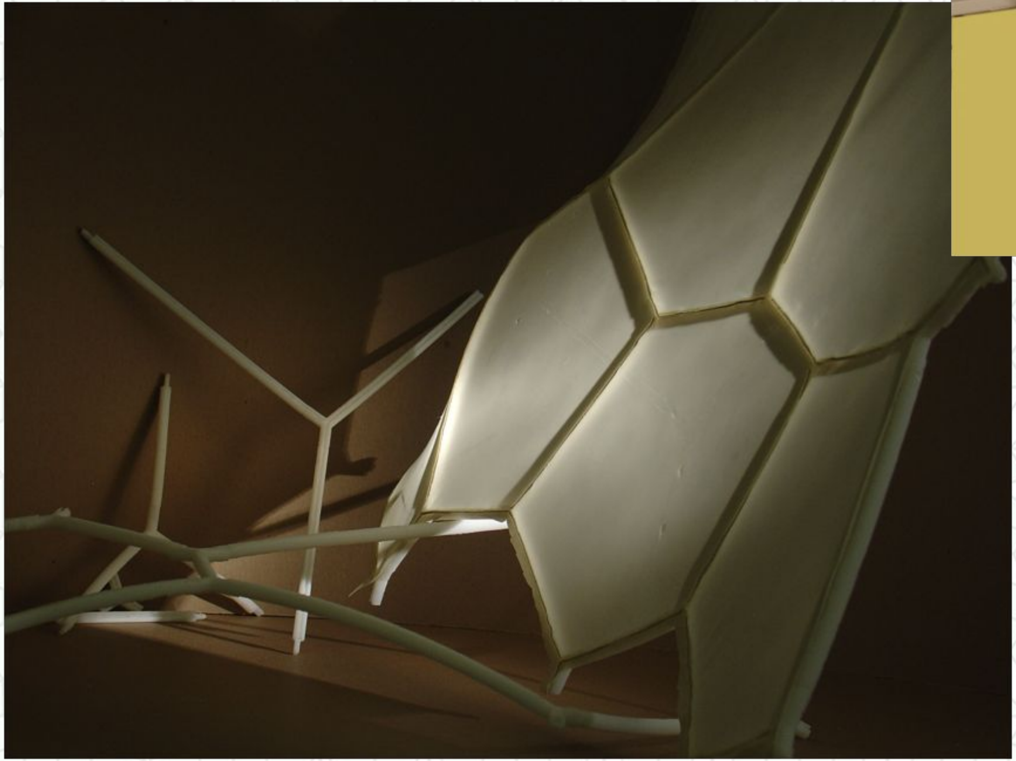
Mark Goulthorpe, MIT, dECOi

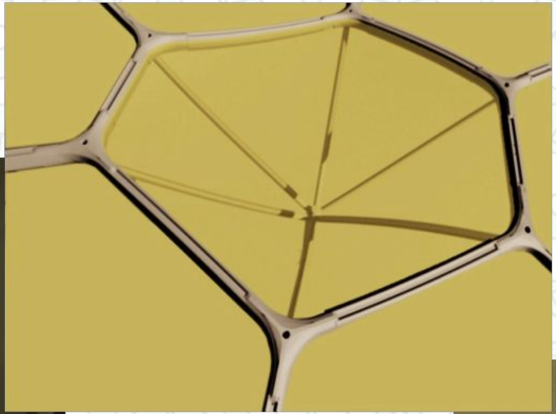
**Great Court at the British Museum**
London, England
Norman Foster and Partners, 2000.

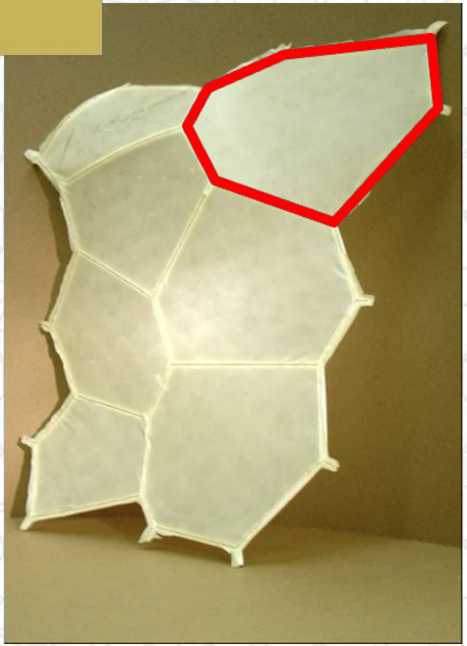


**Chadstone Shopping Center**
Melbourne, Australia
RTKL Associates Inc, 1999.

# “Voronoi Surfaces”





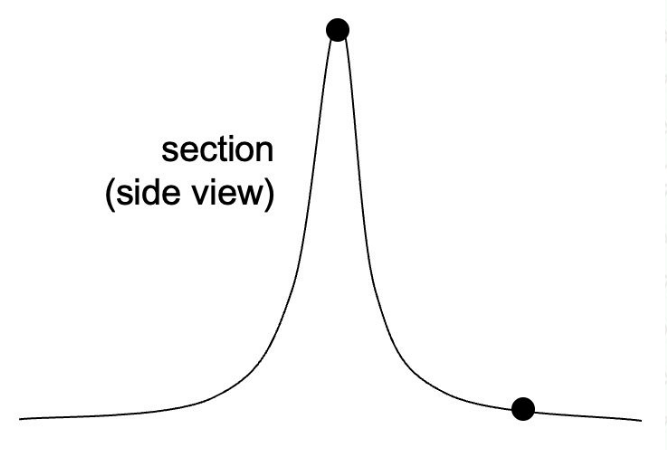


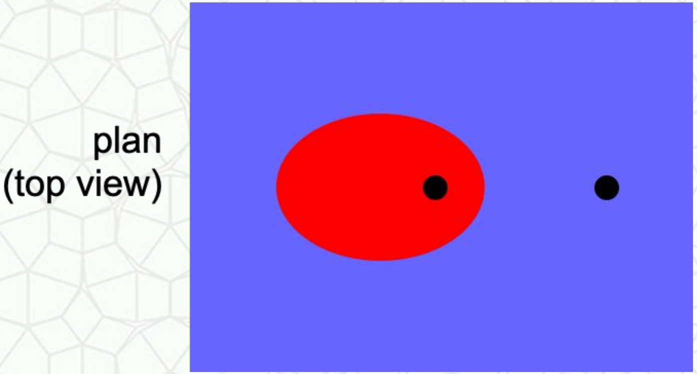
***Unfortunately, the cell vertices are rarely planar!***

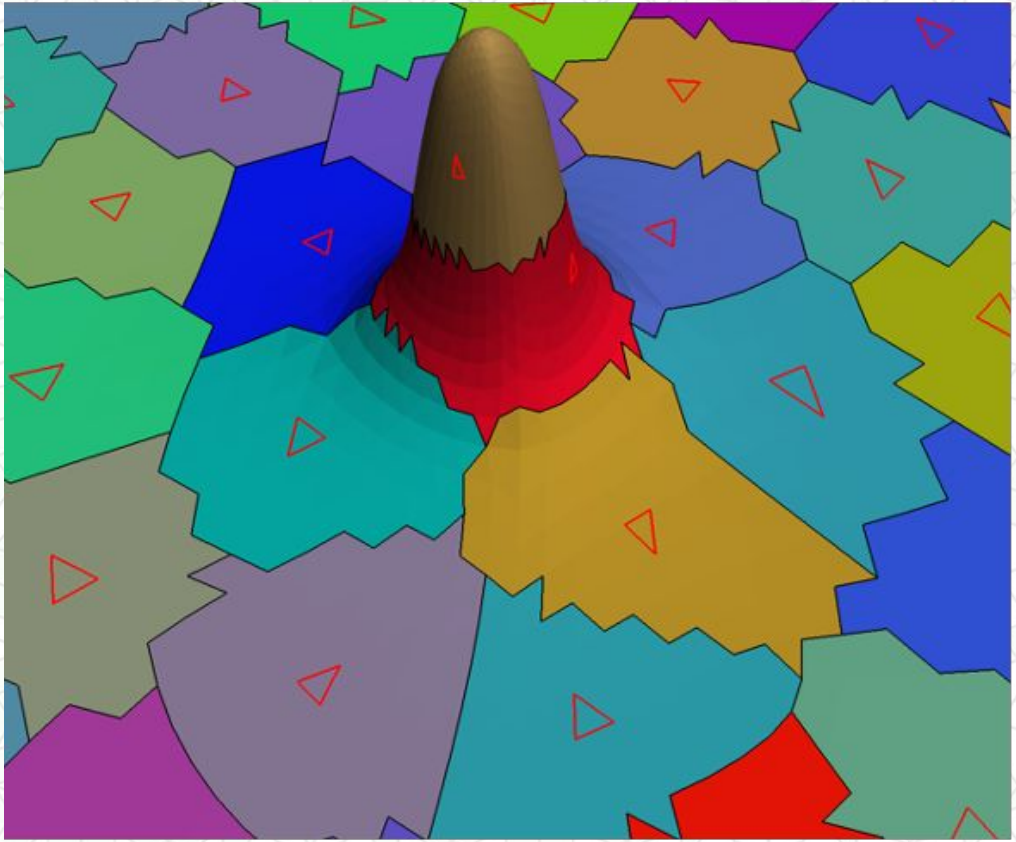
**Mike Powell, MIT**
**Studio Project Fall 2004**

# Voronoi Diagram on a 3D Surface

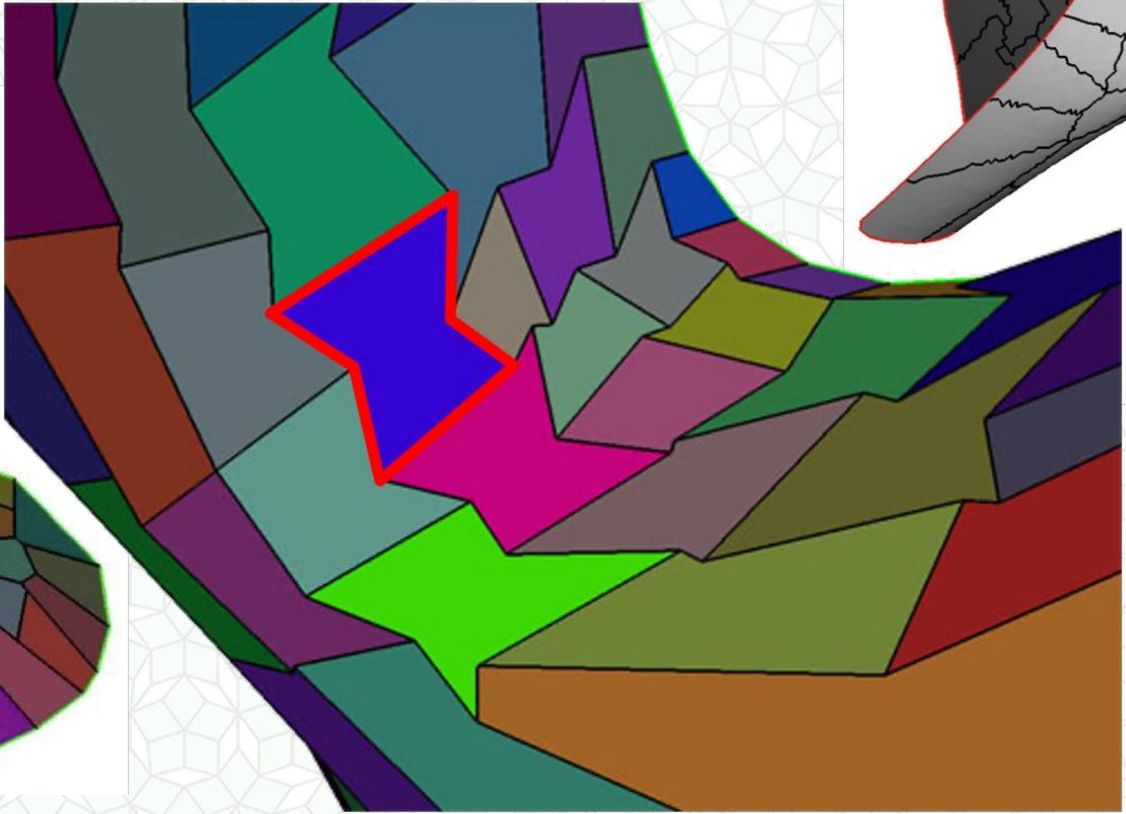
**We’re no longer using *Euclidean Distance!***

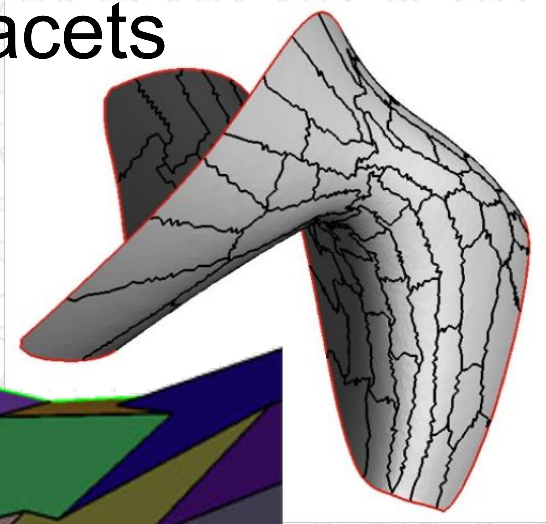


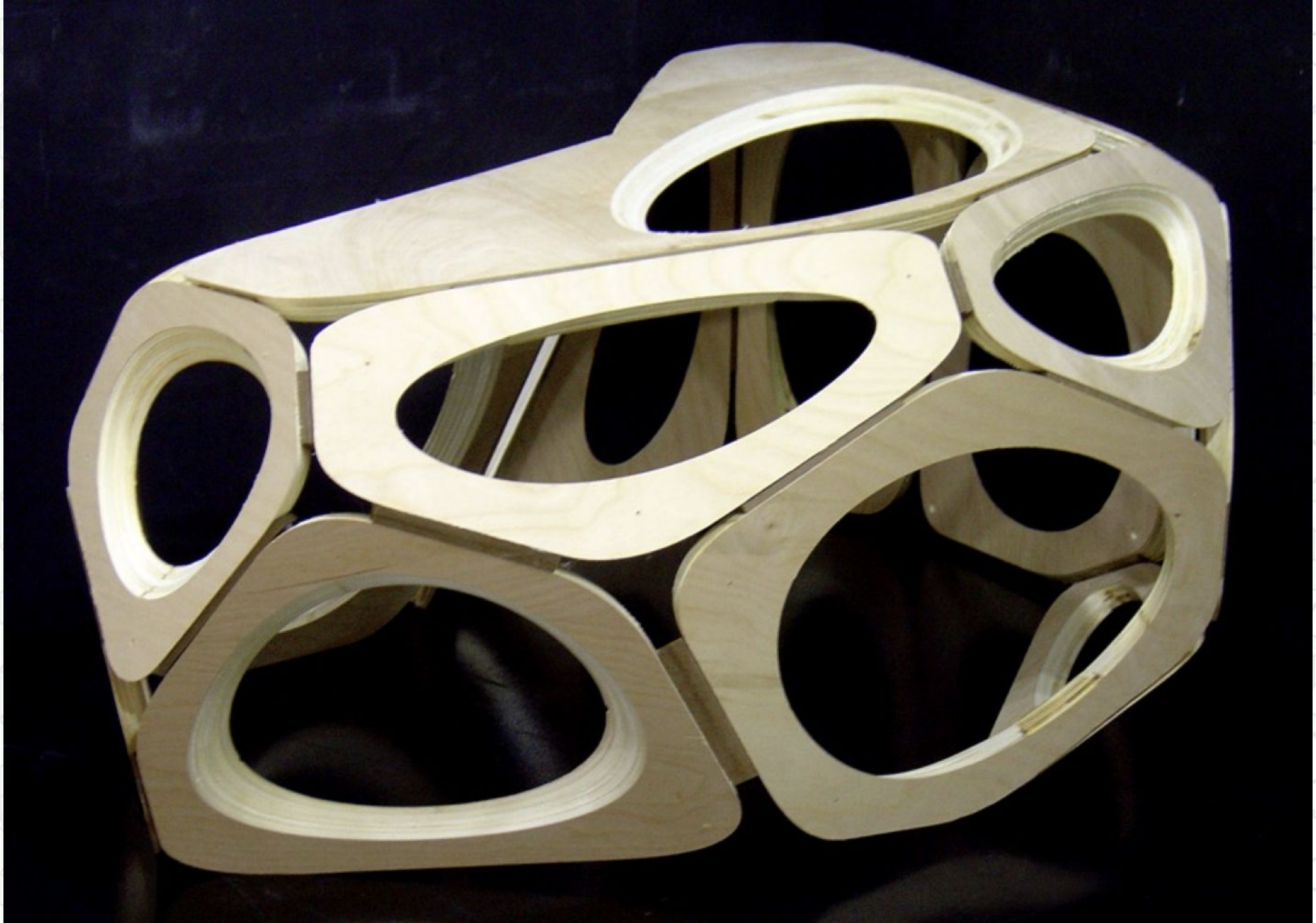
# “Saddle Surface” → Non-Convex Facets



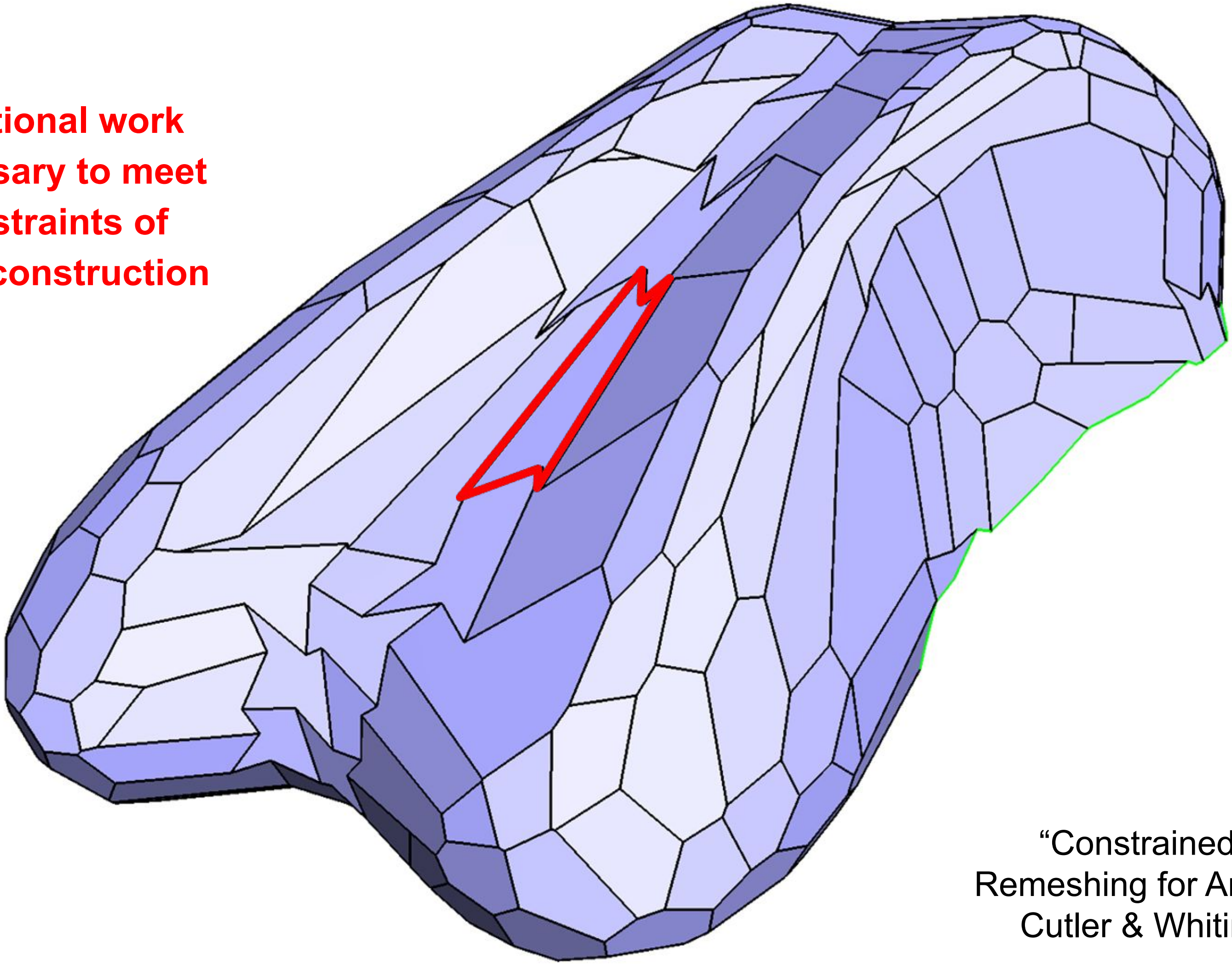


“bowtie” shapes

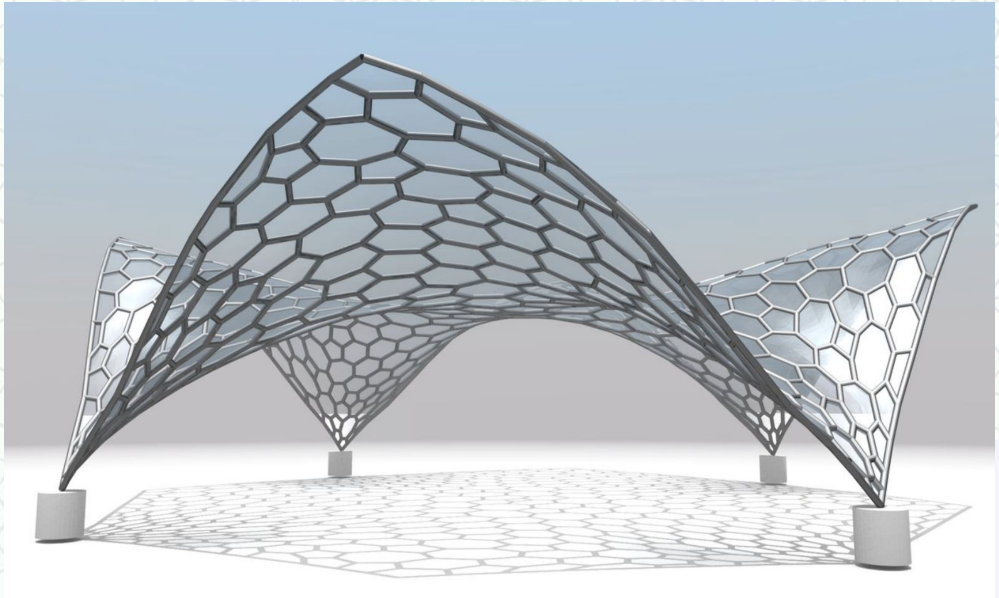
# Fabrication



Additional work necessary to meet constraints of glass construction

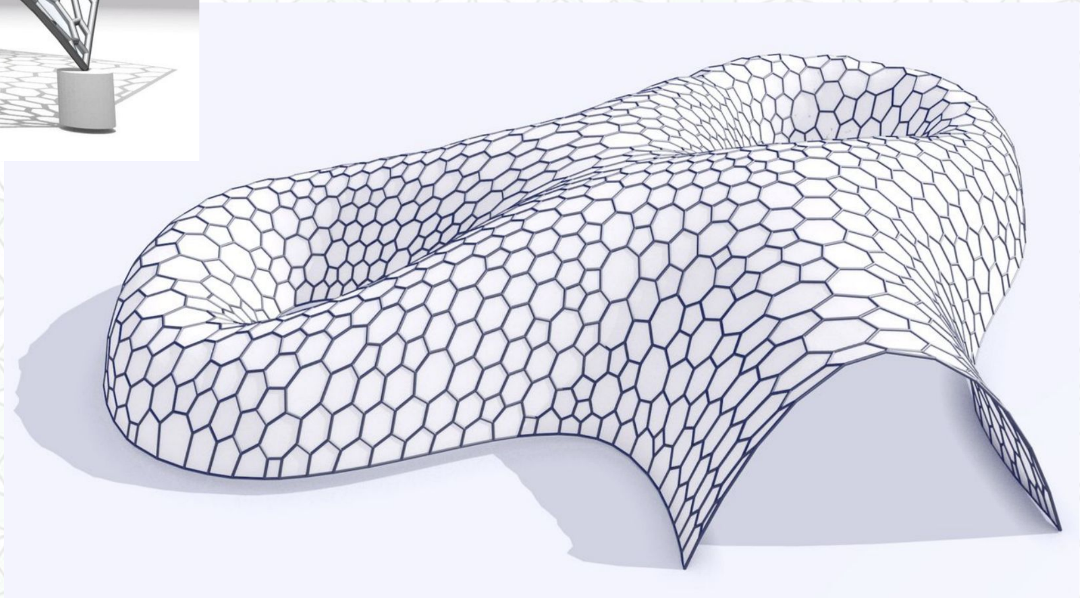


"Constrained Planar Remeshing for Architecture", Cutler & Whiting, 2007

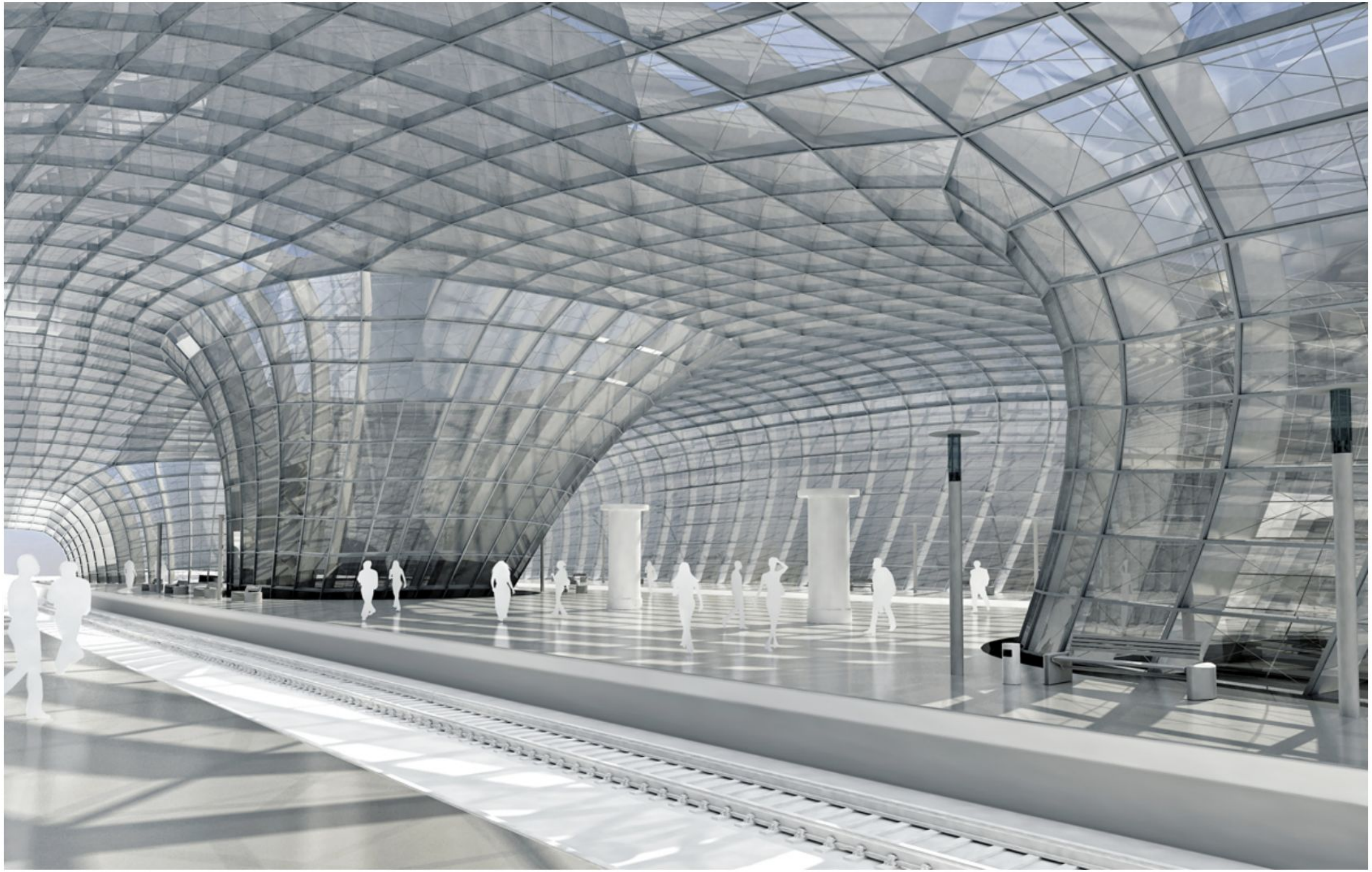


Focus / Goals:

- Structural feasibility / efficiency
- Improve planarity of faces (but not guaranteed)
- Symmetry and similarity in face area, edge length

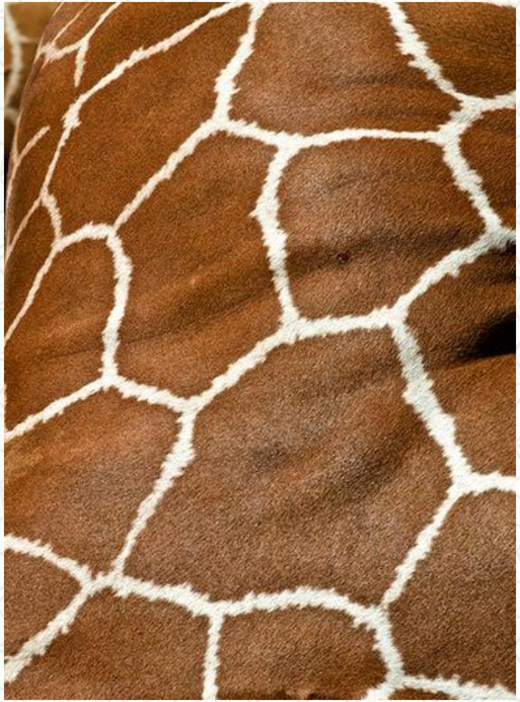


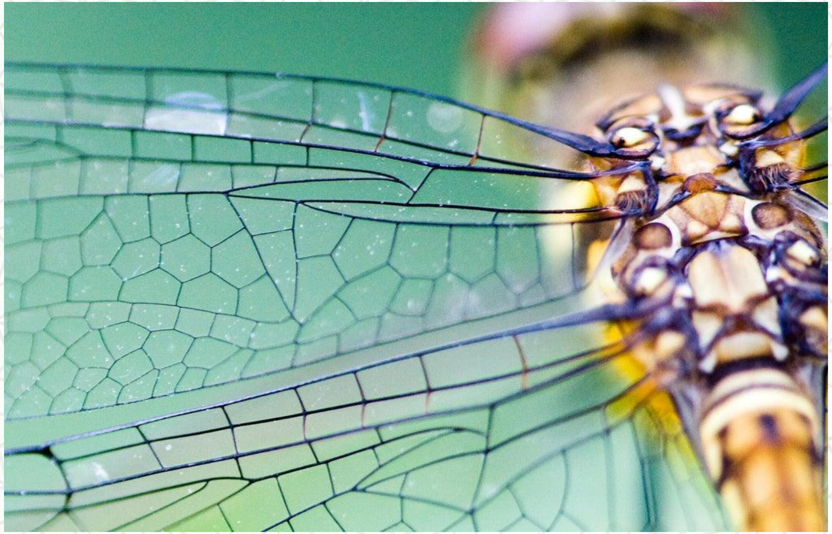
"Voronoi Grid-Shell Structures"
Pietroni, Tonelli, Puppo, Froli, Scopigno, & Cignoni, 2014



“Geometric Modeling with Conical Meshes and Developable Surfaces”
Liu, Pottmann, Wallner, Yang &Wang, SIGGRAPH 2006

# Voronoi Diagram in Nature



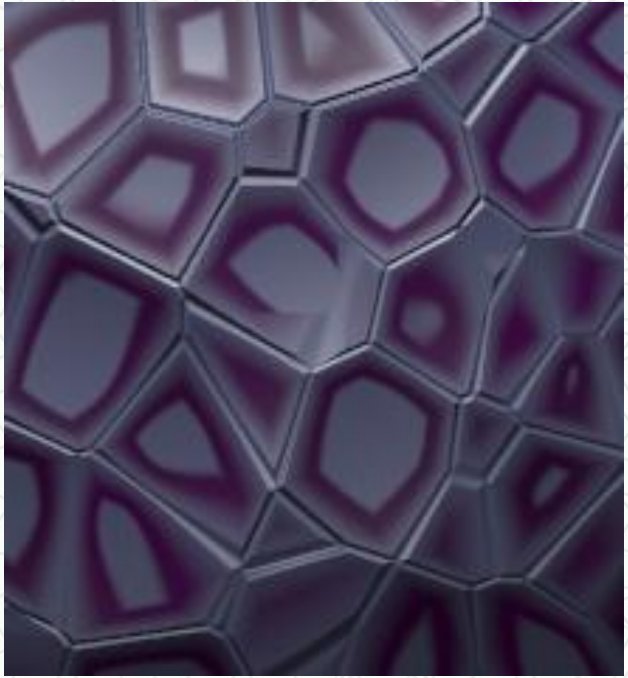


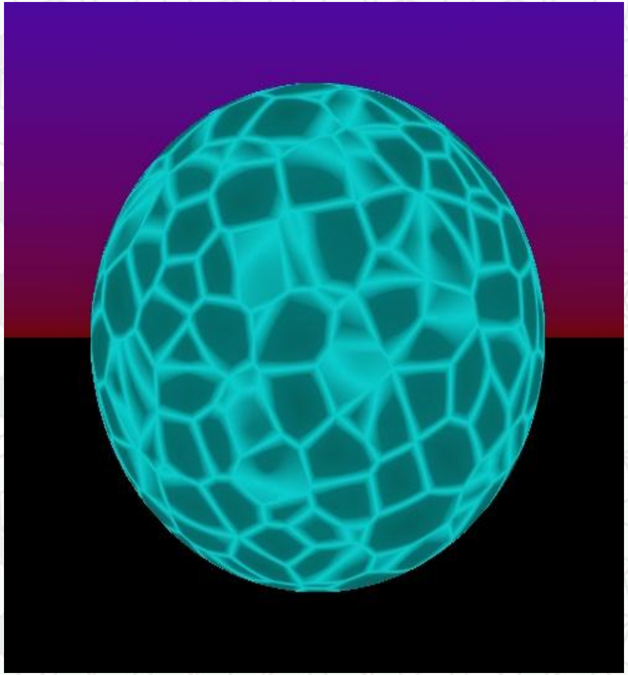
https://spring-of-mathematics.tumblr.com/post/85519358219/the-beauty-of-voronoi-diagram-in-nature-how

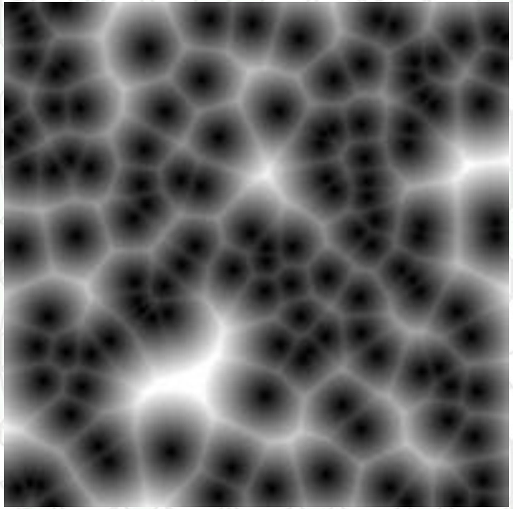
https://blogs.scientificamerican.com/observations/voronoi-tessellations-and-scutoids-are-everywhere/

# Cellular Textures

https://en.wikipedia.org/wiki/Worley_noise#/media/File:Worley.jpg







“A Cellular Texture Basis Function”,
Worley, SIGGRAPH 1996

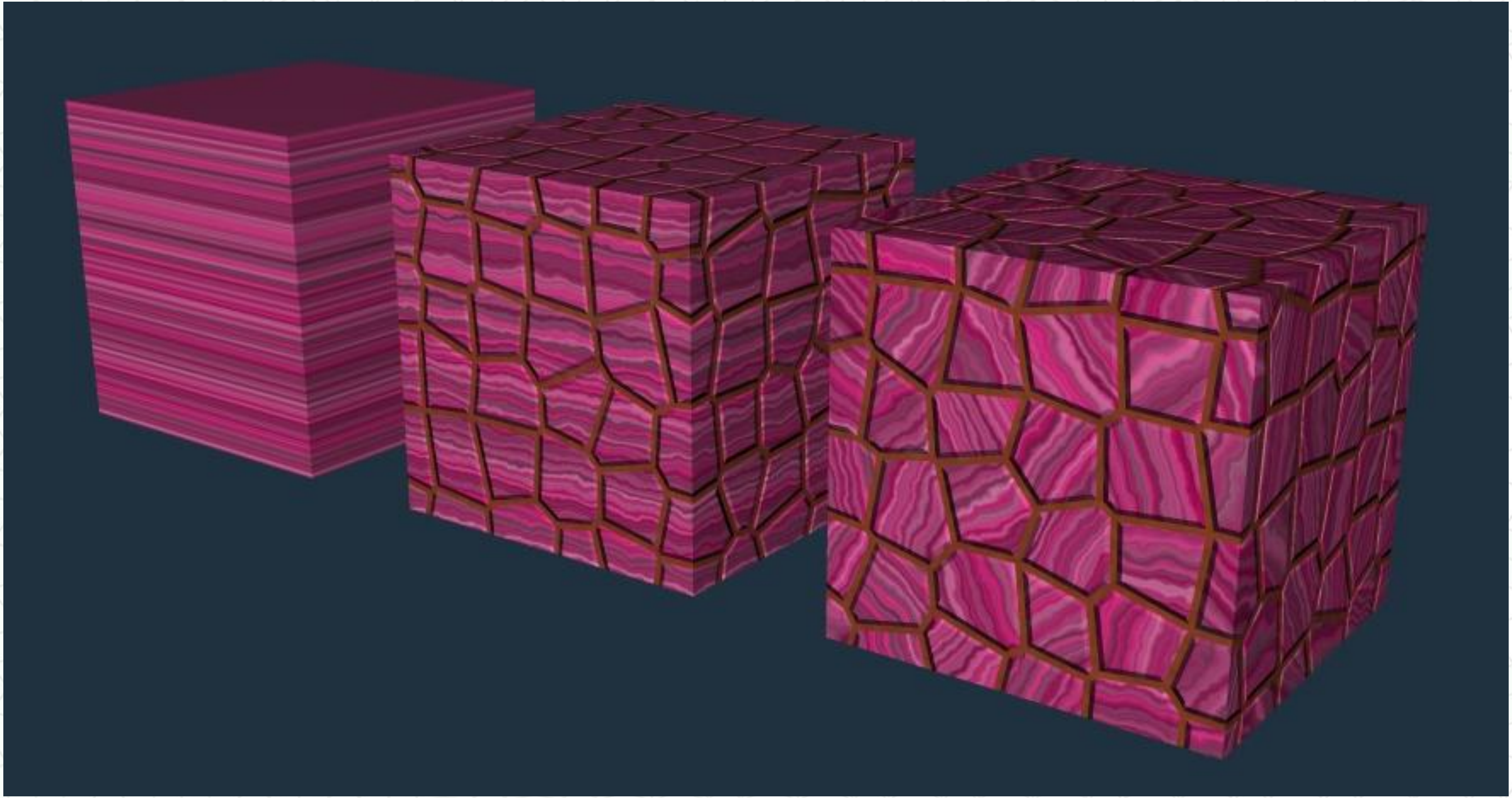


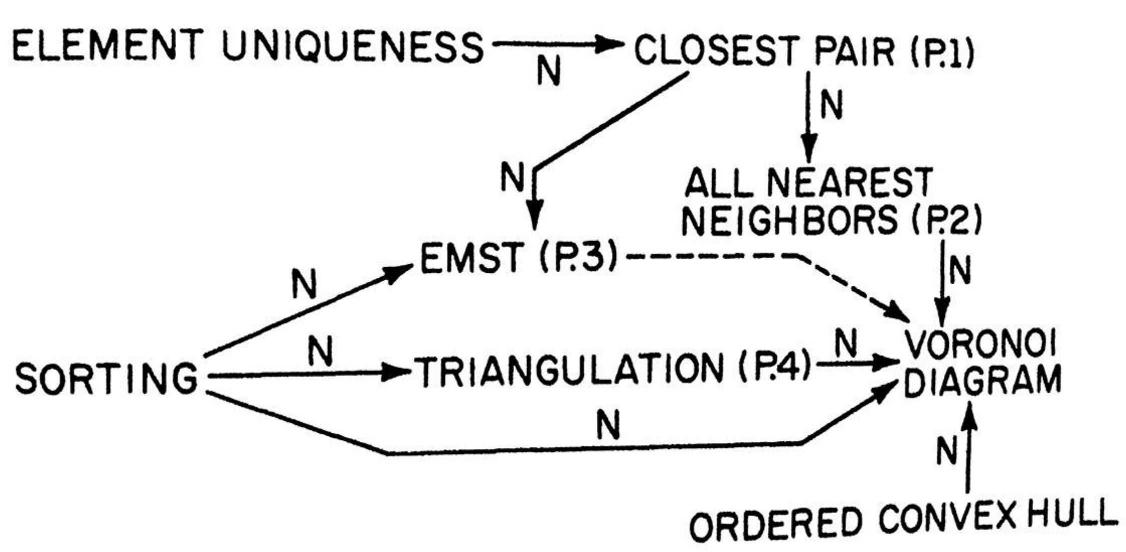
Image by Justin Legakis

# Outline for Today

- Friday Oct 13th: Quiz 1, Logistics
- Homework 6 will be posted tomorrow
- Last Time: More Voronoi Diagrams
- Higher-Order vs Higher Dimension Voronoi Diagrams
- Centroidal Voronoi Diagram
- K-Means Clustering
- Application: Architectural Geometry
- More Spatial Query & Search Problems / Applications
- Reducing Other Problems to the Voronoi Diagram
- Future Topic (after Quiz 1): Delaunay Triangulation

# Problems that Reduce to Voronoi Diagram

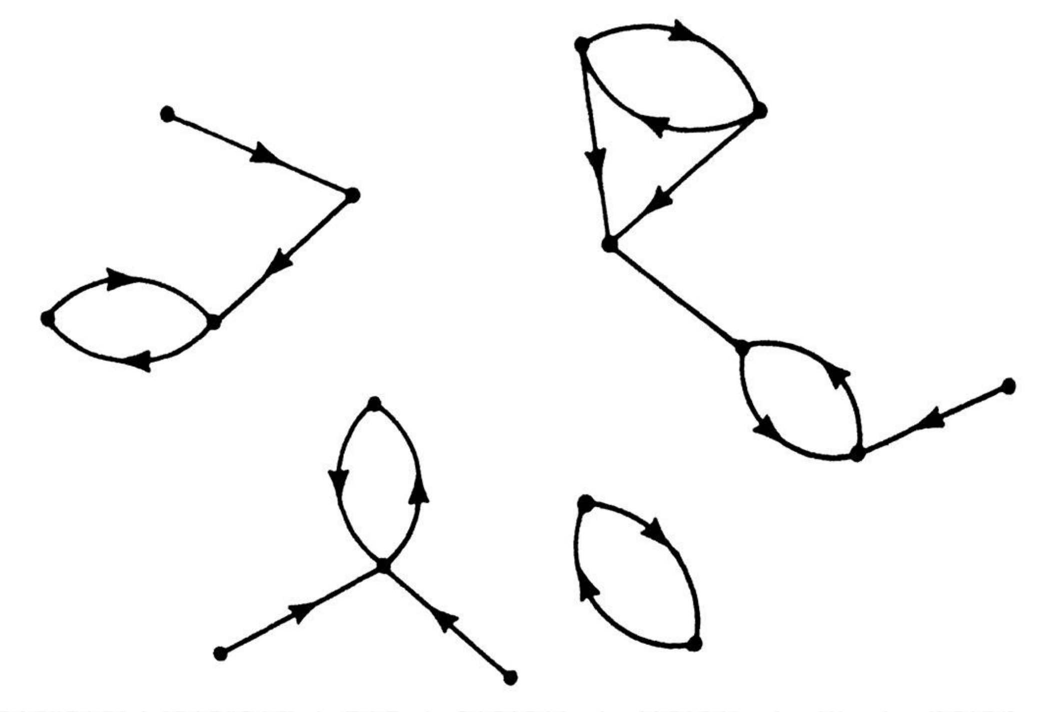
- We can compute the Voronoi Diagram of *n* points in *O(n log n)* time and *O(n)* space.
- These other problems can be computed in *O(n)* additional time if given the Voronoi Diagram.
- *Therefore they are also O(n log n) time and O(n) space.*




*Computational Geometry: An Introduction,*
Preparata & Shamos, 1985, Figure 5.30

# Problem: All Nearest Neighbors

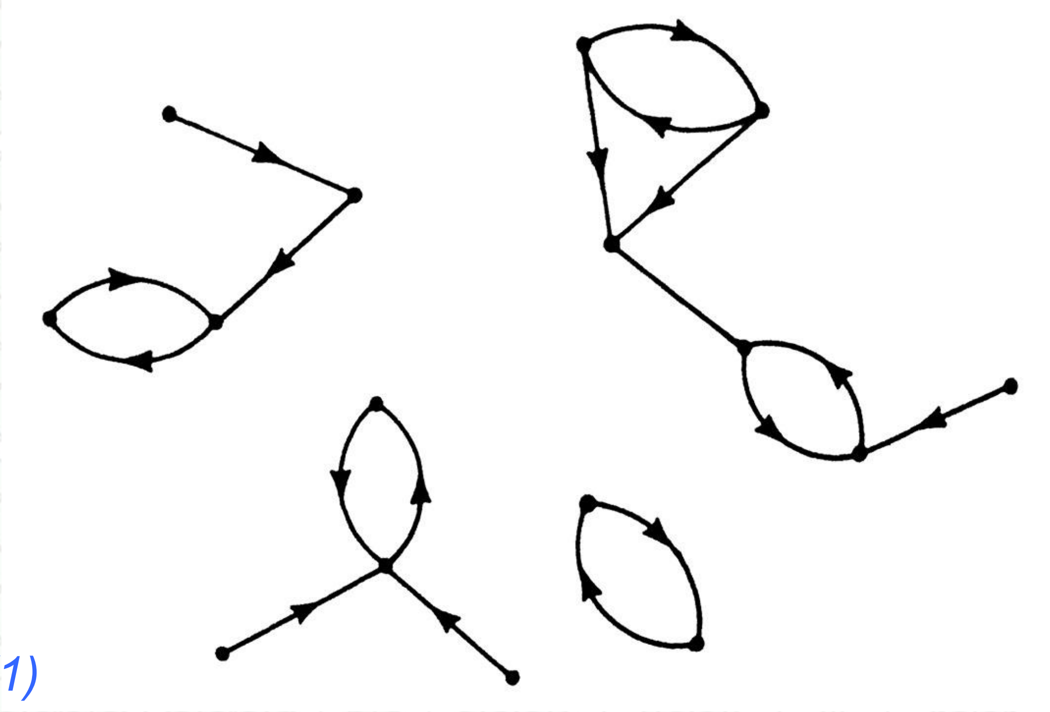
- Connect every point to its nearest point with a directed edge.
- Some points form a *reciprocal pair.*



*Computational Geometry: An Introduction,*
Preparata & Shamos, 1985, Figure 5.1

# Problem: All Nearest Neighbors

- Connect every point to its nearest point with a directed edge.
- Some points form a *reciprocal pair.*
- Simple Algorithm: Compare each point to every other point
- Runtime: $O(n^2)$
- Additional Memory (ignoring input & output): $O(1)$

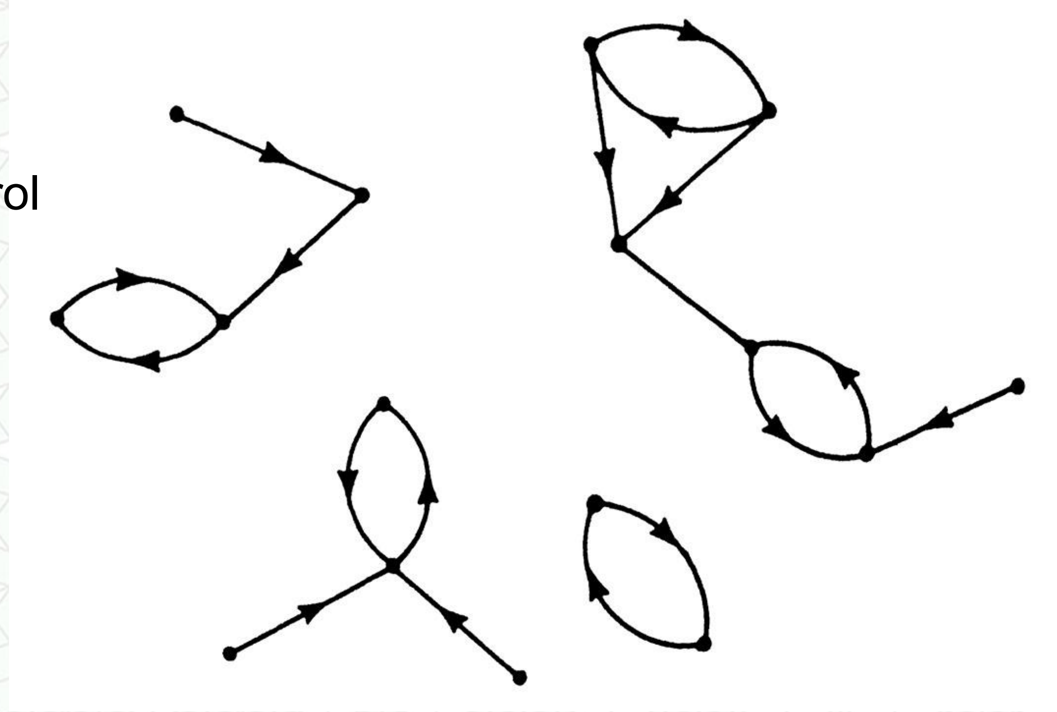


*Computational Geometry: An Introduction*, Preparata & Shamos, 1985, Figure 5.1

***Can we do better? Yes!***

# Problem: Closest Pair

- Which two points are the closest?
- Applications - Collision Detection & Air Traffic Control
- Which two objects have soonest potential for collision?

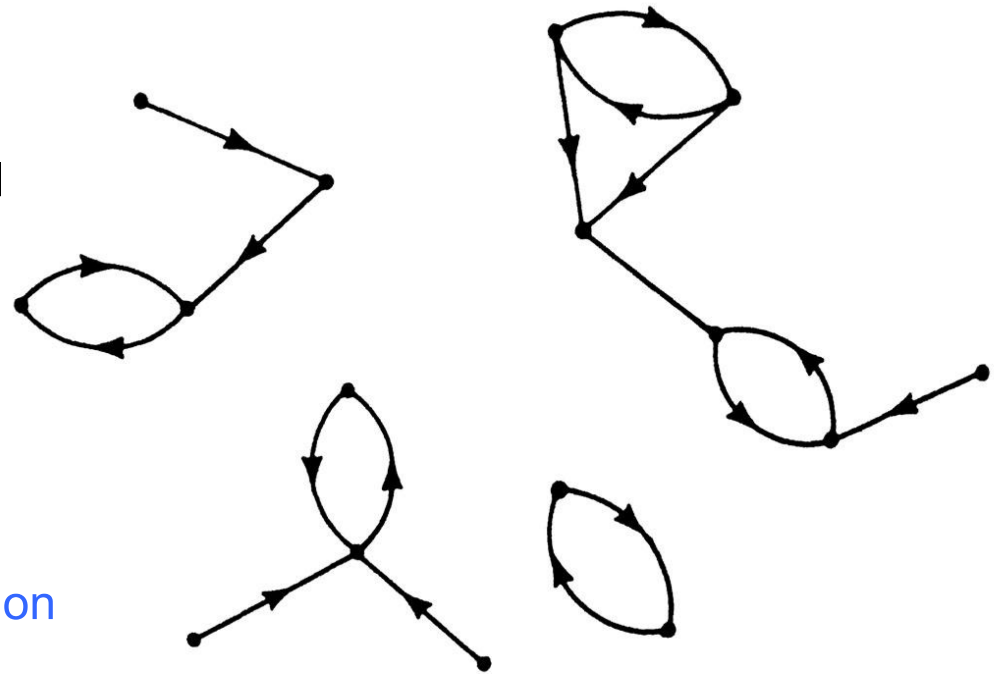


*Computational Geometry: An Introduction,*
Preparata & Shamos, 1985, Figure 5.1

# Problem: Closest Pair

- Which two points are the closest?
- Applications - Collision Detection & Air Traffic Control
- Which two objects have soonest potential for collision?

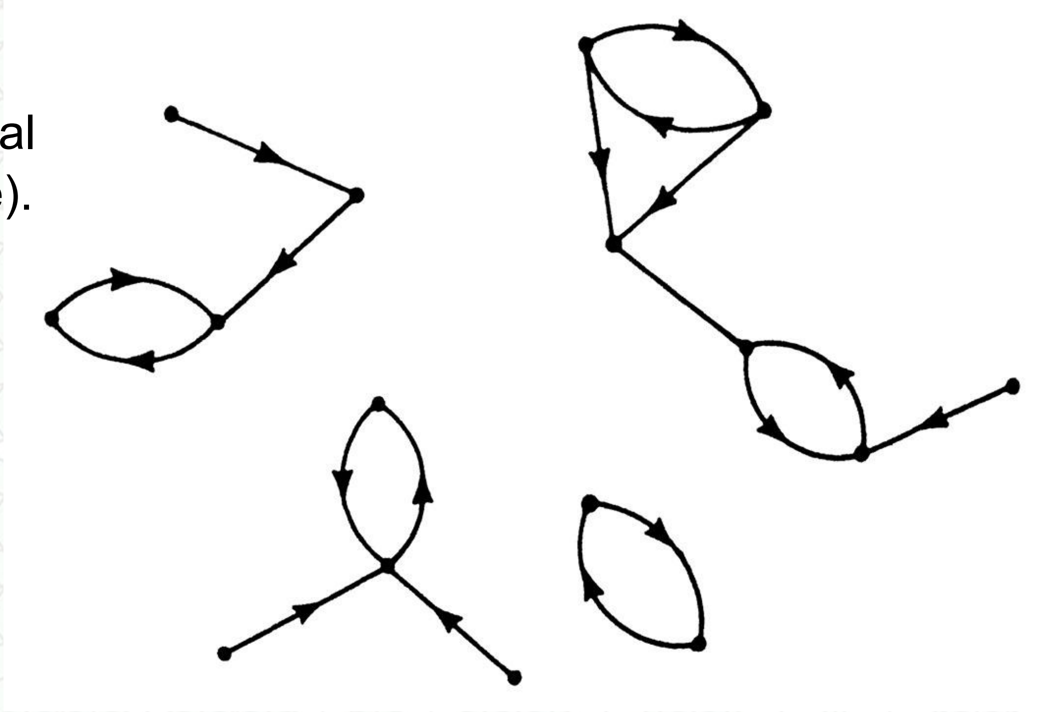
- Algorithm: *O(n)* Linear loop over all edges in the **All Nearest Neighbors** solution to find the shortest edge
- Will be a reciprocal pair



*Computational Geometry: An Introduction,*
Preparata & Shamos, 1985, Figure 5.1

# Problem: Uniqueness

- Given *n* numbers
  (or *n* 2D/3D/etc points),
  decide if any two are identical
  (if not… all items are unique).



*Computational Geometry: An Introduction,*
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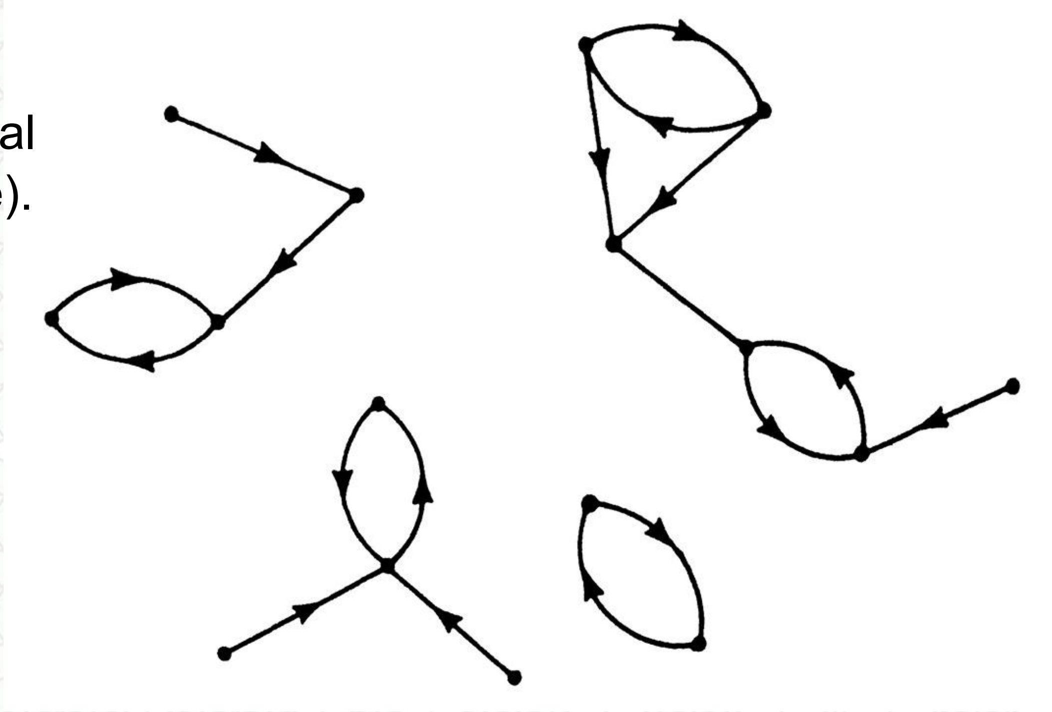
# Problem: Uniqueness

- Given *n* numbers
  (or *n* 2D/3D/etc points),
  decide if any two are identical
  (if not… all items are unique).

- *O(n)* Linear loop over
  all edges in the
  **All Nearest Neighbors**
  solution to check if any
  edges are length zero.



*Computational Geometry: An Introduction,*
Preparata & Shamos, 1985, Figure 5.1

# Problem: Euclidean Minimum Spanning Tree

- Given *n* points
- Draw *n-1* edges to create a tree, connecting all points without creating any cycles.
- Pick edges to minimize the sum of their lengths.

- *Application: Minimize cost of physical power/data lines*

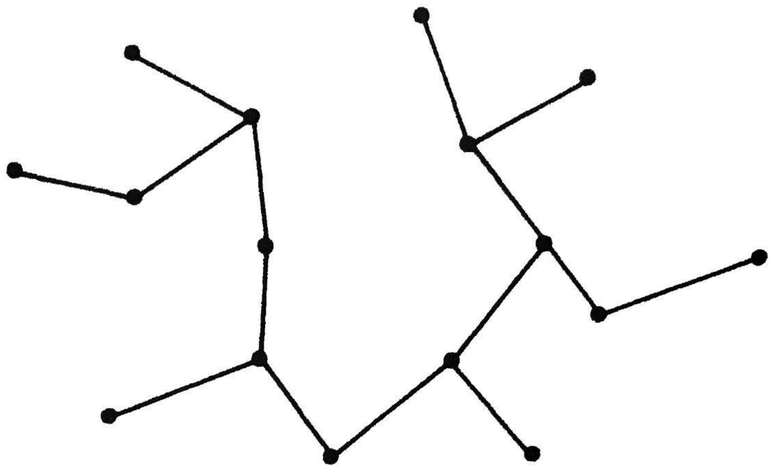


Figure 5.2 A minimum spanning tree on a planar point set.

*Computational Geometry: An Introduction,*
Preparata & Shamos, 1985, Figure 5.2

# A Different Problem: Minimum Spanning Tree

***General (non-Euclidean) MST from Graph Theory***

- Each edge has a weight, *not necessarily the Euclidean distance between two points*
- Worst case, may have $m = n^2$ edges to consider
- Runtime $O(m \log n)$
  $O(n^2 \log n)$ worst case

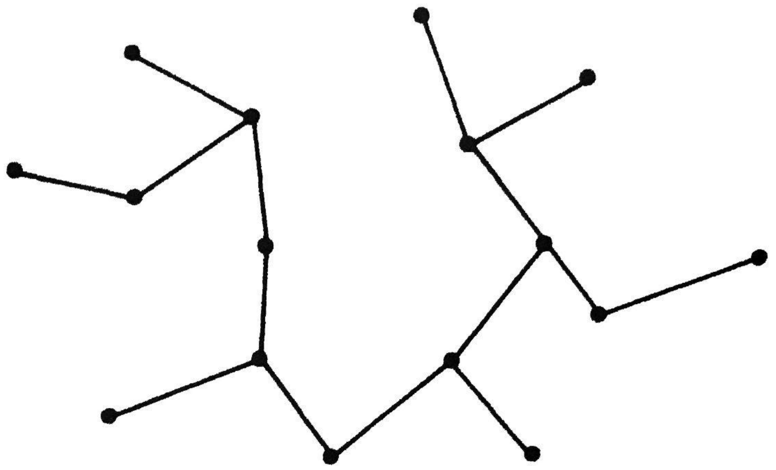


Figure 5.2 A minimum spanning tree on a planar point set.

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# A Different Problem: (Euclidean) Steiner Tree

- If allowed to add additional points – so-called Steiner Points
- Minimize sum of Euclidean distance edge lengths
- *Computing the Steiner Tree is NP Complete / NP hard!*

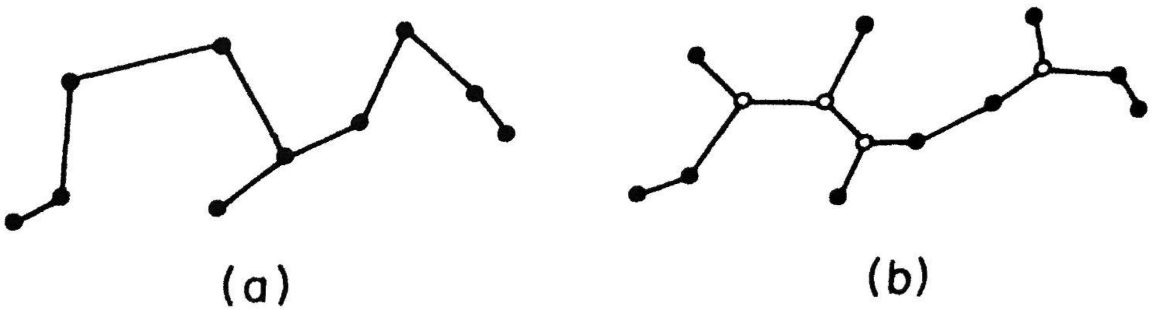


Figure 5.3 A Steiner Tree (b) may have smaller total length than the MST (a).

*Computational Geometry: An Introduction,*
Preparata & Shamos, 1985, Figure 5.3

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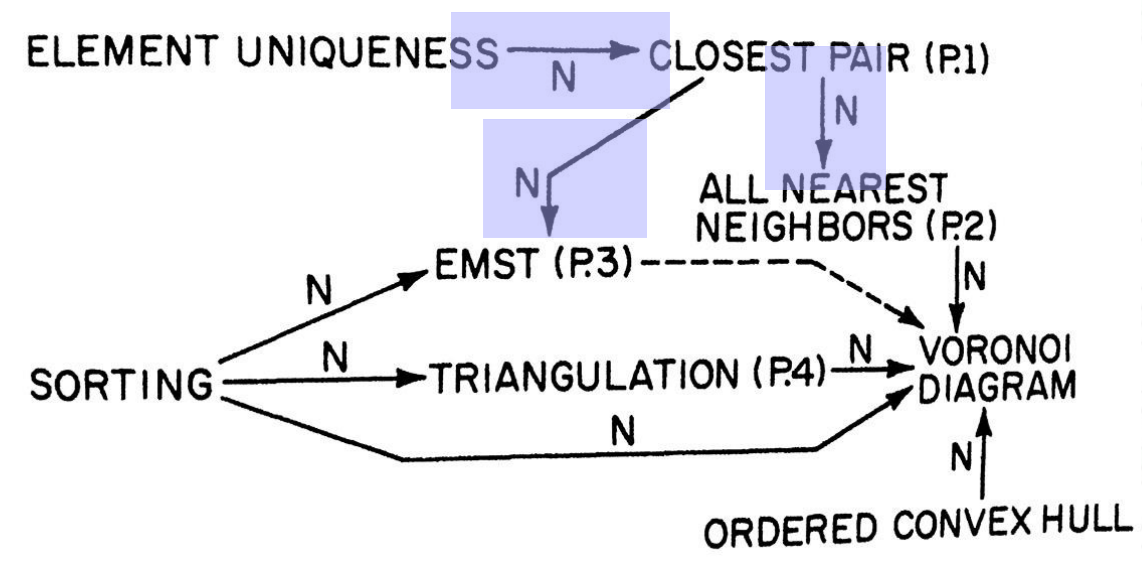


*Computational Geometry: An Introduction,*
Preparata & Shamos, 1985, Figure 5.30

- *Therefore they are also O(n log n) time and O(n) space.*

# Reduce Convex Hull to Voronoi Diagram

- Theorem: Voronoi polygon $V_i$ is unbounded if and only if Voronoi site *i* is on the convex hull of all sites. (proved in Preparata & Shamos)

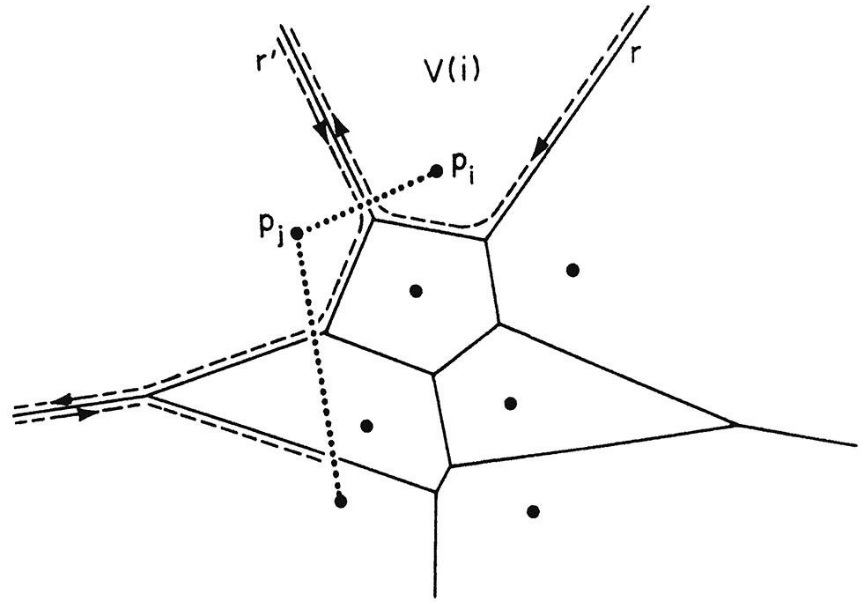


Figure 5.31 Construction of the convex hull from the Voronoi diagram.

*Computational Geometry: An Introduction,*
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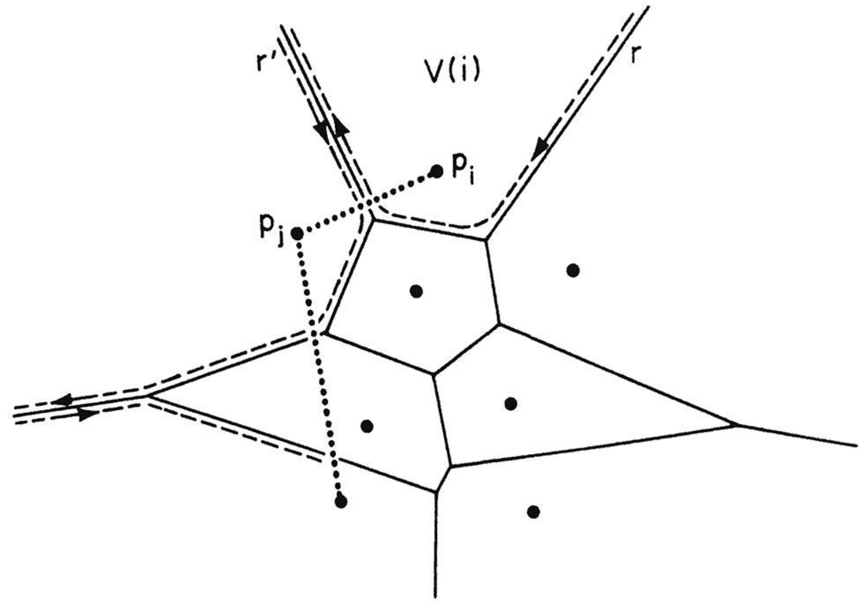
# Reduce Convex Hull to Voronoi Diagram

- Theorem: Voronoi polygon $V_i$ is unbounded if and only if Voronoi site *i* is on the convex hull of all sites. (proved in Preparata & Shamos)

- *O(n)* to convert **Voronoi Diagram** solution to Convex Hull:
    - Start with any unbounded cell
    - Walk edges clockwise to find adjacent unbounded cell
    - Voronoi sites will trace convex hull in counter-clockwise order

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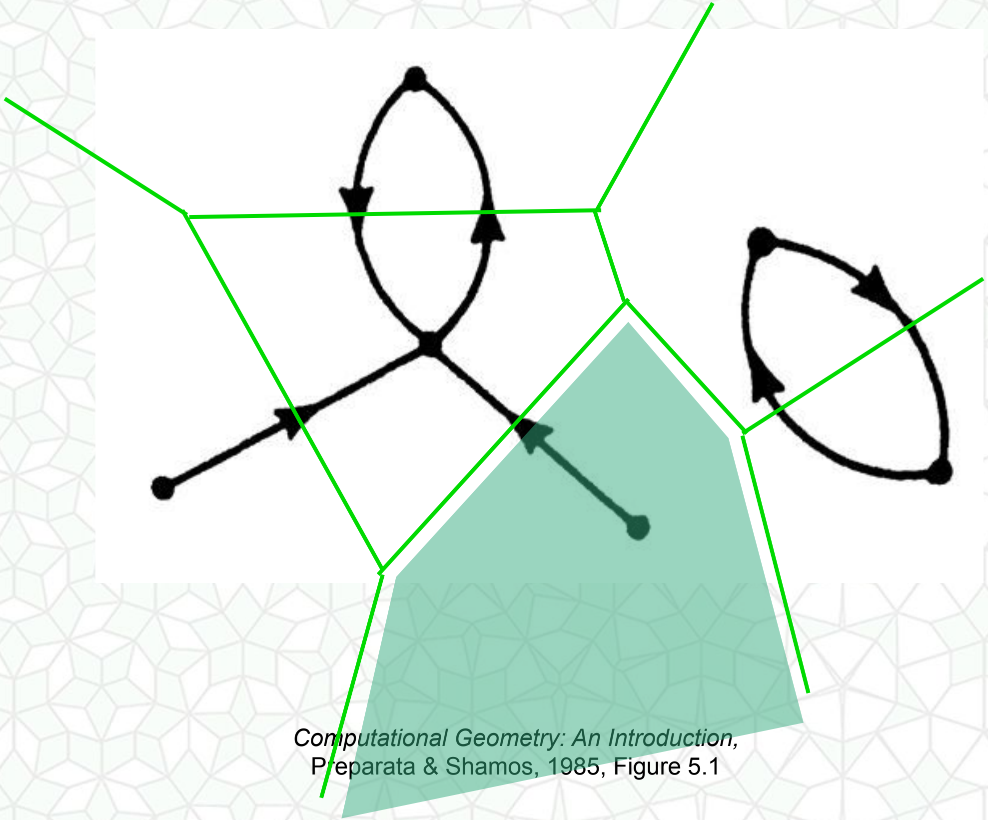
# Reduce All Nearest Neighbors to Voronoi Diagram

- For *n* Voronoi Sites
- By Euler's formula: $F + V = E + 2$
  - # of Voronoi edges ≤ $3n-6$
- Theorem: Every nearest neighbor in the set of Voronoi sites defines an edge of a Voronoi polygon.
  (proved in Preparata & Shamos)



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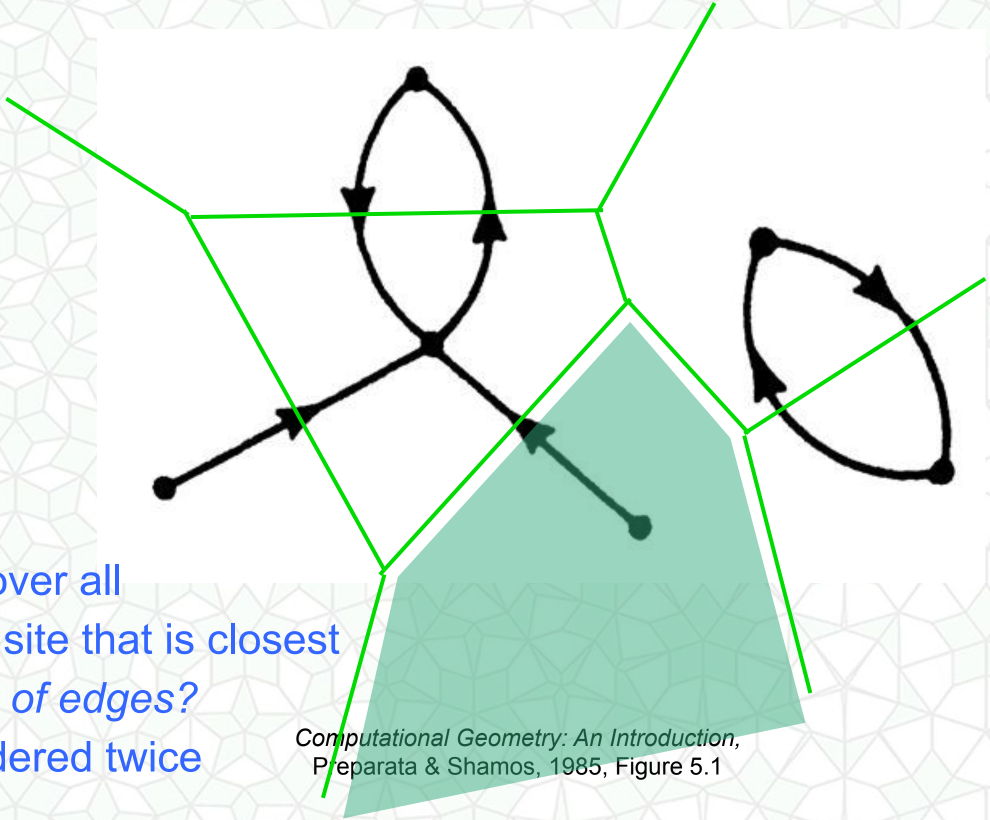
  (proved in Preparata & Shamos)

- $O(n)$ to convert **Voronoi Diagram** solution to All Nearest Neighbors
  - For every Voronoi polygon, loop over all Voronoi edges, & select adjacent site that is closest
  - *What if some cells have a huge # of edges?* Every Voronoi edge will be considered twice



*Computational Geometry: An Introduction*, Preparata & Shamos, 1985, Figure 5.1

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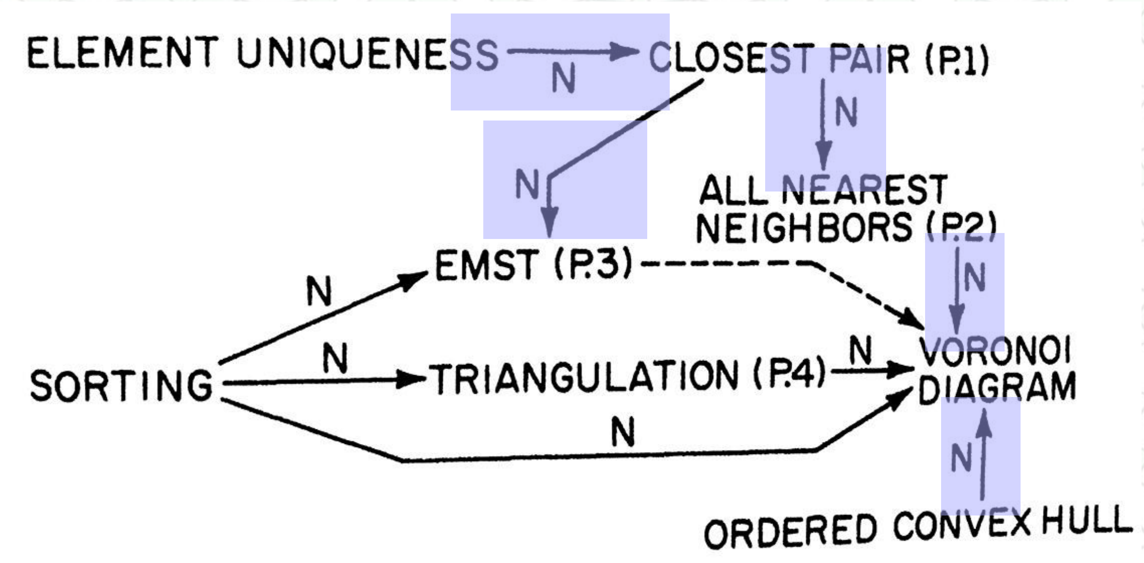
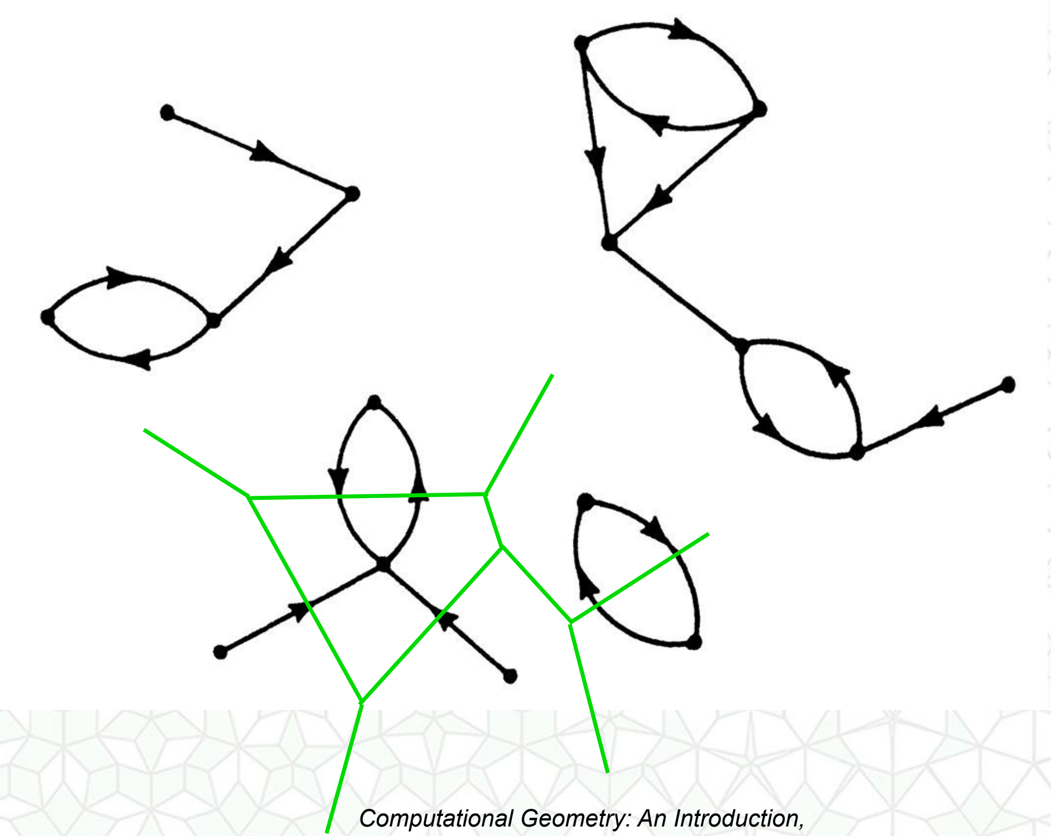



*Computational Geometry: An Introduction*, Preparata & Shamos, 1985, Figure 5.30

- *Therefore they are also O(n log n) time and O(n) space.*

# Reduce EMST to Voronoi Diagram

- All Nearest Neighbors will have one or more cycles reciprocal pair(s)
    - Remove one edge from each cycle
- All Nearest Neighbors may be disconnected
    - *A forest of trees*
- *Core challenge: Find and add shortest edge between disconnected trees in the forest…*



*Computational Geometry: An Introduction,*
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# Kruskal’s and Prim’s Algorithm for MST

*Did you cover this in Introduction to Algorithms and/or Graph Theory?*

- Kruskal’s - *O(E log E)*
  - maintain a set of trees
  - find the shortest edge that merges two trees
  - repeat until there is only a single tree
- Prim’s - *O(E + V log V)*
  - maintain *one* tree, and all unconnected vertices
  - find the shortest edge from the tree to an unconnected vertex
  - repeat until there are no unconnected vertices

*For a general (non-Euclidean) MST, we may have to consider O($n^2$) edges worst case.*
*For the Euclidean MST, we only need to consider the O(n) Voronoi/Delaunay edges.*

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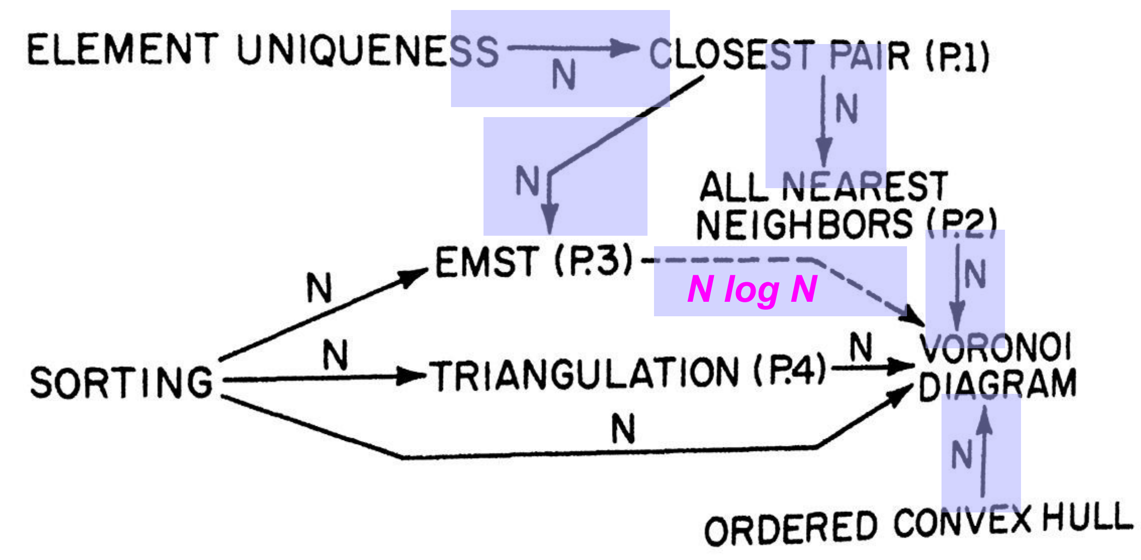


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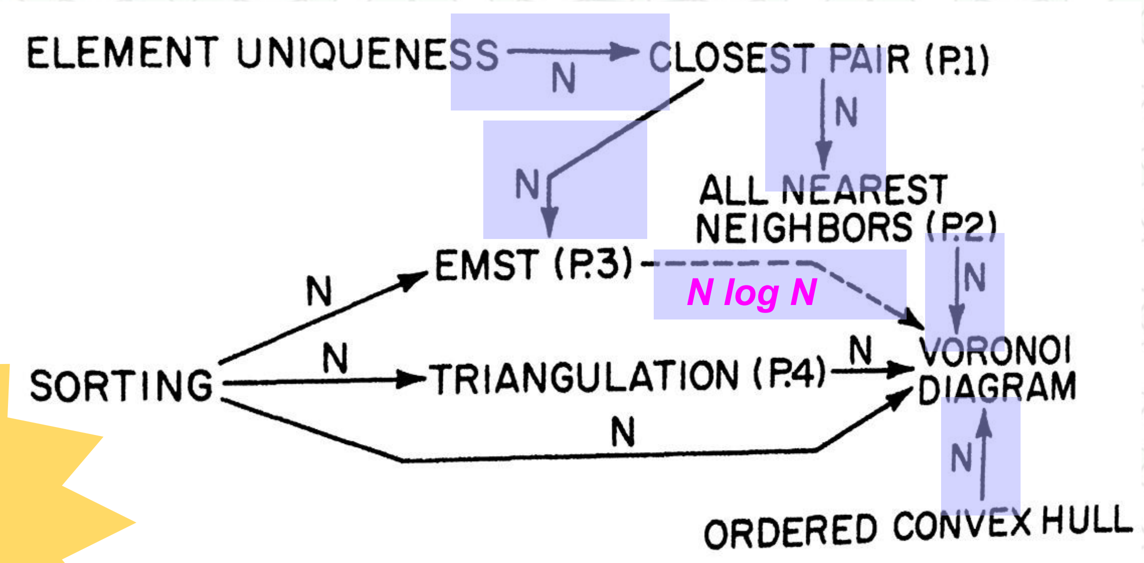
# Problems that Reduce to Voronoi Diagram

- We can compute the Voronoi Diagram of *n* points in *O(n log n)* time and *O(n)* space.
- The… oth…

Vo…

Because UNIQUENESS and SORTING are lower bound Ω(n log n)... VORONOI DIAGRAM is also Ω(n log n)

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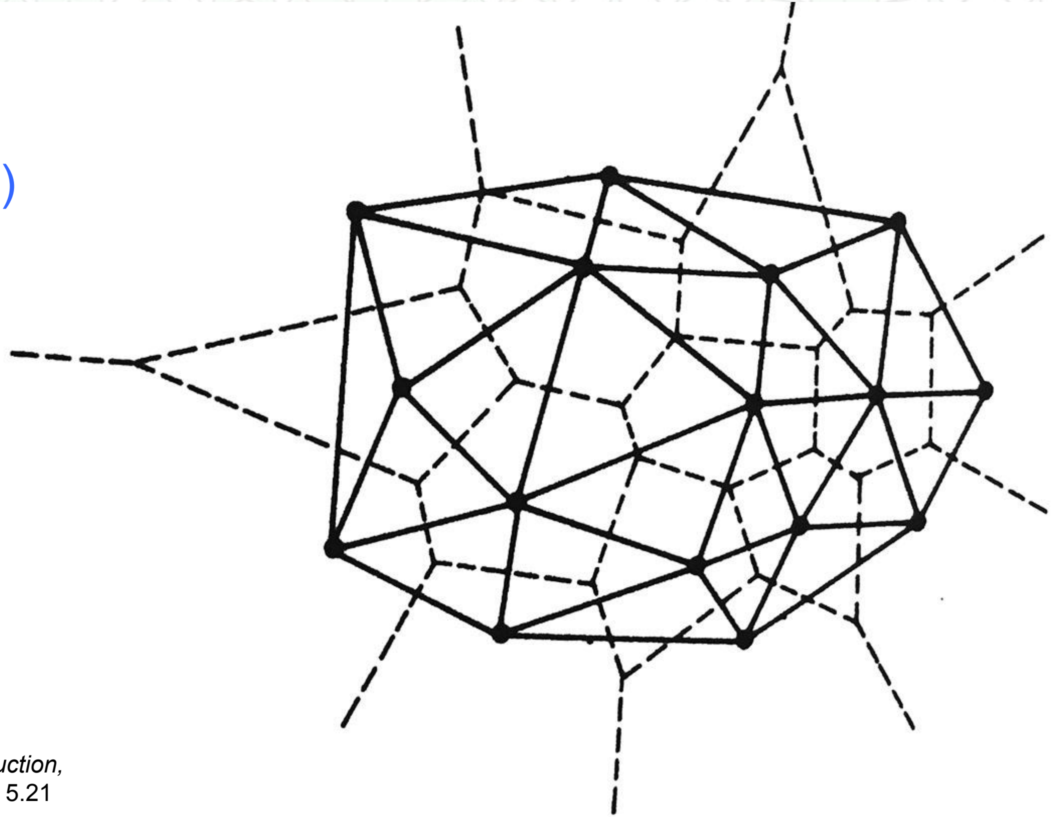
# Next Time: Delaunay Triangulation!

- The Voronoi Diagram (VD) *is the dual of the* Delaunay Triangulation (DT)
- Every Voronoi Site is a face in Voronoi Diagram and a vertex in the DT
- Every Voronoi Edge is an edge in the DT
- Every Voronoi Vertex is a triangle in the DT



*Computational Geometry: An Introduction,*
Preparata & Shamos, 1985, Figure 5.21