CSCI 4560/6560 Computational Geometry

`https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/F23/`

Lecture 14: Delaunay Triangulations, part 1

Outline for Today

- Homework 5 Questions?
- Final Project: Brainstorming Ideas & Partner Matching
- Last Time: Duality & Arrangements
- Motivation: Interpolation & Terrain Height Maps
- Graph vs. Planar Graph vs. Plane Graph
- Triangulation & Angle-Optimal Triangulation
- Thale’s Theorem & Inscribed Angle Theorem
- Brute Force Construction of Angle-Optimal Triangulation
- Duality: Voronoi Diagram & Delaunay Triangulation
- Next Time: More Delaunay Triangulations!

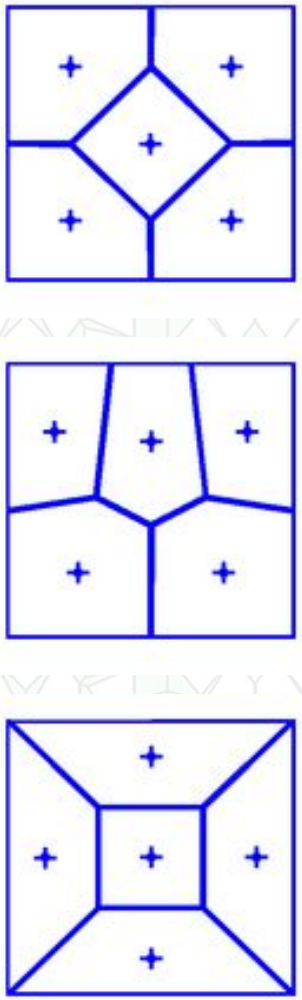
Centroidal Voronoi Diagram

- *What if we could place all of the grocery stores?*
- Where should we place the grocery stores so that they are centrally located for all of their customers?
- But if you change the position of the store, the closest store will change for some customers…

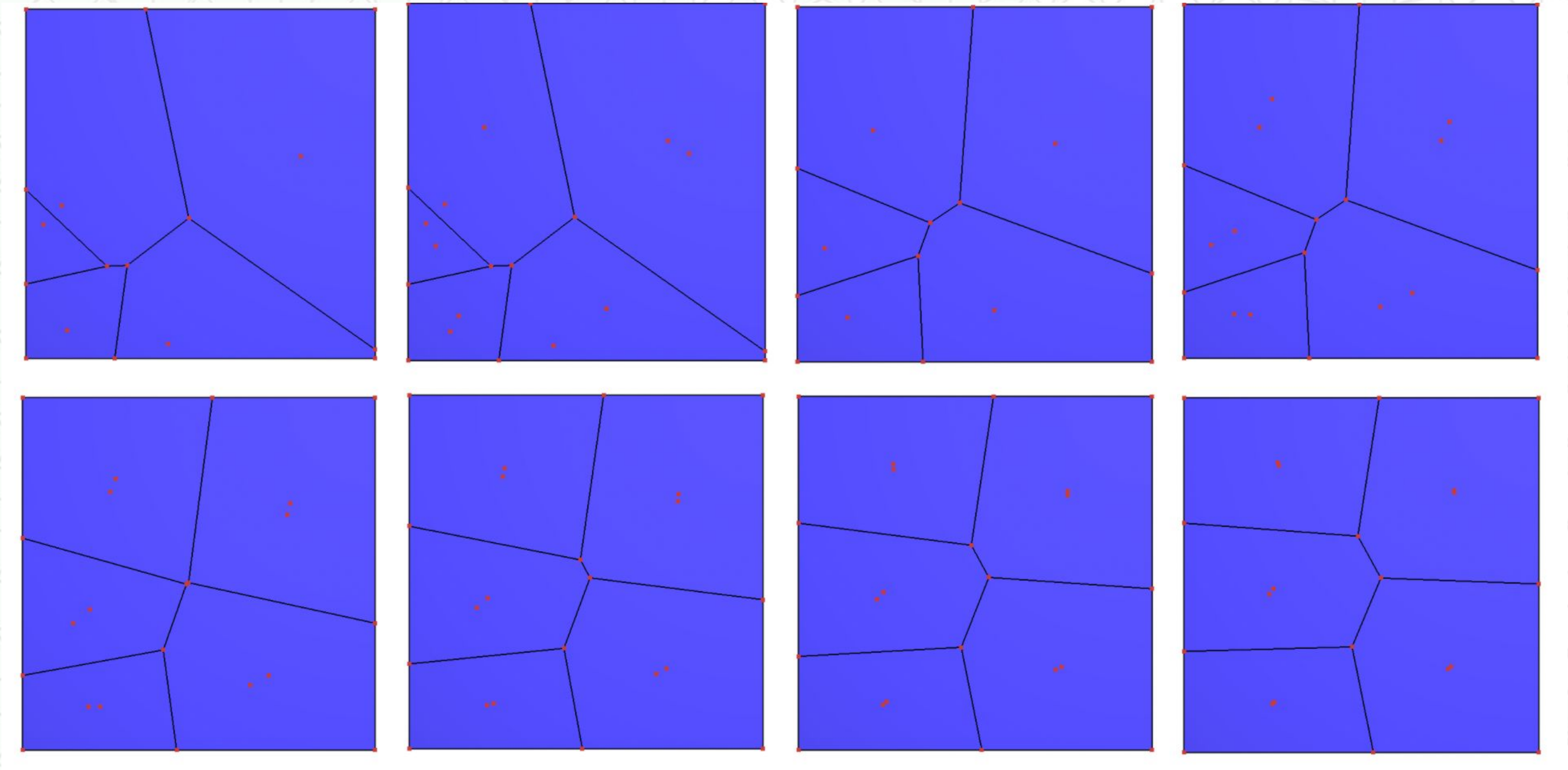
- Points are at the center of mass of their cell
- Constructed using k-means clustering / Lloyd's algorithm - an iterative relaxation algorithm

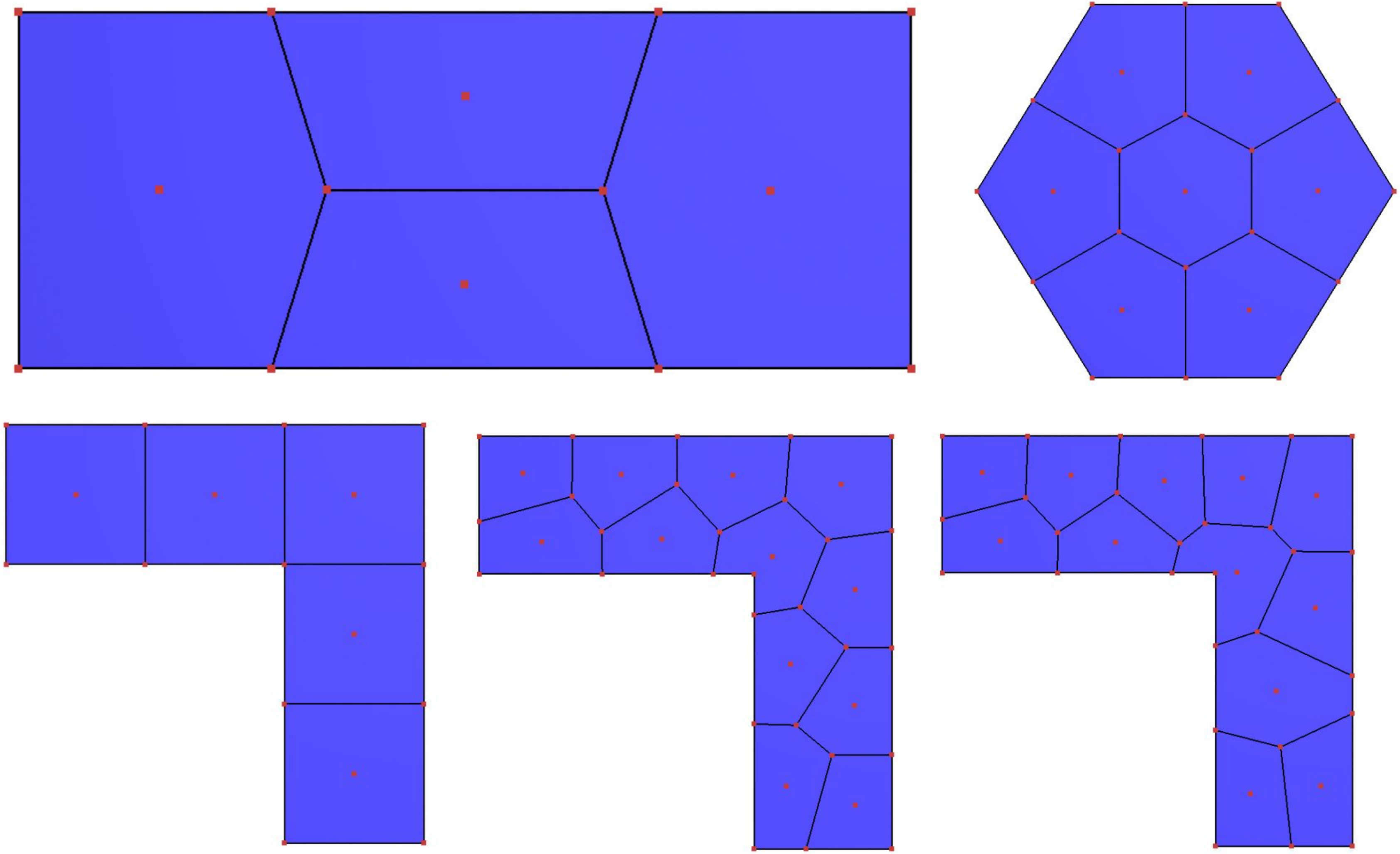
- *Note: May be multiple solutions!*

https://en.wikipedia.org/wiki/Centroidal_Voronoi_tessellation



Homework 5 Questions?





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Final Project Brainstorming

- Each student posts two different final project ideas on the forum
- For each idea:
  - Briefly describe the idea, your motivation for it, and an example of the potential result
  - What is the technical implementation/theory challenge?
- Can be an Individual Project or a Team of 2
- Have you already decided on one idea? Which one?
- Do you already have a partner? Who? (even if you have chosen an idea and/or a partner everyone must post 2 different ideas)
- Due Monday 10/23 @ 11:59pm

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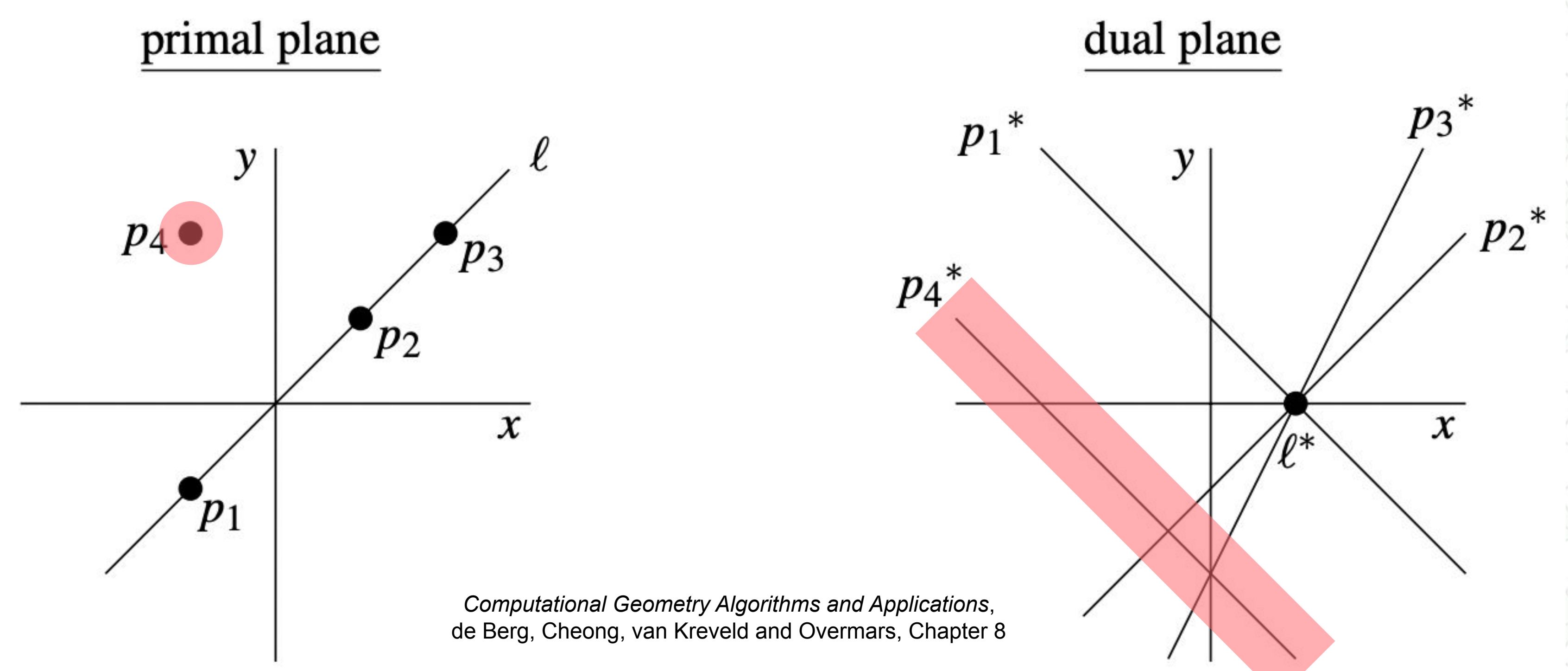
Duality: Points ↔ Lines

Point p: (p_x,p_y) in primal plane ↔ Line p^*: $y = \overset{\text{slope}}{p_x} x - \overset{\text{y-intercept}}{p_y}$ in dual plane



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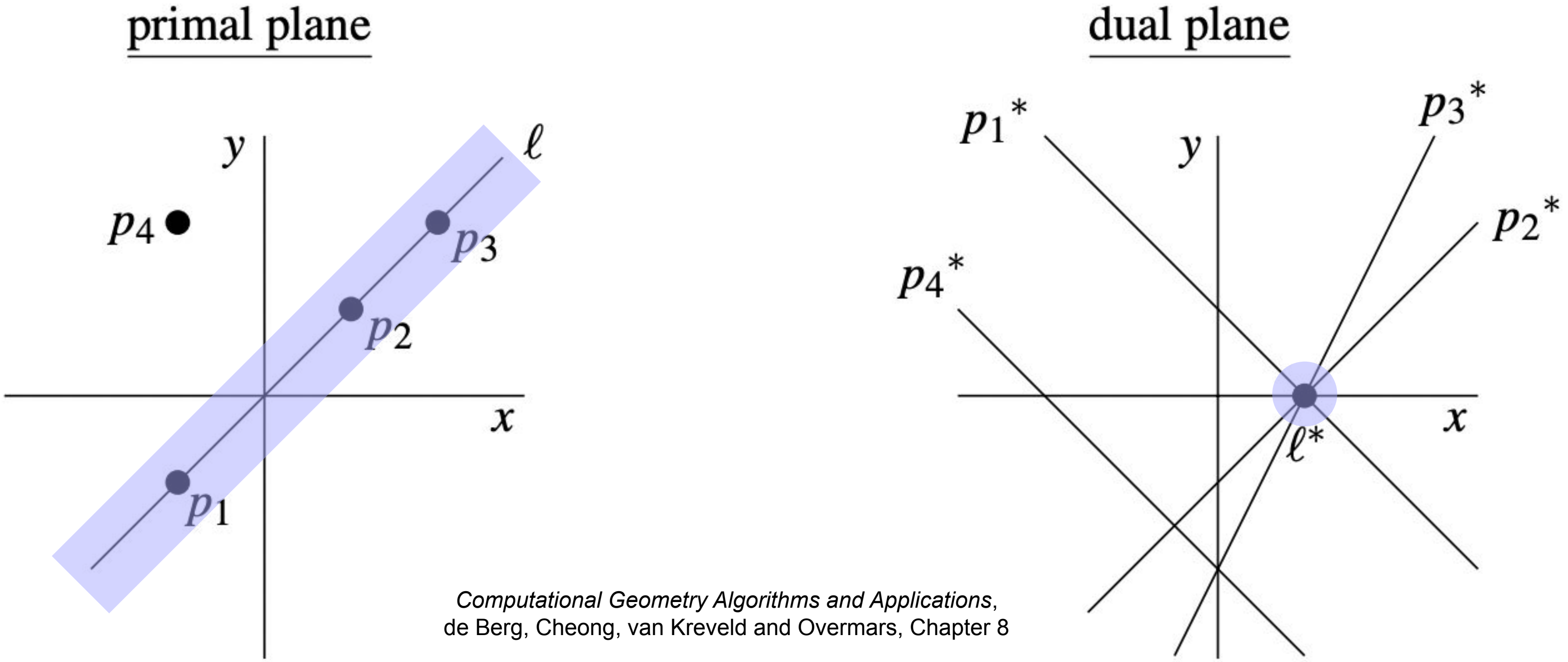
Duality: Points ↔ Lines

Line ℓ: $y = mx + b$ in primal plane ↔ Point ℓ^*: (m,b) in dual plane

slope **y-intercept**



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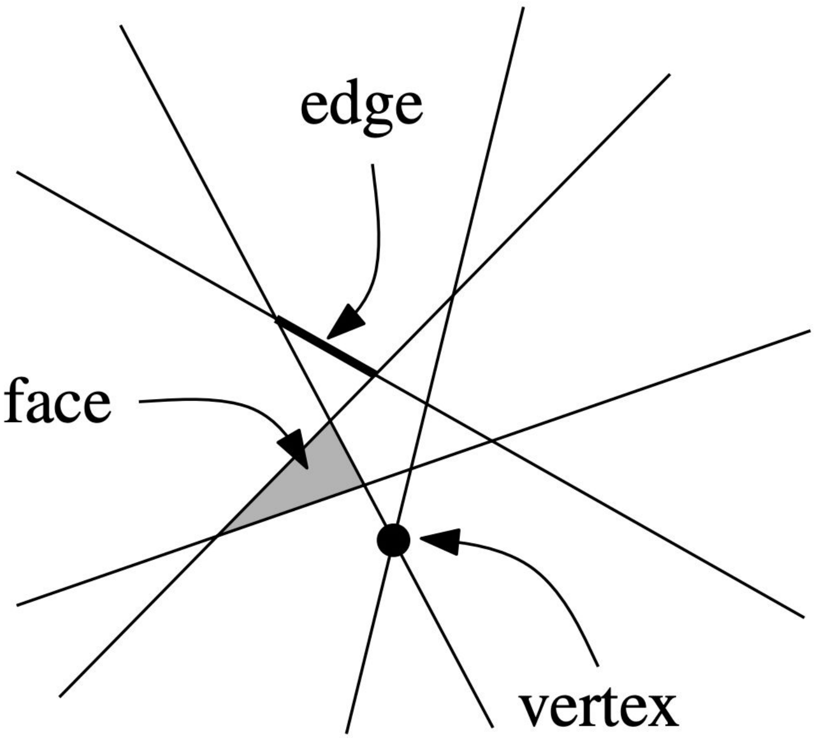
Complexity of an Arrangement of Lines

- A collection of *n* lines in the plane
- How many vertices?
  - $n * (n-1) / 2$
- How many edges?
  - n^2
- How many faces?
  - $n^2/2 + n/2 + 1$

Or fewer if not a *simple arrangement*

  - 3 or more lines intersect at a point, or
  - 2 or more lines are parallel



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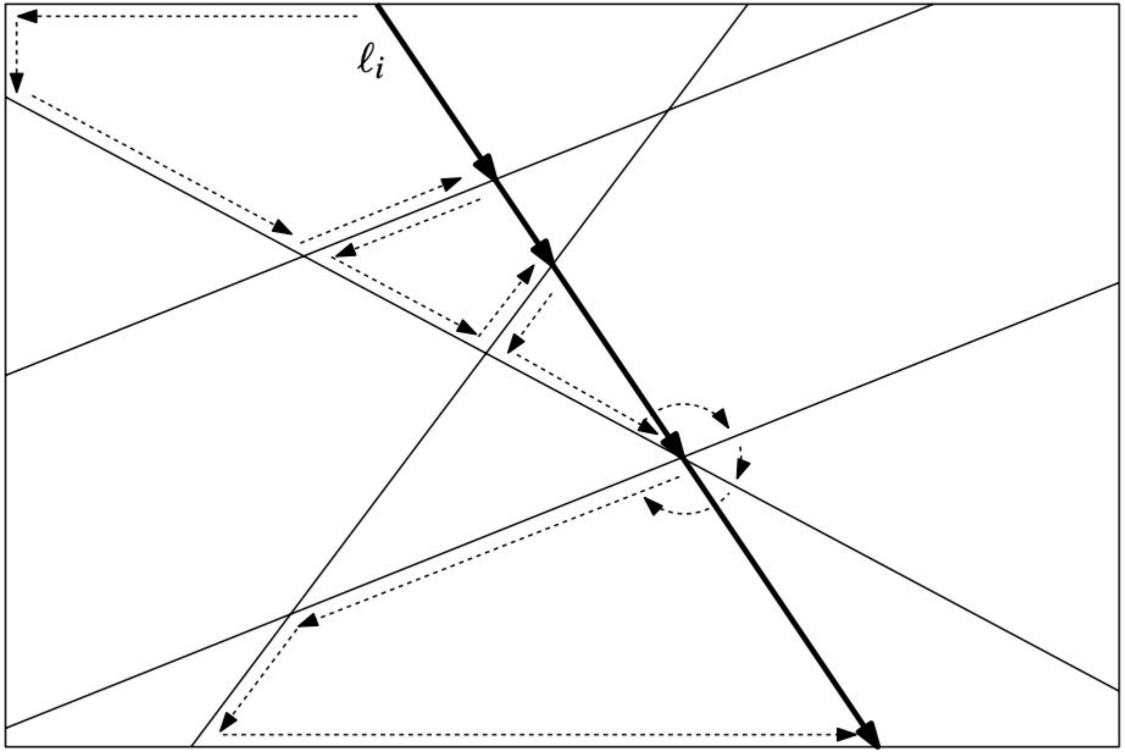
Construct an Arrangement

- Insert the lines one at a time
- Intersect the line with the bounding box
- Cut edge into two new edges
- Cut face into two new faces
- Walk the edges of the face to find the next face

Line arrangements (& their computation) are quadratic…



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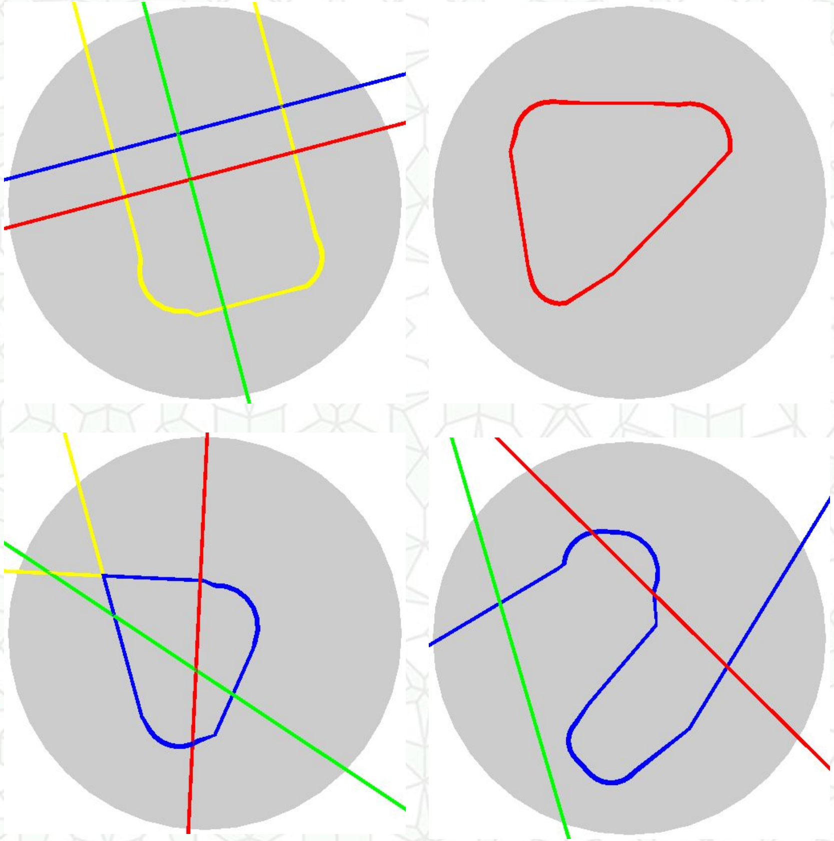
A Modified “Line Arrangement”

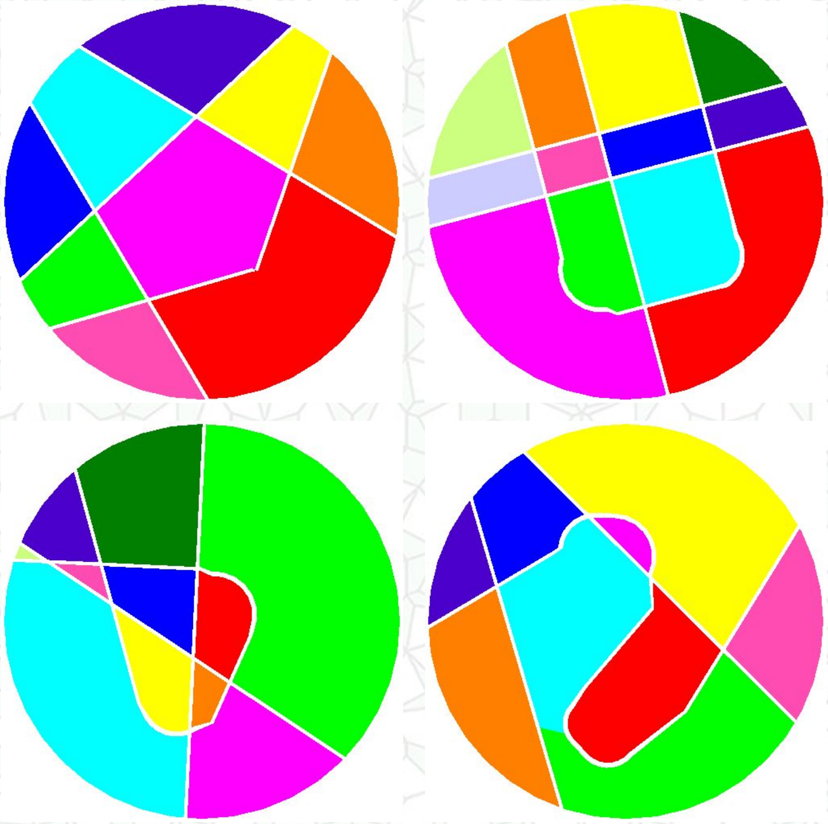
- In addition to infinite straight lines, a “wall chain” may:
    - Bend or be curved!
    - Be a closed loop!
    - Cross itself!
    - Cross another wall chain more than once!



Halfspace Zones & Enclosure

- For *n* wall chains
- *For simplicity, assume they are infinite straight lines*
- We will have $O(n^2)$ faces/cells in the arrangement
- Each face/cell can be "interior" or "exterior"

- $\rightarrow 2^{O(n^2)}$ *possible buildings*
  Not feasible to check all of them!!



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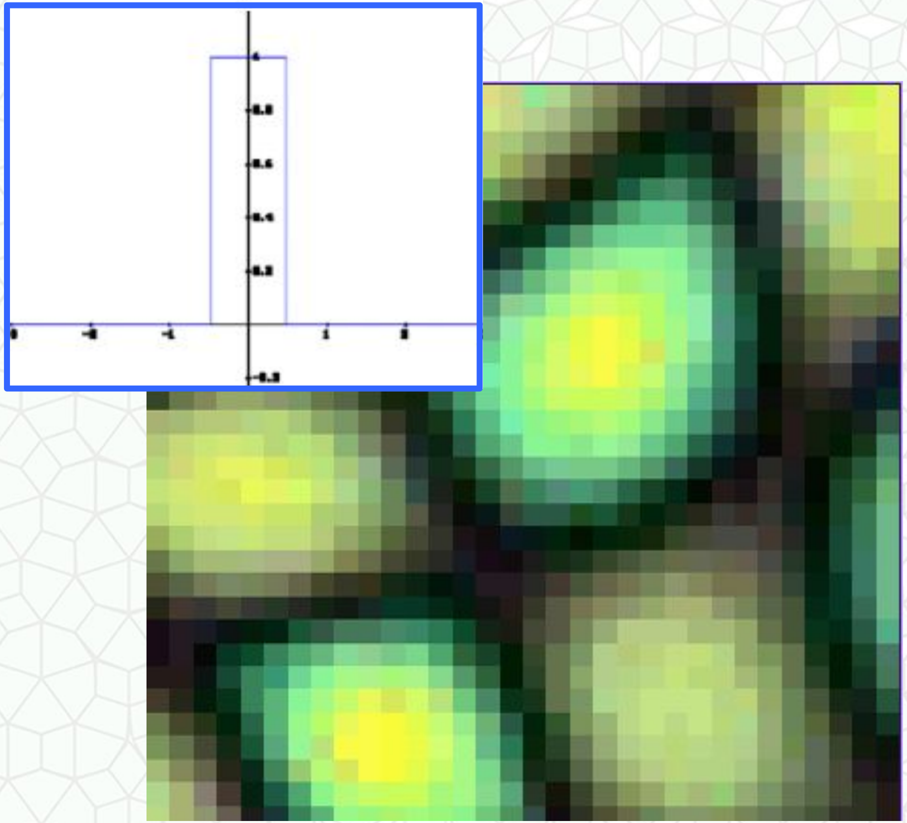
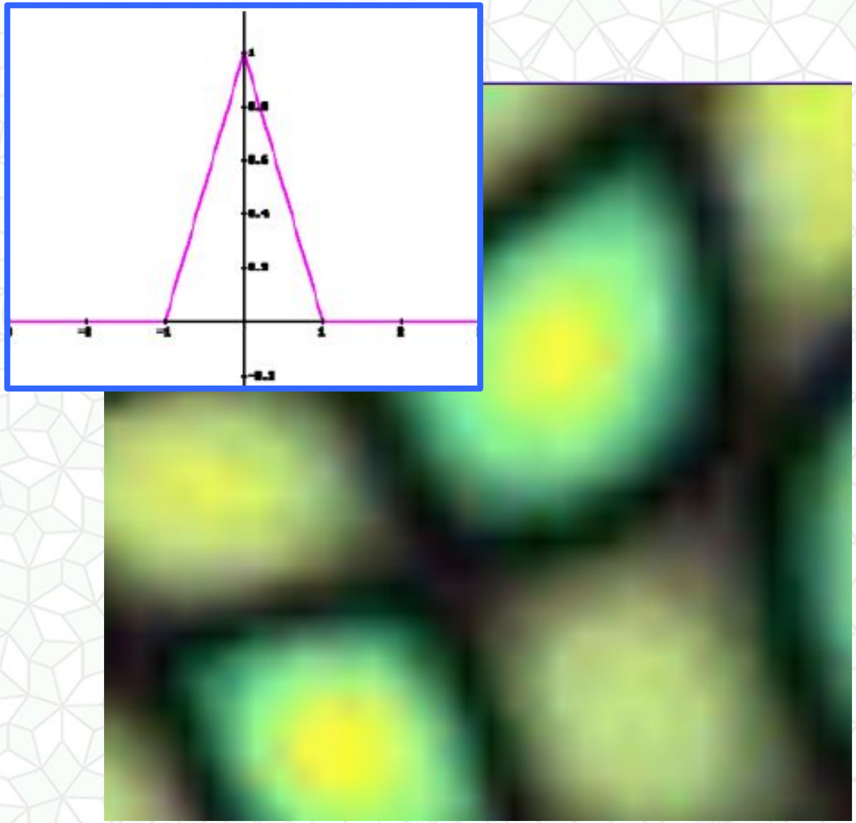
Nearest Neighbor vs. Bi-Linear Interpolation





Bilinear Interpolation

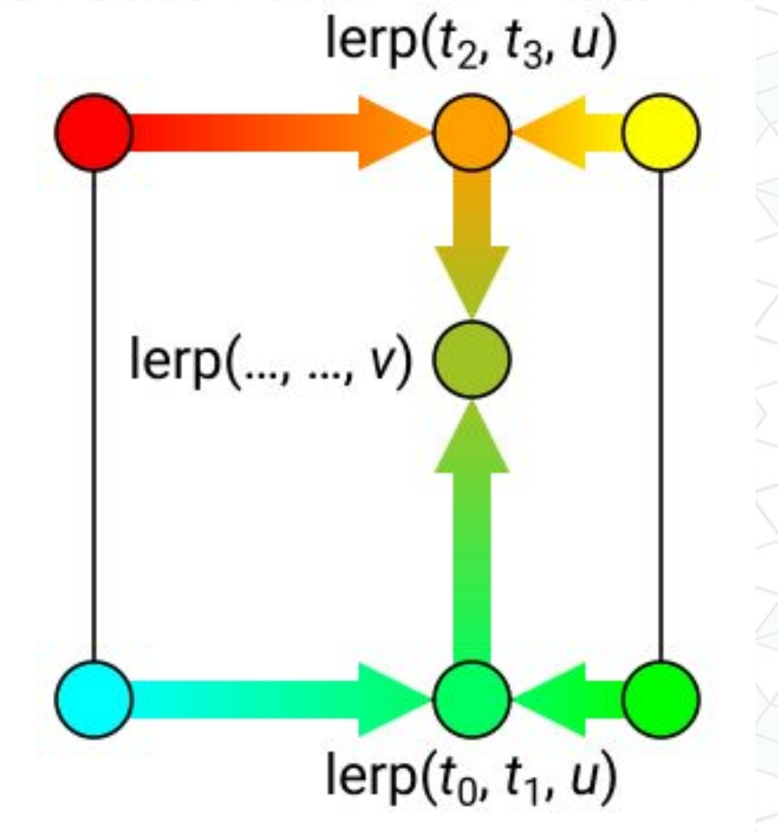
- Think about one axis at a time (doesn't matter which axis)
- Calculate *u*, the fraction of the distance along the horizontal axis, ***e.g., u=0.65***
- Then calculate the top & bottom averages:

*orange = (1-u)*red + u*yellow*

*bluegreen = (1-u)*cyan + u*green*

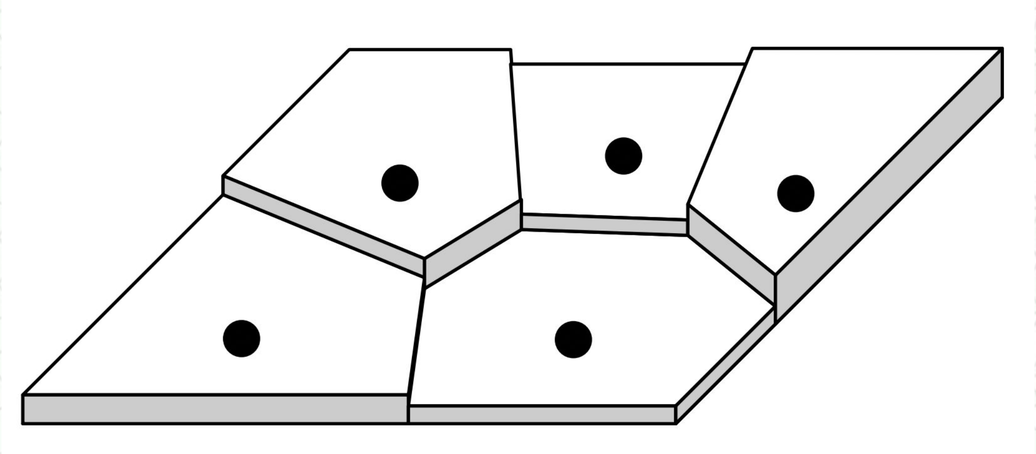
- Calculate *v*, the fraction of the distance along the vertical axis, ***e.g., v=0.6***
- Then calculate the final average:

*pukegreen = (1-v)*bluegreen + v*orange*

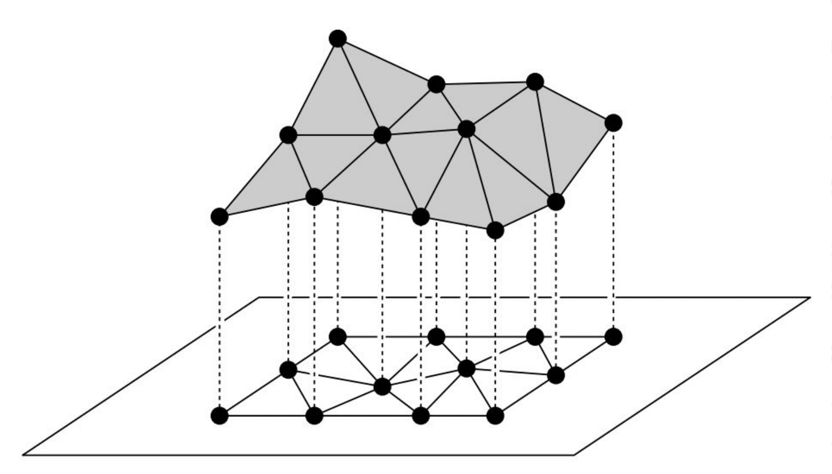


http://reedbeta.com/blog/quadrilateral-interpolation-part-2/

Motivation: Terrain Height Map



Nearest Neighbor



Bi-Linear Interpolation

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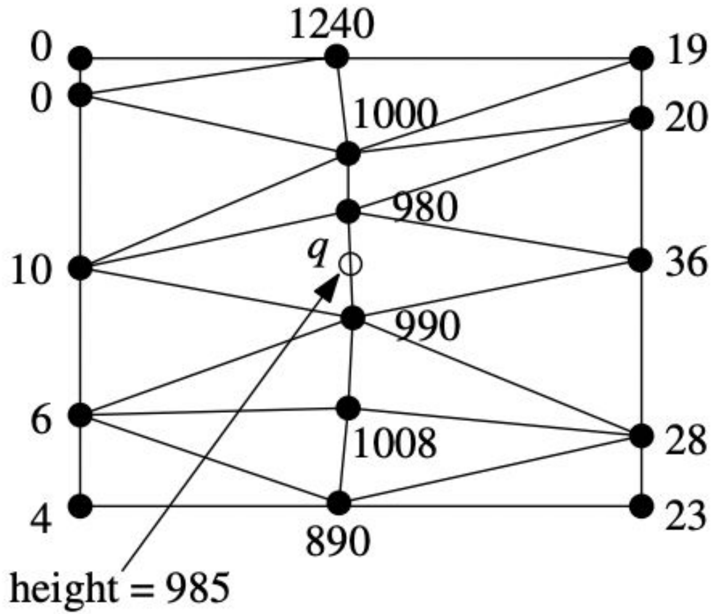
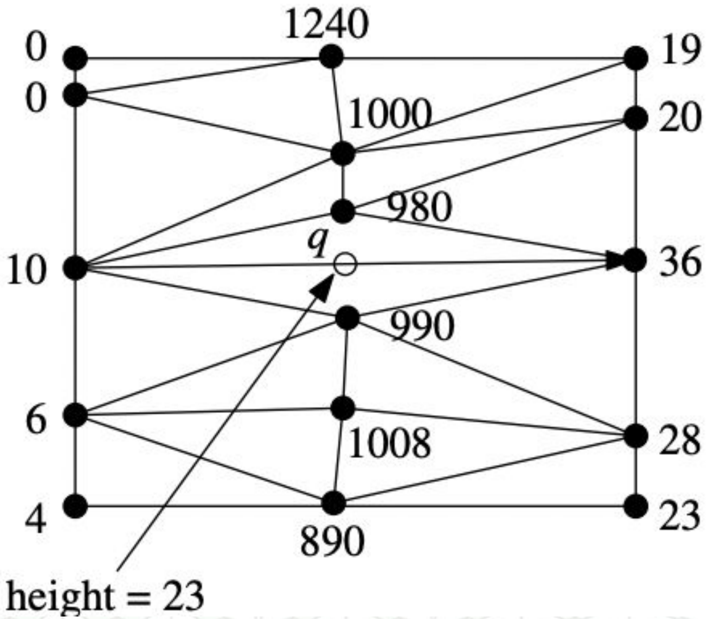
Not all Triangulations are the same!

this triangulation is better



this triangulation is worse



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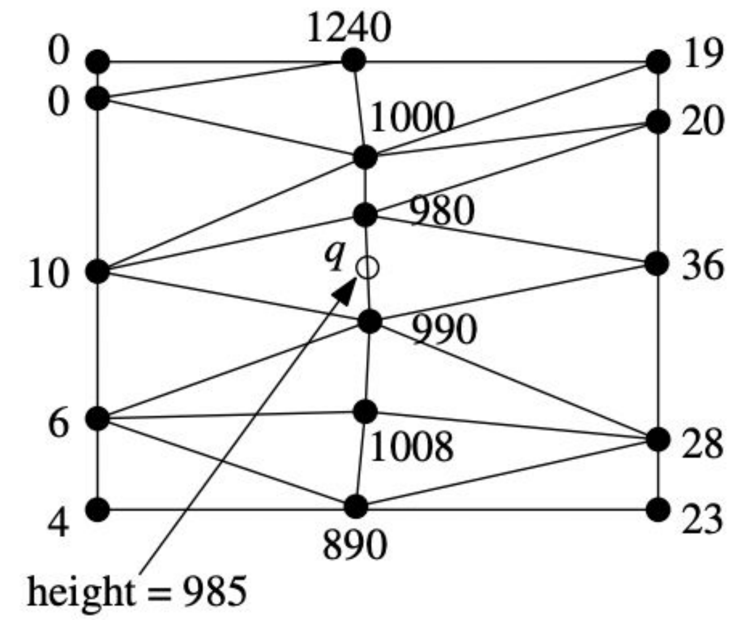
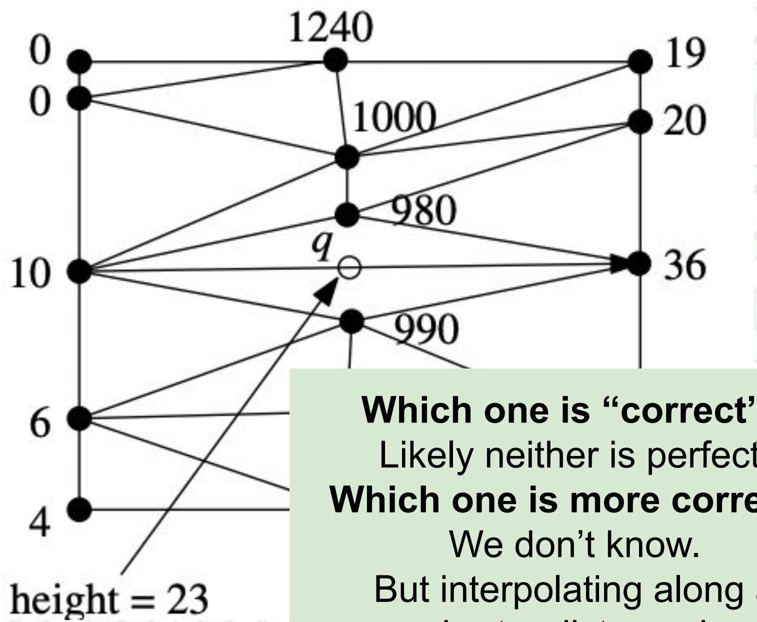


this triangulation is worse



Which one is "correct"?
Likely neither is perfect.
Which one is more correct?
We don't know.
But interpolating along a shorter distance is probably more accurate.

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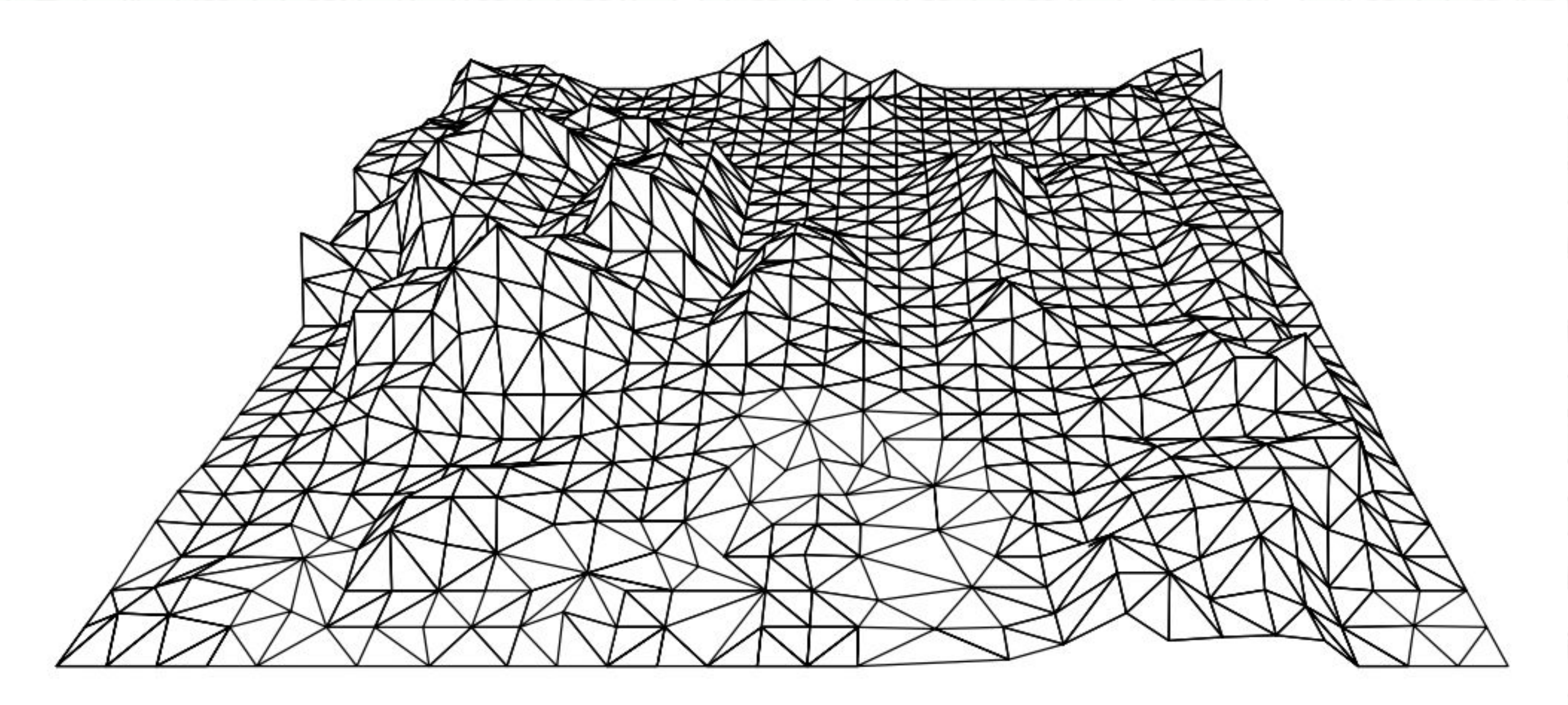
Motivation: Terrain Height Map



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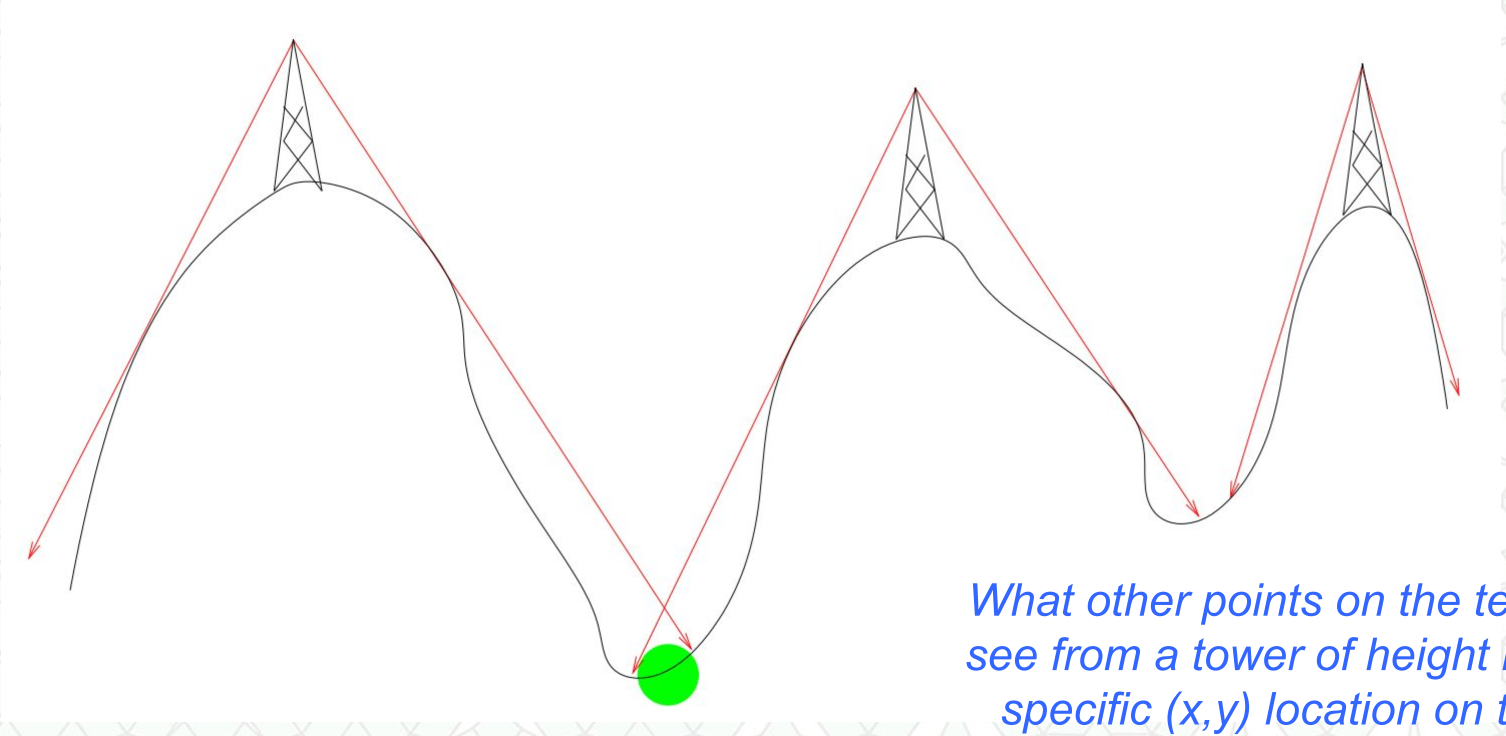
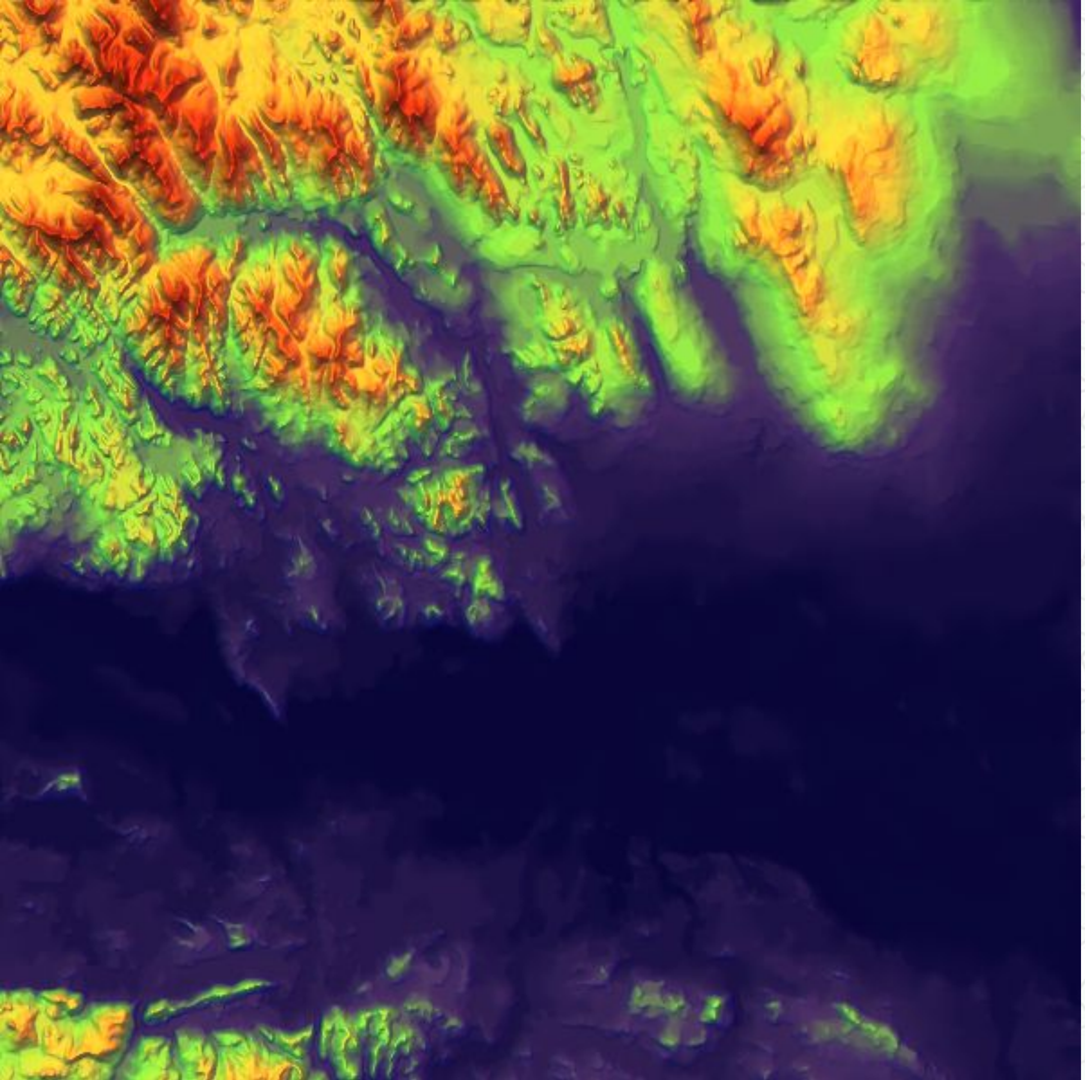
“Siting Observers on Terrain”

W. Randolph Franklin, RPI ECSE, 2004



What other points on the terrain can we see from a tower of height h placed at a specific (x,y) location on the terrain?



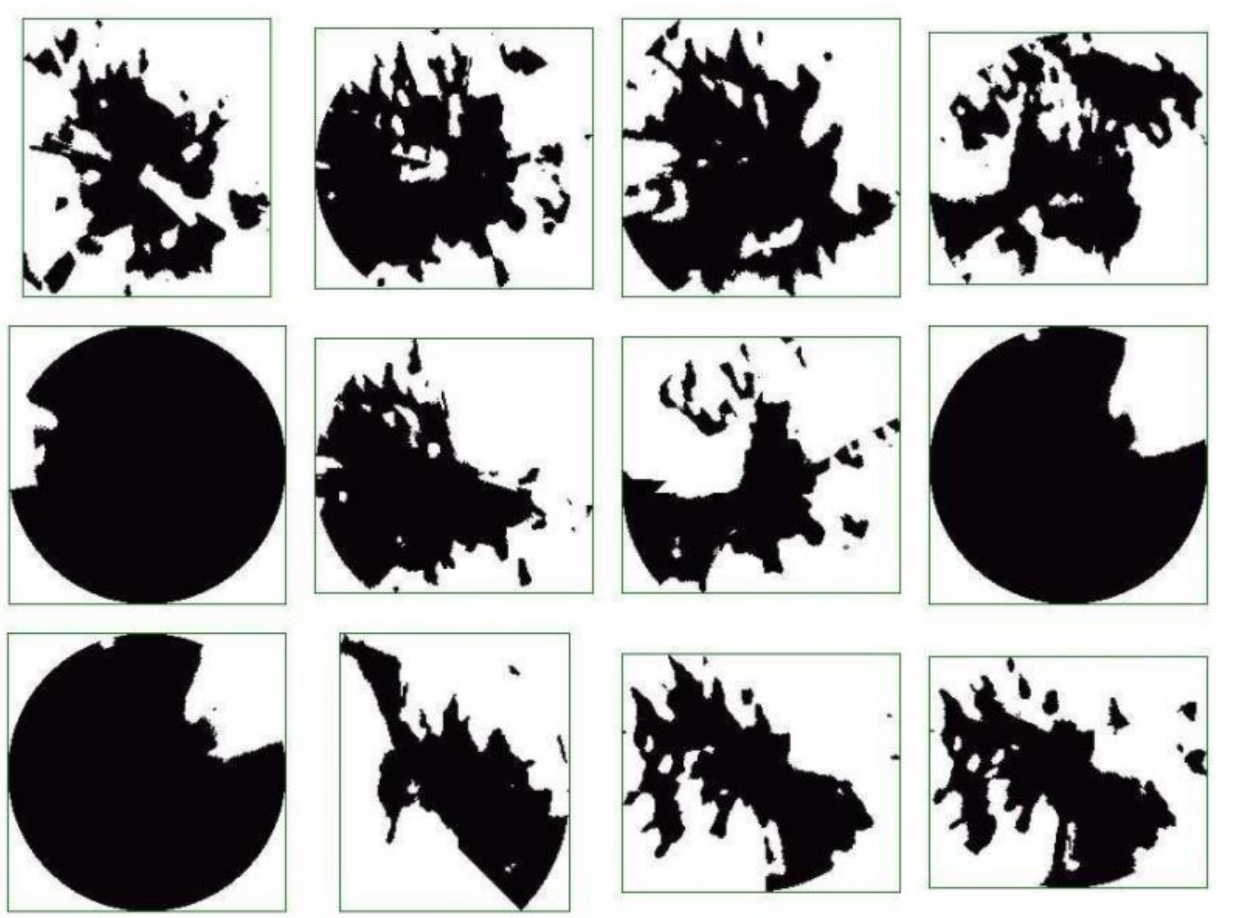
Terrain Height Visualization

red = higher elevations

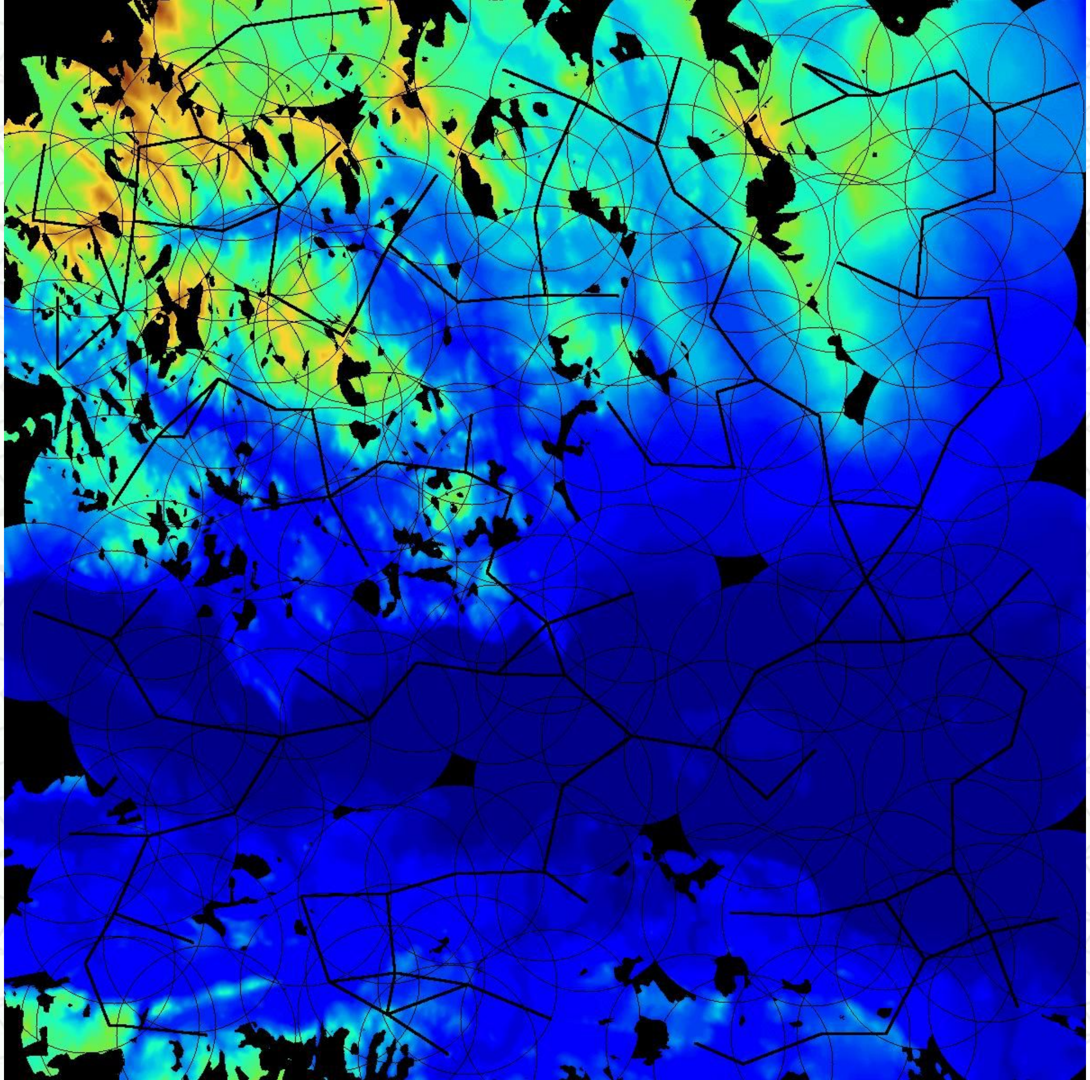
blue = lower elevations

Slide from W. Randolph Franklin

- Observers have a specified maximum straight line sight distance
- Some observer placements see more (black)
- Regions that are white are occluded or too far from observer



Slide from W. Randolph Franklin



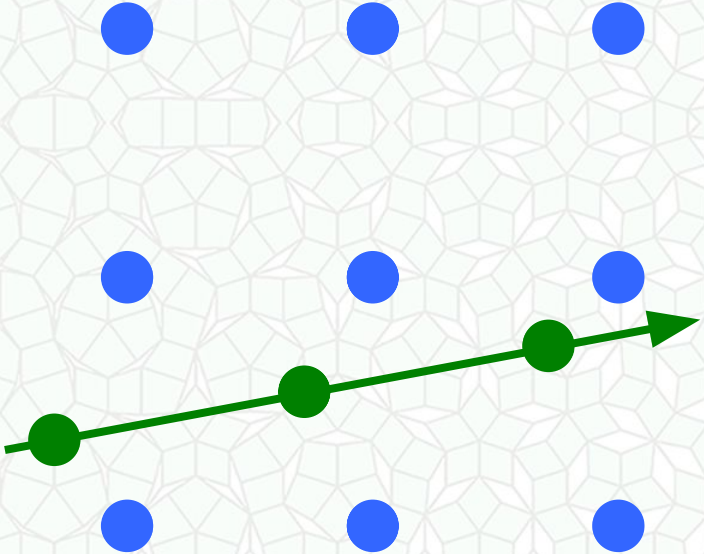
- Place *k* observers to maximize coverage
- Additional constraint: The observers must also be connected by line-of-sight

Slide from W. Randolph Franklin

Incorrect Interpolation

Regular grid of height samples

Query for occlusions along sight line



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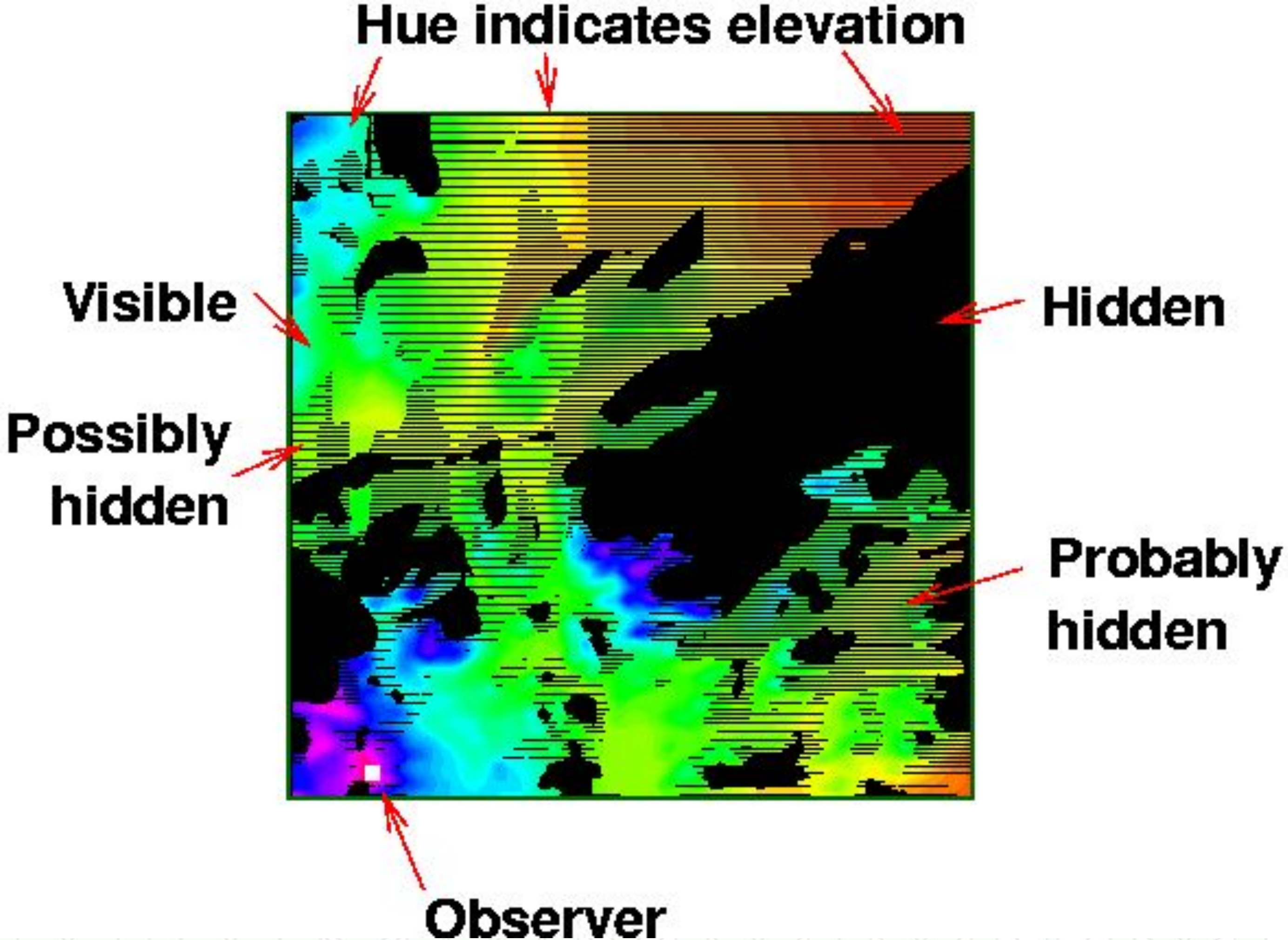

Possibly Hidden /
Probably Hidden:
If height is changed
by epsilon, the
visibility flips!

The visibility of one half of the points in uncertain!

Slide from W. Randolph Franklin

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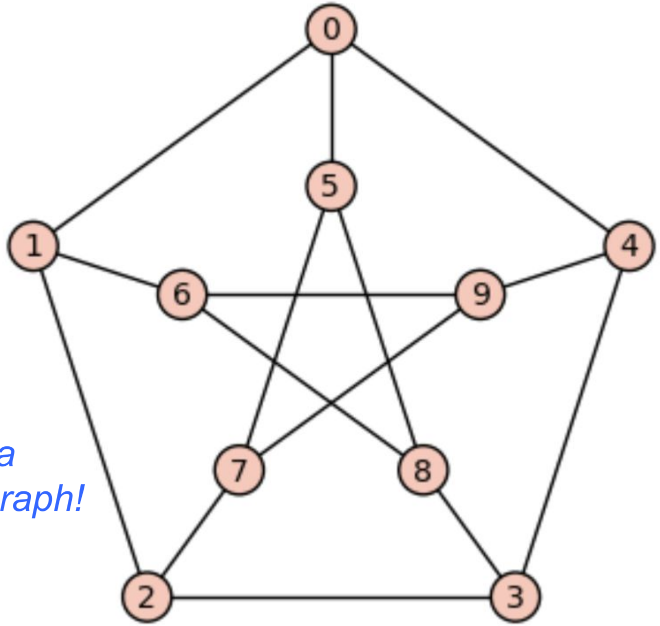
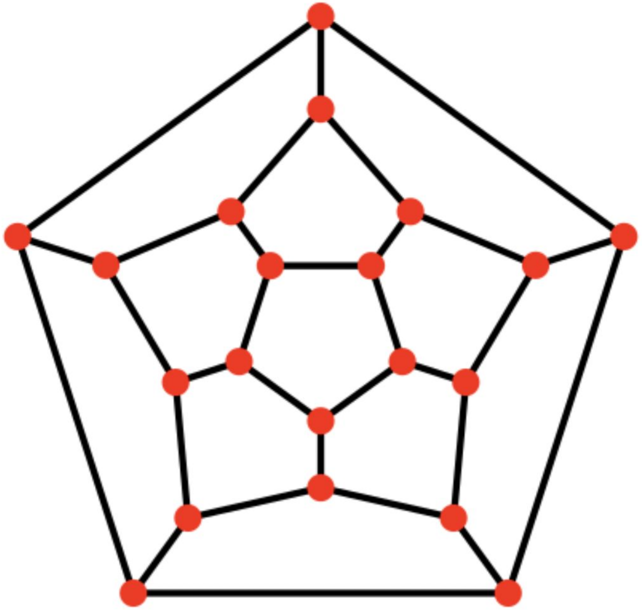
Definition: Planar Graph vs. Plane Graph

Planar Graph: A graph that can be arranged/drawn in 2D without edge crossings

Plane Graph: An embedding, a 2D drawing of a graph without edge crossings



not a planar graph!



https://en.wikipedia.org/wiki/Planar_graph

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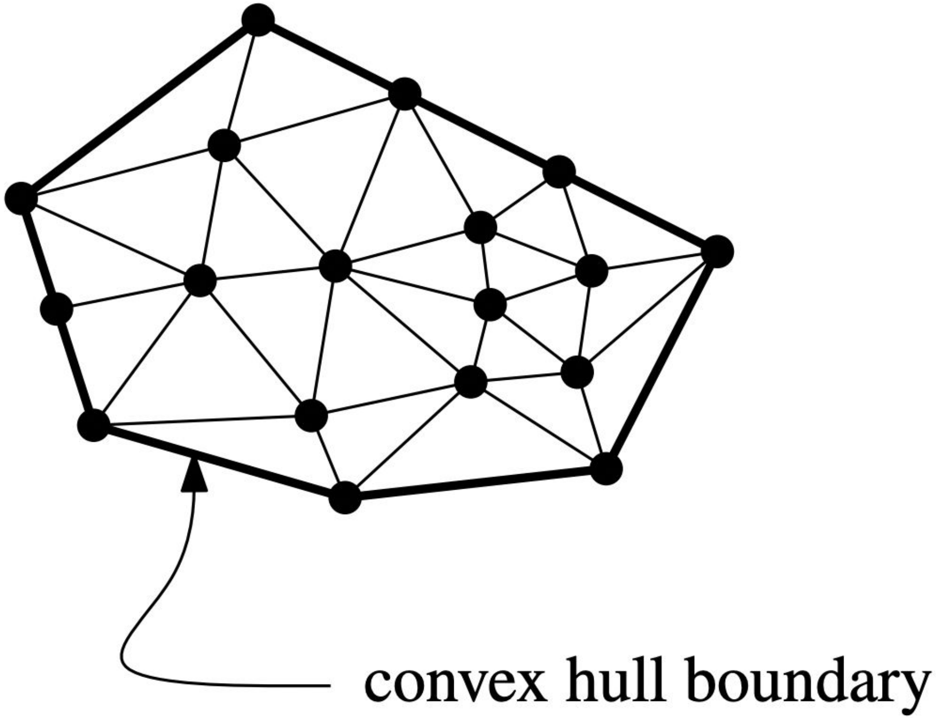
Definition: Point Set Triangulation

- A triangulation is a *Maximal Planar Subdivision* of a vertex set
- No edge connecting two vertices can be added without destroying planarity
- Every face will have 3 vertices



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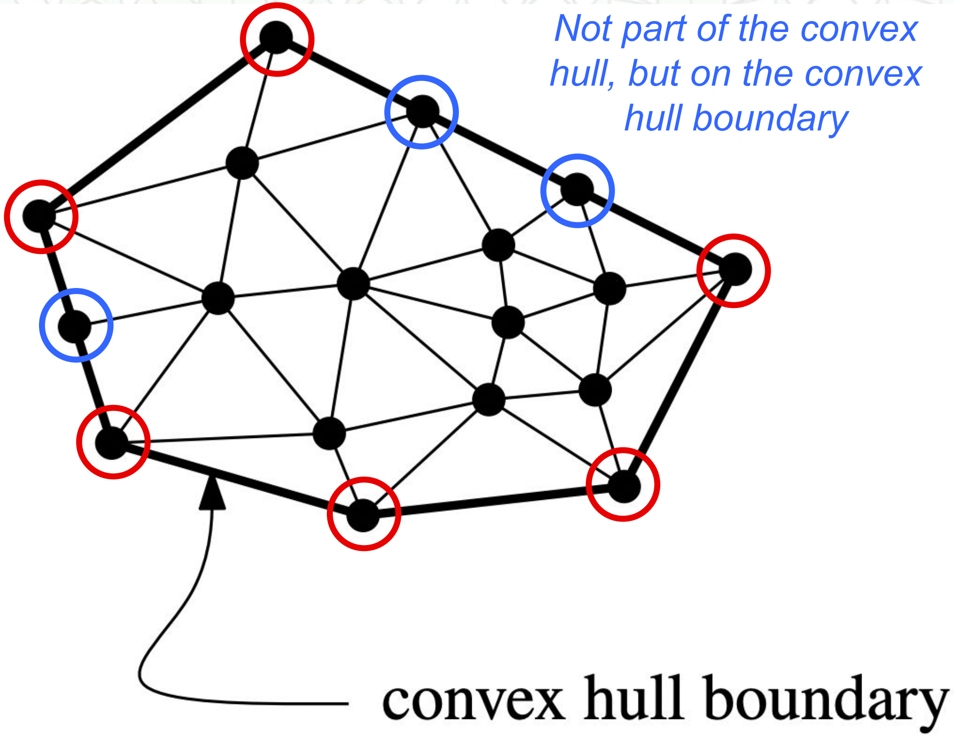
Face/Edge/Vertex Count of a Triangulation

- For $n = 18$ vertices
- With $k = 9$ vertices on the convex hull boundary
- The *unbounded face* has all of the vertices on the convex hull boundary
- Euler's formula: $n - n_e + n_f = 2$
- Every bounded face has 3 edges (each shared with another face)
    - $2 * n_e = 3 * (n_f - 1) + k$
    - *# edges:* $n_e = 3n - k - 3 = 42$
    - *# triangles:* $n_f - 1 = 2n - 2 - k = 25$



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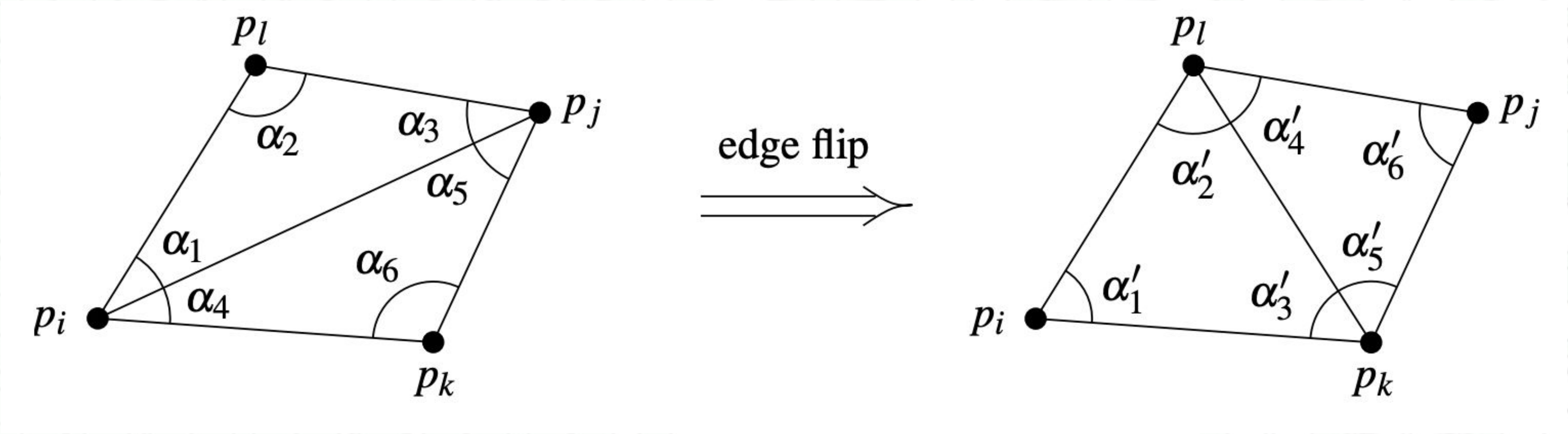
Definition: Angle-Optimal Triangulation

- We want to maximize the smallest angle
- Consider replacing each edge between two triangles with the edge connecting the other vertices of those two triangles (only possible if the combined area of the two triangles is convex)



- Edge p_ip_j is said to be *illegal* if: $\min_{1 \leqslant i \leqslant 6} \alpha_i < \min_{1 \leqslant i \leqslant 6} \alpha'_i$

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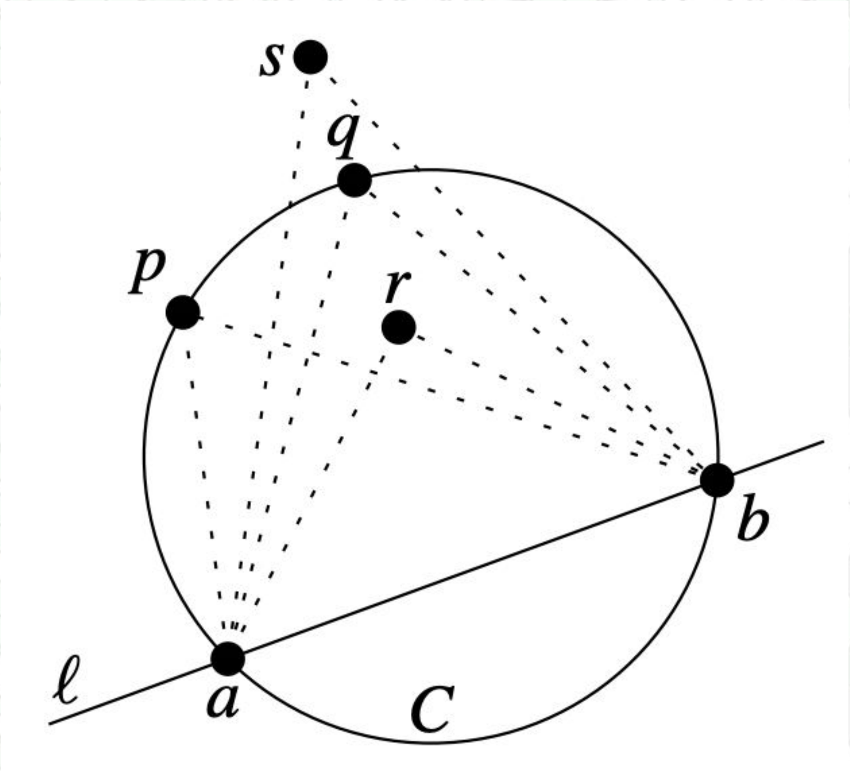
Relationship: Angles & Circumscribed Circle

Thales Theorem: Let *C* be a circle, ℓ a line intersecting *C* in points *a* and *b*, and *p*, *q*, *r*, and *s* points lying on the same side of ℓ. Suppose that *p* and *q* lie on *C*, that *r* lies inside *C*, and that *s* lies outside *C*. Then

$$\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb$$



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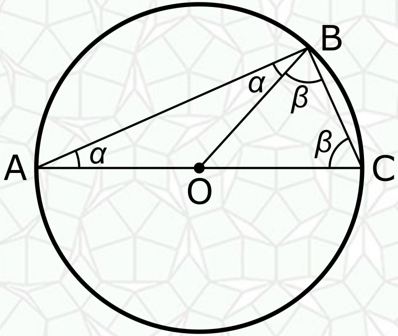
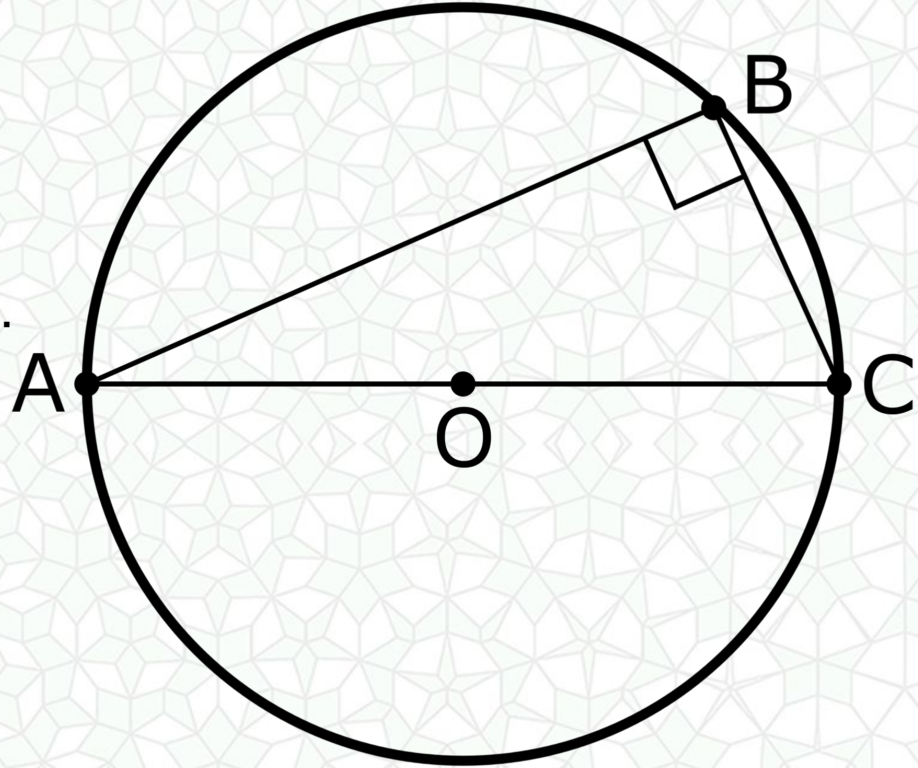
Thale’s Theorem

If A, B, and C lie on a circle, and AC is a diameter, then the angle at B (the angle ABC) is a right angle.

Dissection proof: The sum of the angles of a triangle is 180°

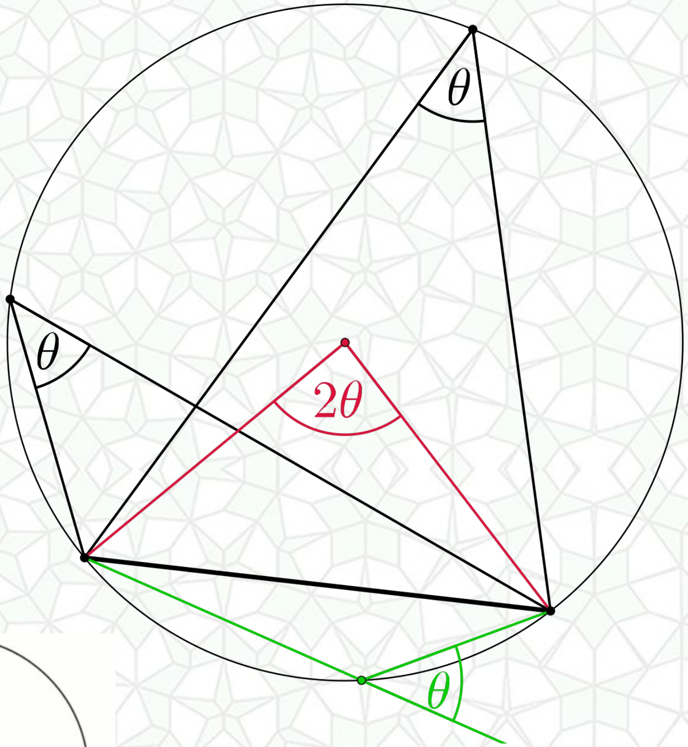


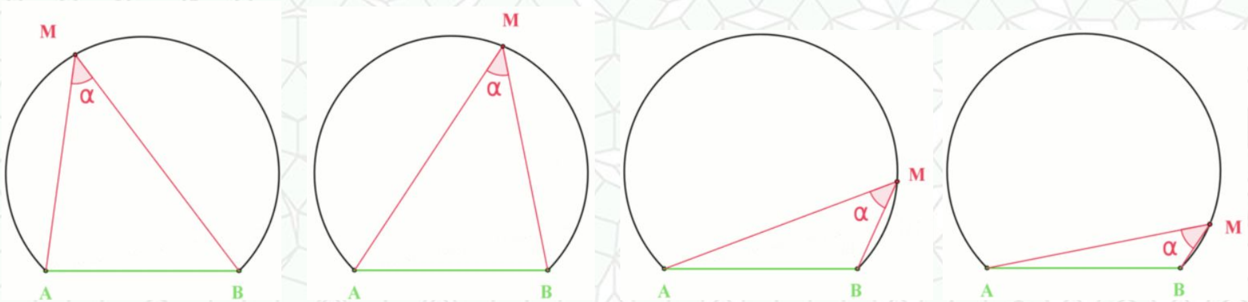


https://en.wikipedia.org/wiki/Thales%27s_theorem

Inscribed Angle Theorem

The inscribed angle θ is half of the central angle 2θ that subtends the same arc on the circle. The angle θ does not change as its vertex is moved around on the circle.



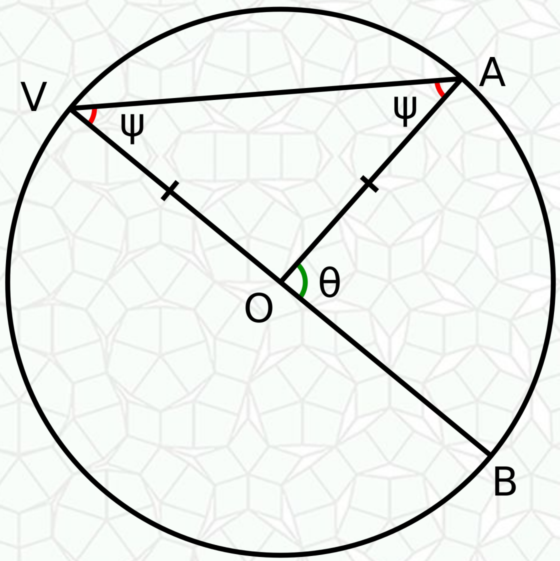


https://en.wikipedia.org/wiki/Inscribed_angle#Theorem

Inscribed Angle Theorem

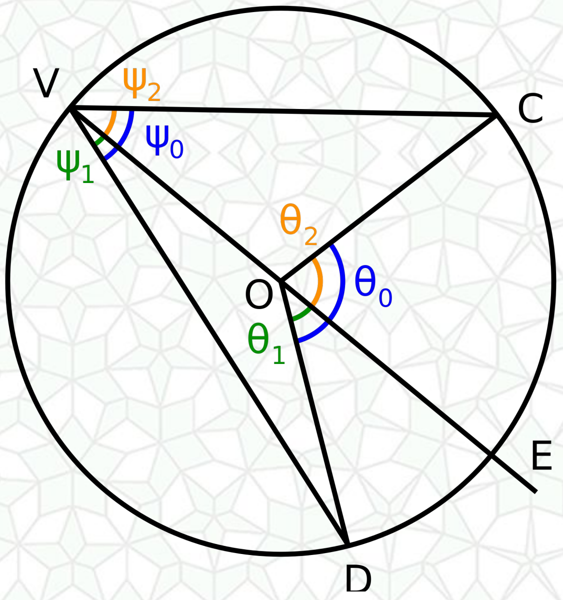
Proof:

Where 1 chord is a diameter



Proof:

General Case

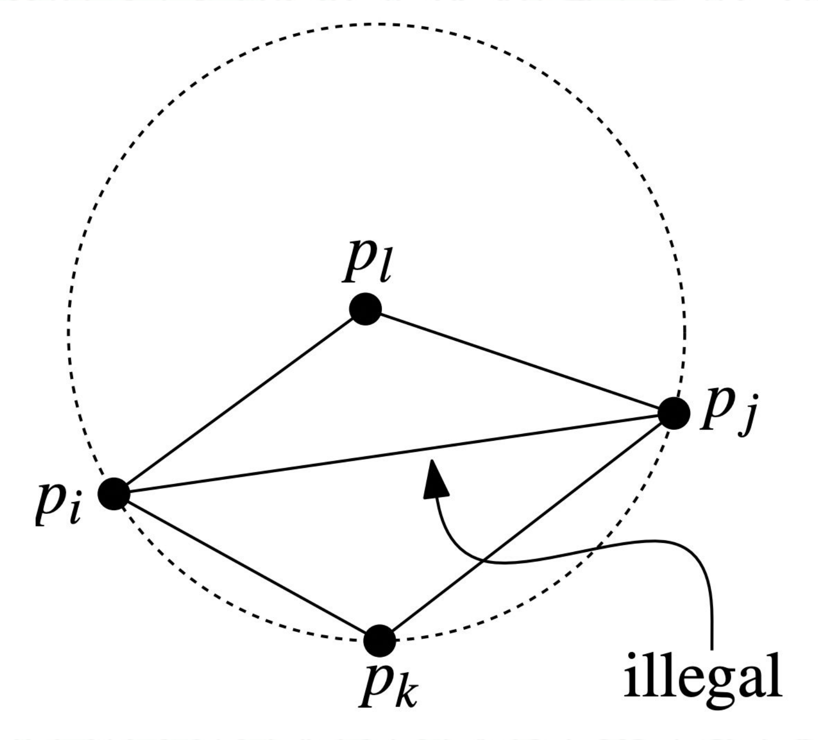


https://en.wikipedia.org/wiki/Inscribed_angle#Theorem

Definition: Angle-Optimal Triangulation

- We want to maximize the smallest angle.

- An edge is *illegal* only if the other vertex of the neighboring triangle is inside the circumscribed circle.



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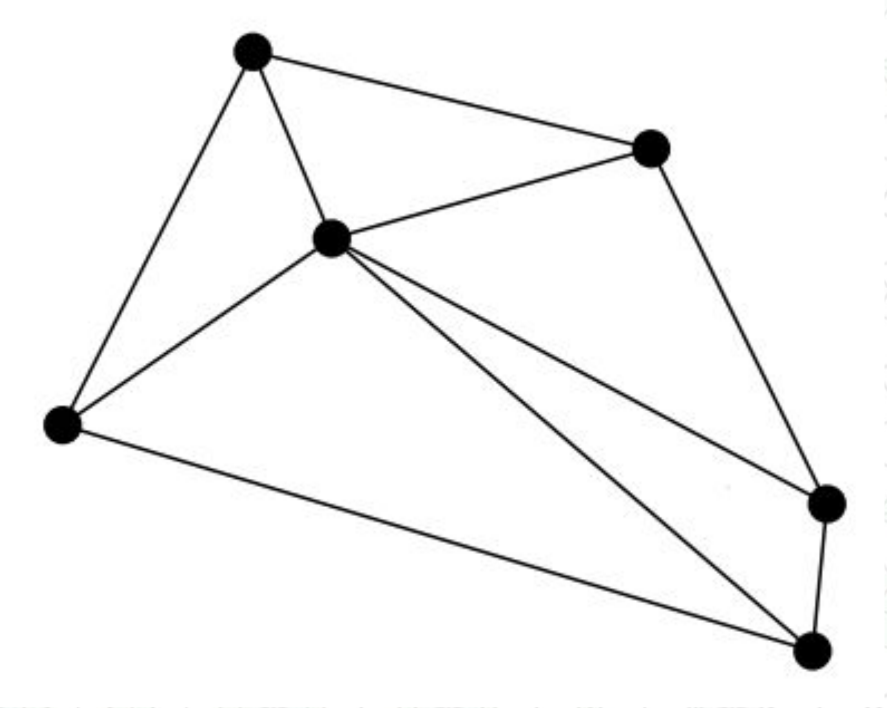
Constructing an Angle-Optimal Triangulation

Brute Force:



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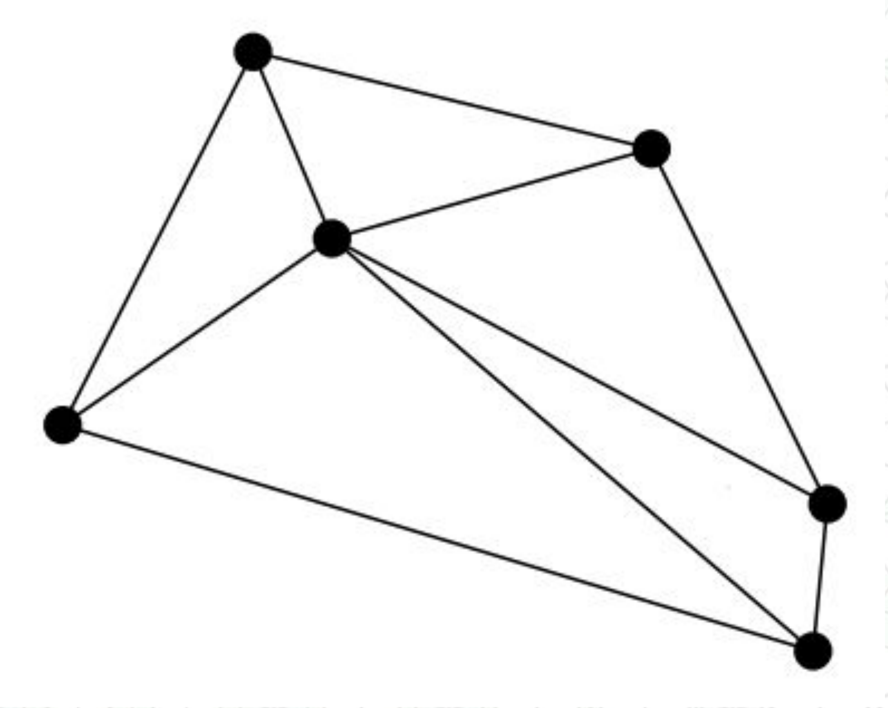
Constructing an Angle-Optimal Triangulation

Brute Force:

- Try all combinations of 3 vertices
- Construct the circumscribed circle
- If no other vertex is inside of that circle, keep it
- Only works if no more than 3 vertices are on the circle
- Analysis?



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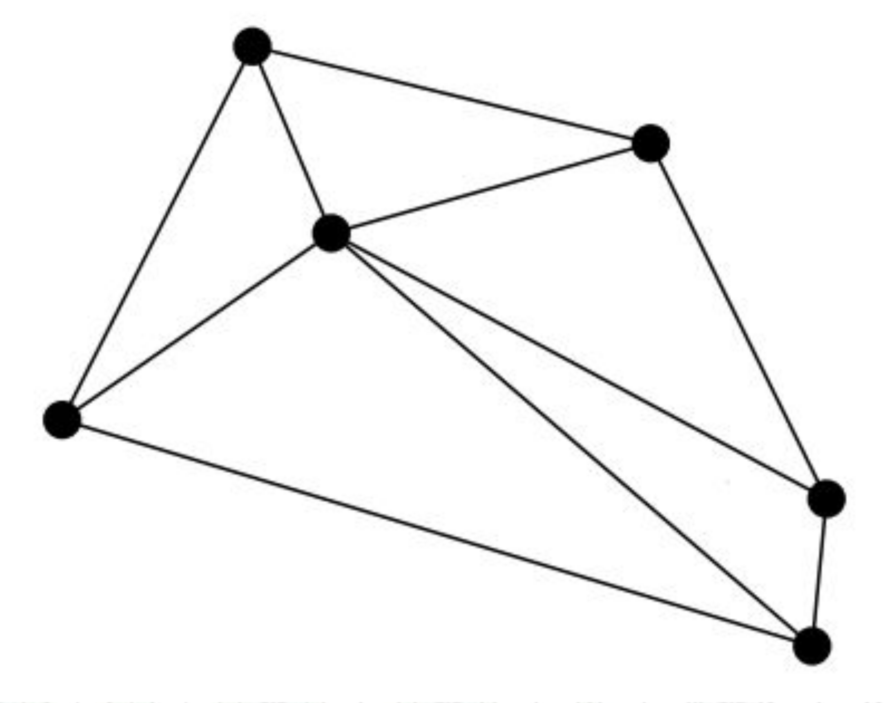
Constructing an Angle-Optimal Triangulation

Brute Force:

- Try all combinations of 3 vertices
  $\binom{n}{3} \rightarrow O(n^3)$
- Construct the circumscribed circle
  $\rightarrow O(1)$
- If no other vertex is inside of that circle, keep it
  $\rightarrow O(n)$
- Only works if no more than 3 vertices are on the circle
- Analysis?
  $\rightarrow O(n^4)$ overall



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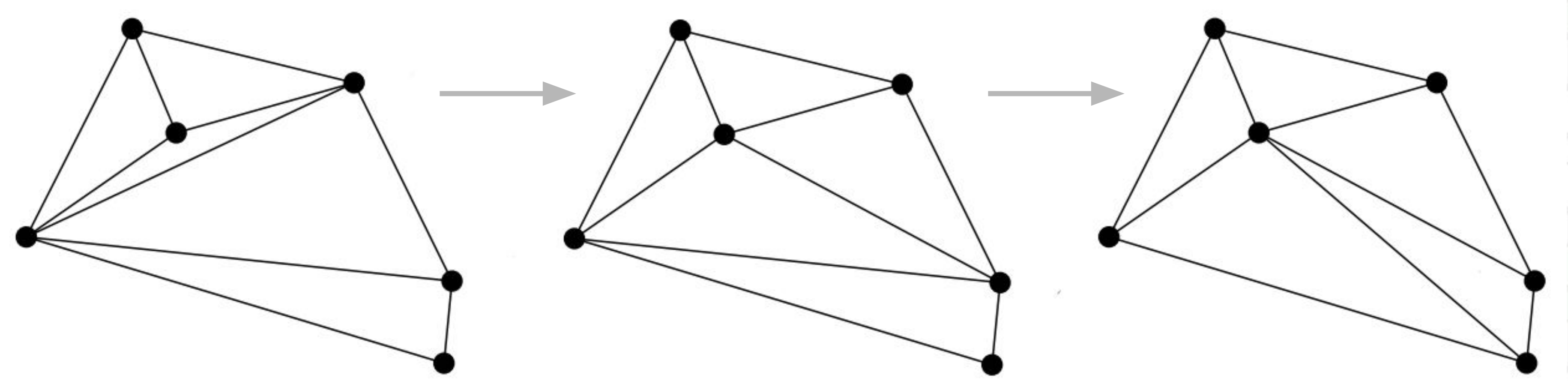
de Berg, Cheong, van Kreveld and Overmars, Chapter 9

Constructing an Angle-Optimal Triangulation

- Start with any triangulation = a maximal planar subdivision
- Check to see if any edge is illegal, if so flip it
- Repeat until every edge is legal



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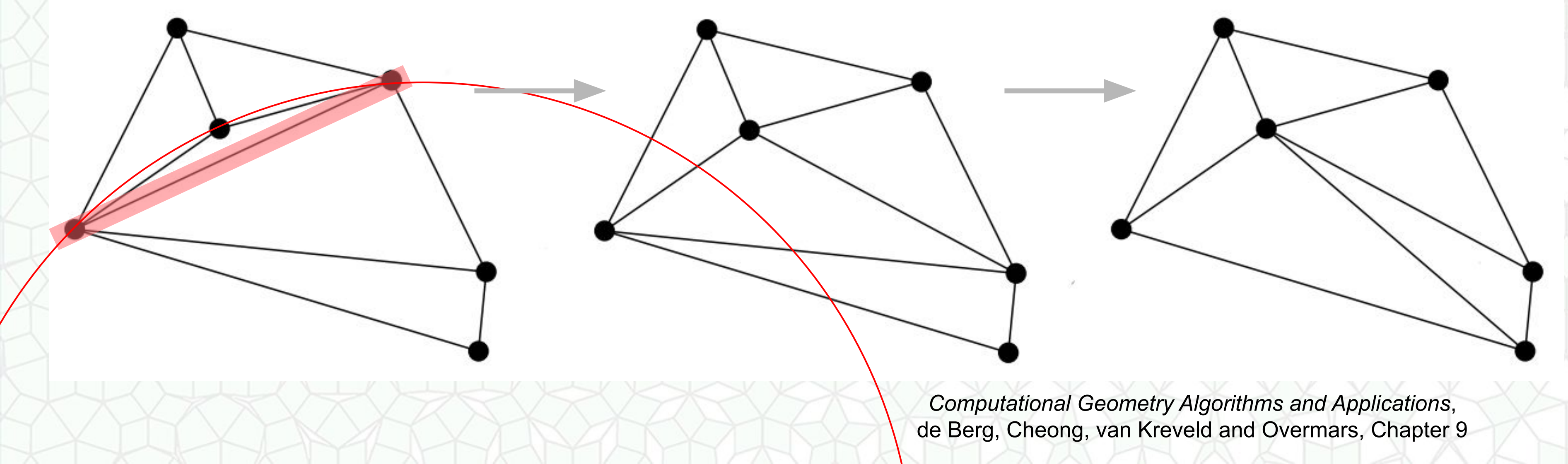

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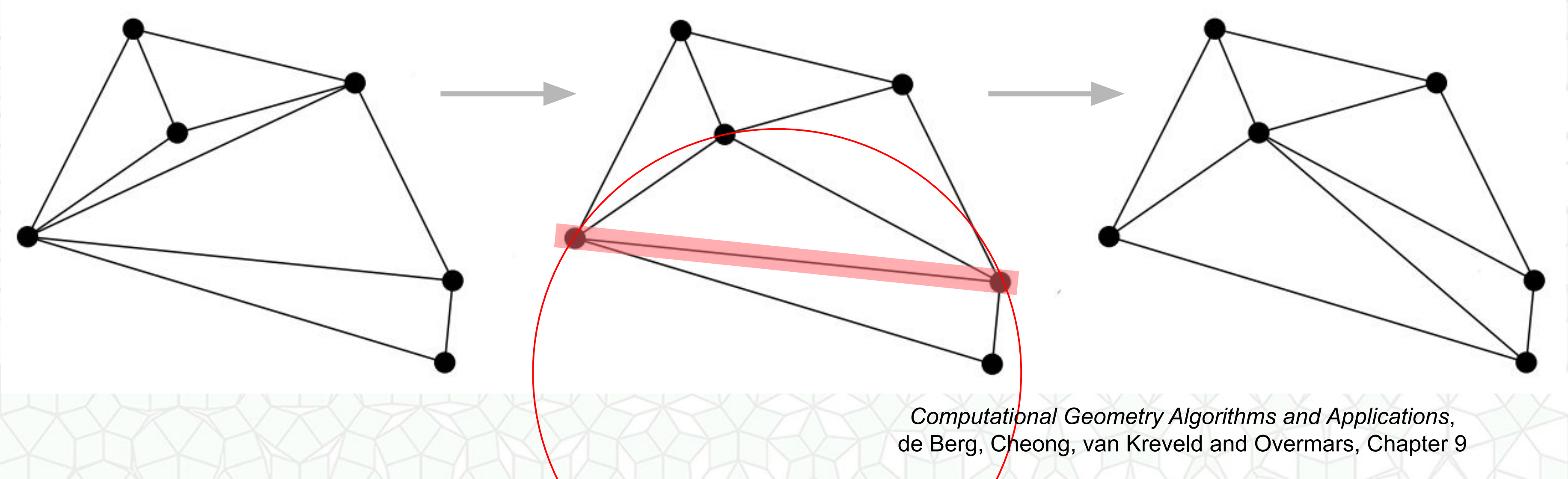

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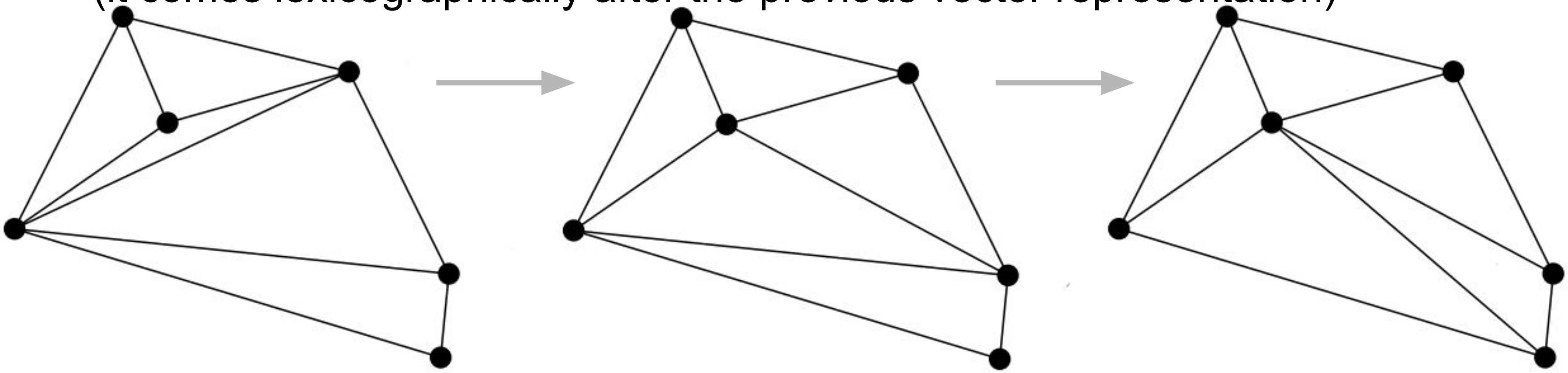
Guaranteed to Terminate? Yes!

- Create a sorted vector of all of the angles of every triangle
  vector length = 3 * # of triangles
- Each edge flip replaces one of the smaller angles
- New sorted vector representation is the same up to that angle..
  (it comes lexicographically after the previous vector representation)



[5, 5, 20, 30, 30, 40, 70, 50, 50, 50, 90, 90, 100, 100, 170]

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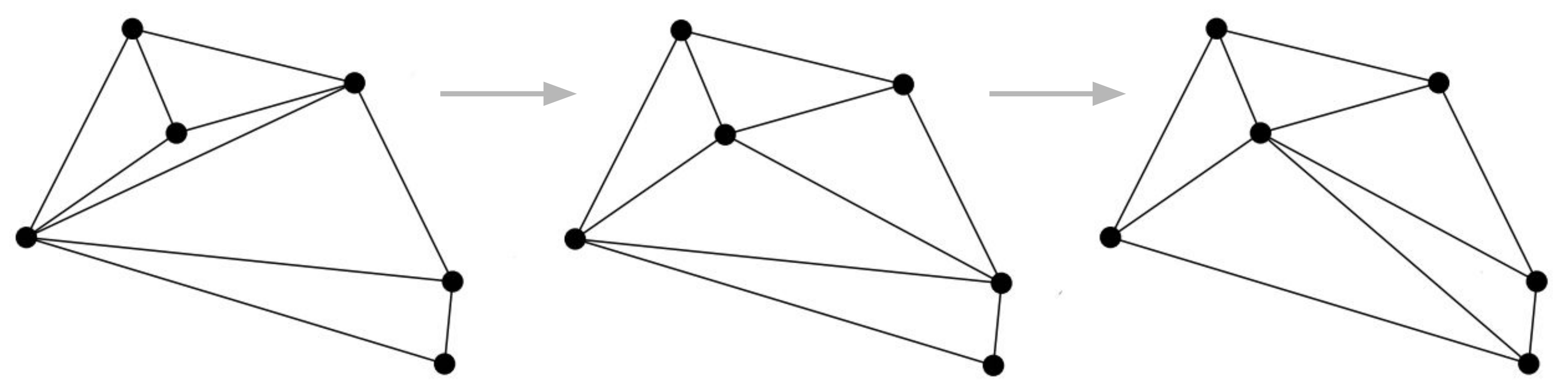
Converge to Optimal & Unique Solution?

- Yes!
  If the vertices are in general position
  … *if no 4 vertices lie on the same circumscribed circle*



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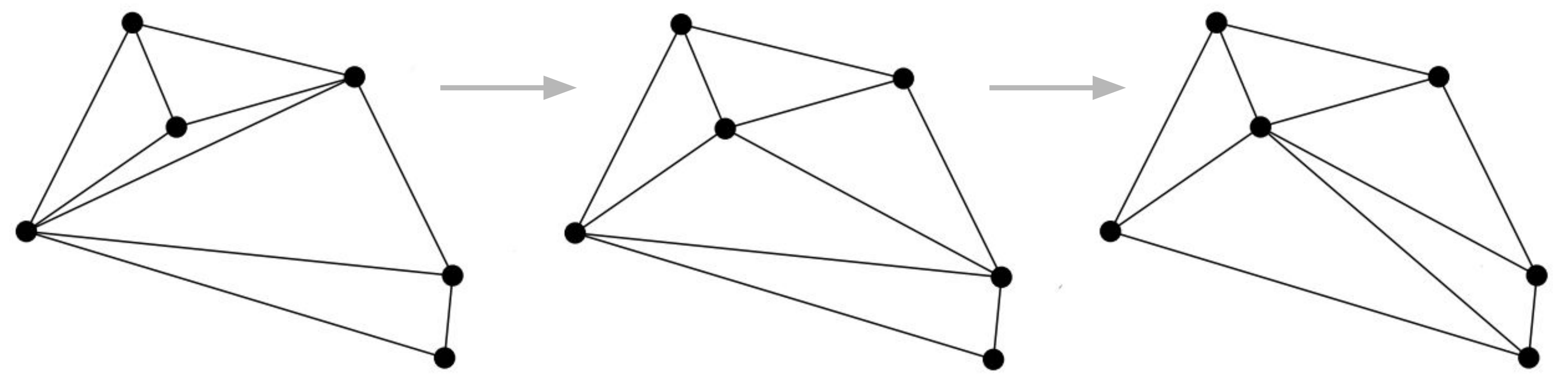
Analysis of Incremental Flipping Algorithm

```
while (true)
    loop over all edges in the mesh
        if the edge is illegal
            swap edge       & continue outer loop
    if all edges are legal
        break outer loop
```



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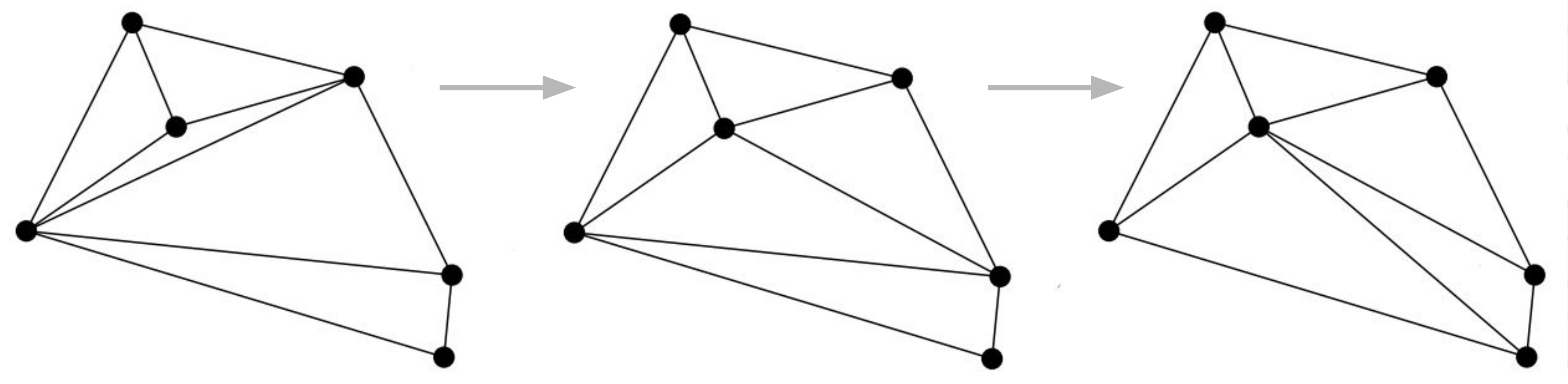
Analysis of Incremental Flipping Algorithm

```
while (true)
    loop over all edges in the mesh →O(n)
        if the edge is illegal →O(1)
            swap edge →O(1) & continue outer loop
    if all edges are legal
        break outer loop
```

How many times do we execute outer loop?



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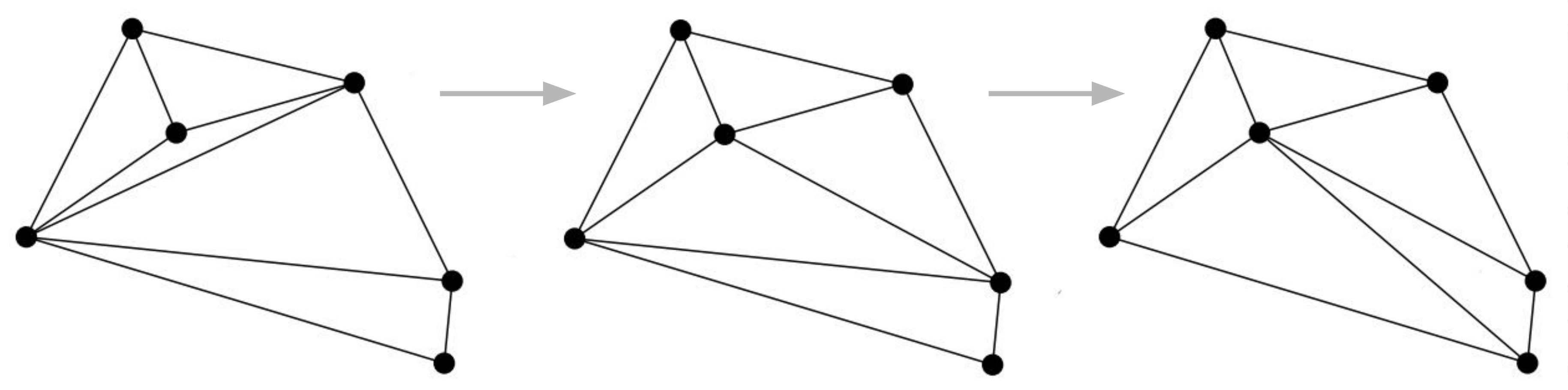
Analysis of Incremental Flipping Algorithm

- We know we won't loop/repeat the same triangulation… but
- Given n points, and k points on the convex hull,
  each triangulation has $2n - k - 2$ triangles
- How many different triangulations? It might be exponential! → $O(c^n)$



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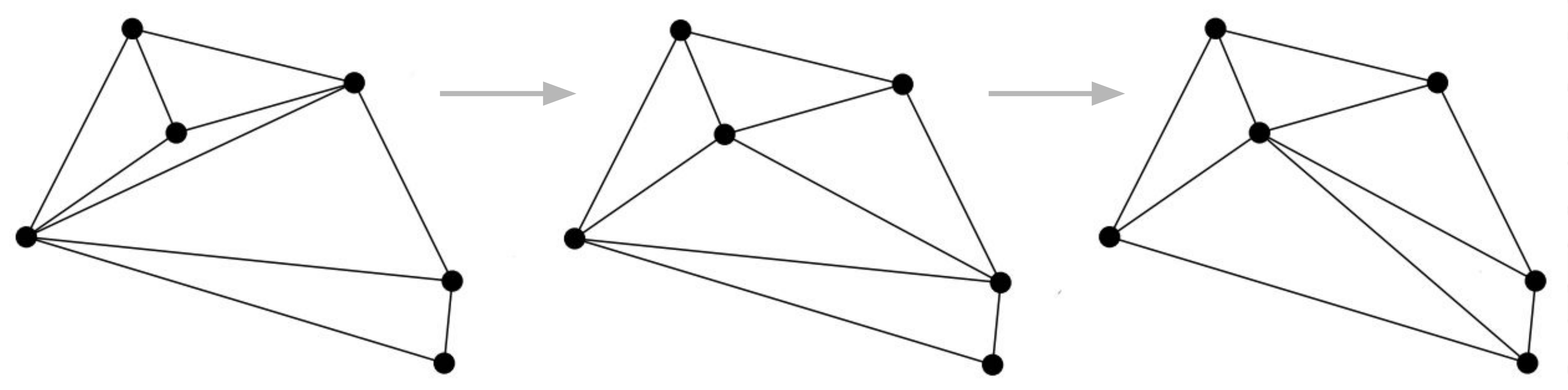
Analysis of Incremental Flipping Algorithm

- It's slow! Could it be $O(n * c^n)$????
- Can we do better? Yes!

  (more next lecture)



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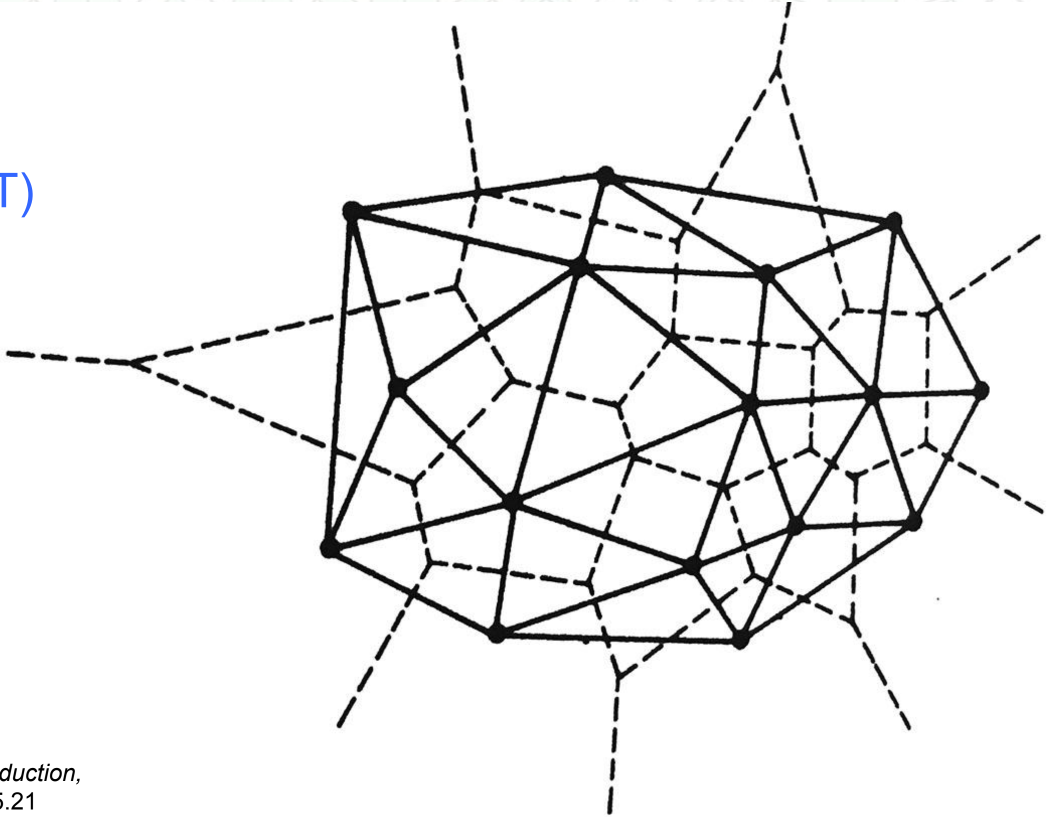
Outline for Today

- Homework 5 Questions?
- Final Project: Brainstorming Ideas & Partner Matching
- Last Time: Duality & Arrangements
- Motivation: Interpolation & Terrain Height Maps
- Graph vs. Planar Graph vs. Plane Graph
- Triangulation & Angle-Optimal Triangulation
- Thale’s Theorem & Inscribed Angle Theorem
- Brute Force Construction of Angle-Optimal Triangulation
- Duality: Voronoi Diagram & Delaunay Triangulation
- Next Time: More Delaunay Triangulations!

Dual: Voronoi Diagram & Delaunay Triangulation

- The Voronoi Diagram (VD) *is the dual of the* Delaunay Triangulation (DT)
- Every Voronoi Site is a face in Voronoi Diagram and a vertex in the DT
- Every Voronoi Edge is an edge in the DT
- Every Voronoi Vertex is a triangle in the DT

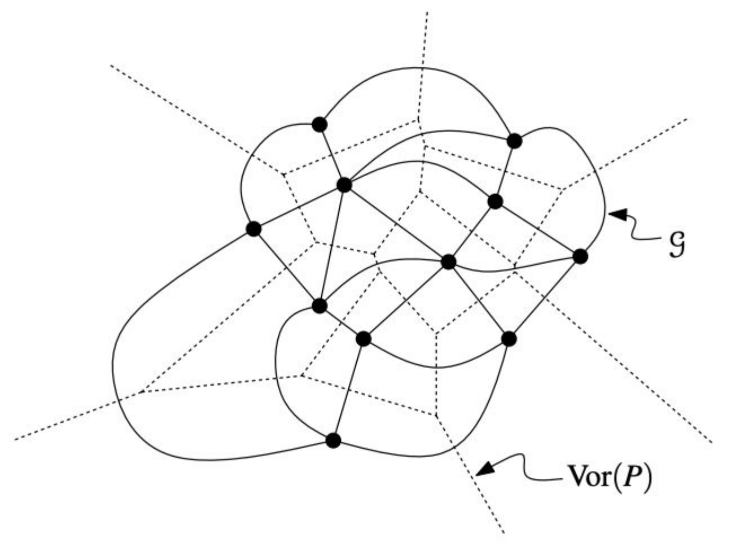


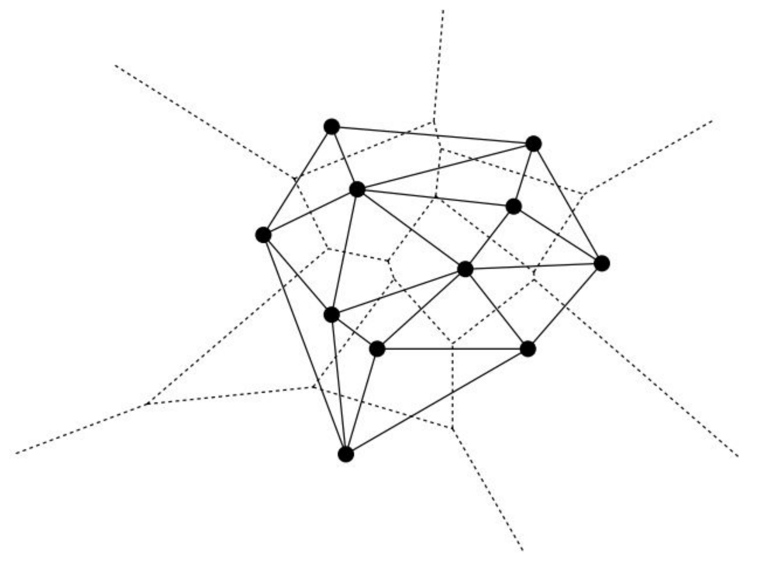
Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.21

Dual Graph of the Voronoi Diagram

Dual Graph: Has an arc connecting two Voronoi Sites for every edge between neighboring cells in the Voronoi Diagram.

Delaunay Graph: Straight line embedding of the Dual Graph of the Voronoi Diagram.





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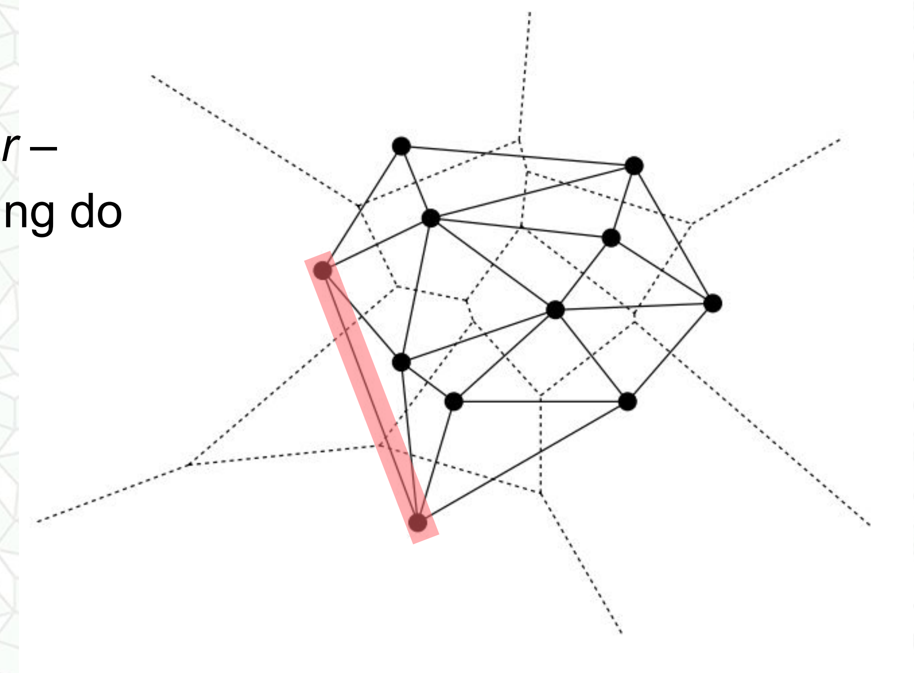
Delaunay Graph

- NOTE: Straight line edges of the embedding *may not cross* their corresponding Voronoi edge.
- But the Delaunay Graph is *planar* – straight line edge of the embedding do not cross (proof in textbook).



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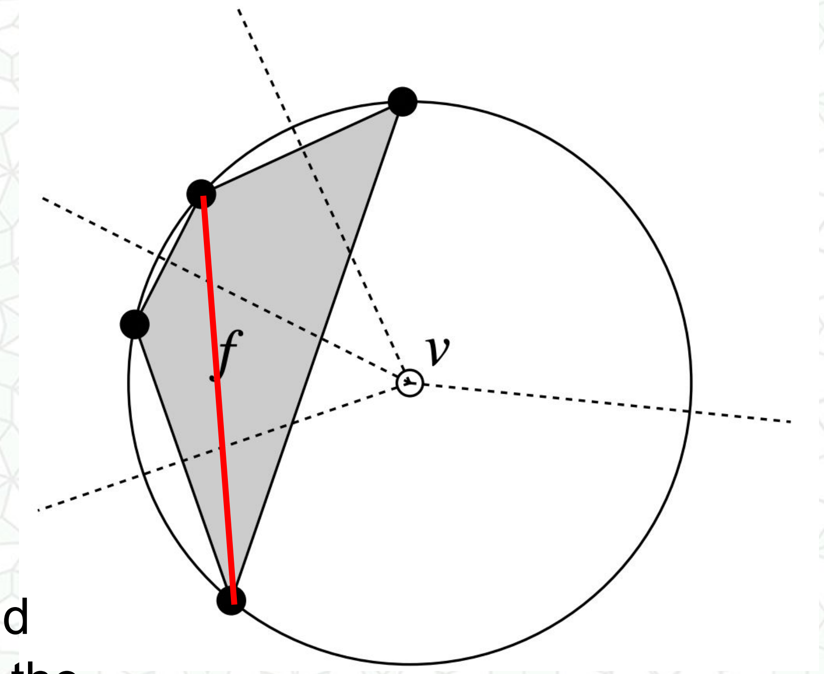
Delaunay Graph vs. Delaunay Triangulation

- If 4 (or more) vertices do lie on the same circumscribed circle
- Voronoi Vertex, *v*, will have degree ≥ 4
- The corresponding face in the Delaunay Graph will have ≥ 4 edges
- This face is guaranteed to be convex
- This face can be trivially triangulated
- Once all of these faces are triangulated, we have a Delaunay Triangulation
- The Delaunay Triangulation is unique and equivalent to the Delaunay Graph only if the vertices are in general position



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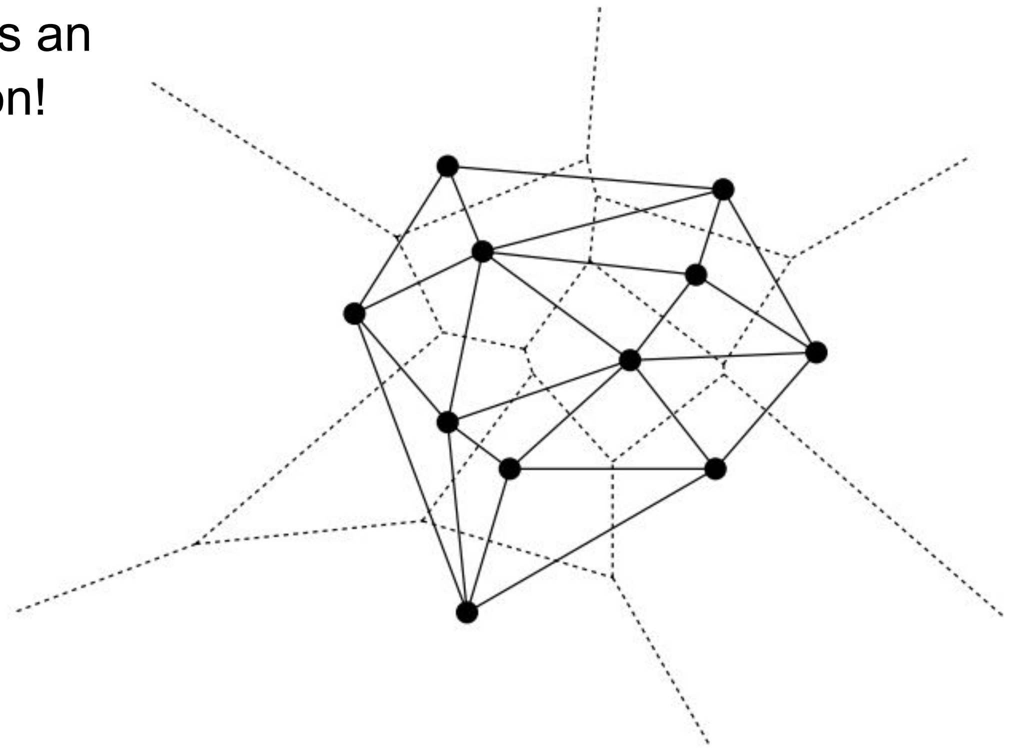
Delaunay Triangulation

- A Delaunay Triangulation is an Angle-Optimal Triangulation!



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