CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/F23/

Lecture 15: More (Delaunay) Triangulations

Outline for Today

- Final Project: Brainstorming & Feedback & Partner Matching
- Last Time / Motivation: Terrain Height Maps
- Polygon Triangulation vs. Point Set Triangulation
- Counting the Number of Triangulations
- Incremental Triangulation by Point Insertion
- Incremental Triangulation by Line Sweep
- Flip Graph & Connectedness of all Triangulations
- Delaunay Construction by Edge Flips
- Randomized Incremental Delaunay Triangulation
- Delaunay Triangulation Construction Analysis Summary
- Next Time: Data Structures for Line Segment Queries

Final Project Brainstorming

- Each student posts two different final project ideas on the forum
- For each idea:
    - Briefly describe the idea, your motivation for it, and an example of the potential result
    - What is the technical implementation/theory challenge?
- Can be an Individual Project or a Team of 2
- Have you already decided on one idea? Which one?
- Do you already have a partner? Who? (even if you have chosen an idea and/or a partner everyone must post 2 different ideas)
- Due Monday 10/23 @ 11:59pm

Final Project Feedback *due Monday Oct 30th*

- Read your classmates ideas posts. Post a response to (at least) 3 other students' posts. Pick one of their proposed ideas and:
  - Ask a detailed question about the project idea,
  - Suggest a specific illustrative example or input dataset,
  - Suggest a specific data structure or algorithm or library to use for the project,
  - Suggest a reference (paper, book, URL, etc.), or
  - Suggest an extension or hybrid project related to your own idea or the idea of another student.
- Let's try to distribute the responses so that everyone gets at least 2 responses of feedback.

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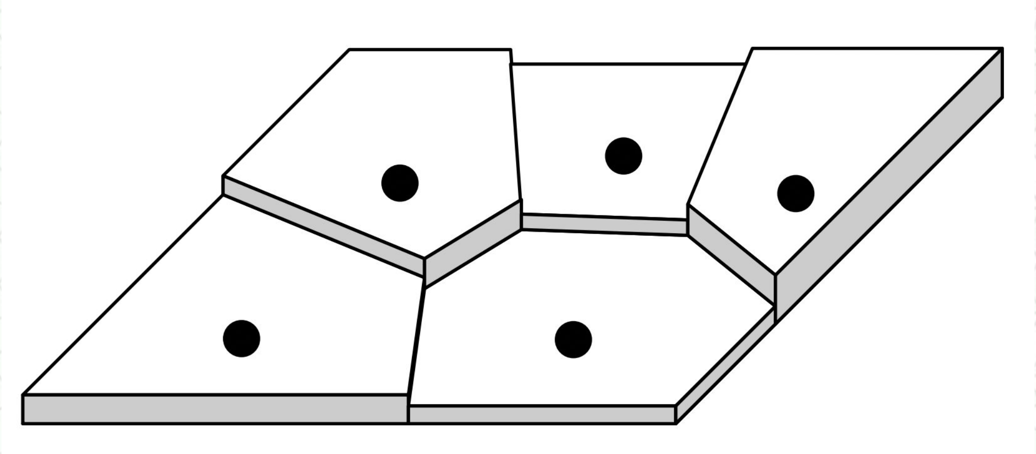
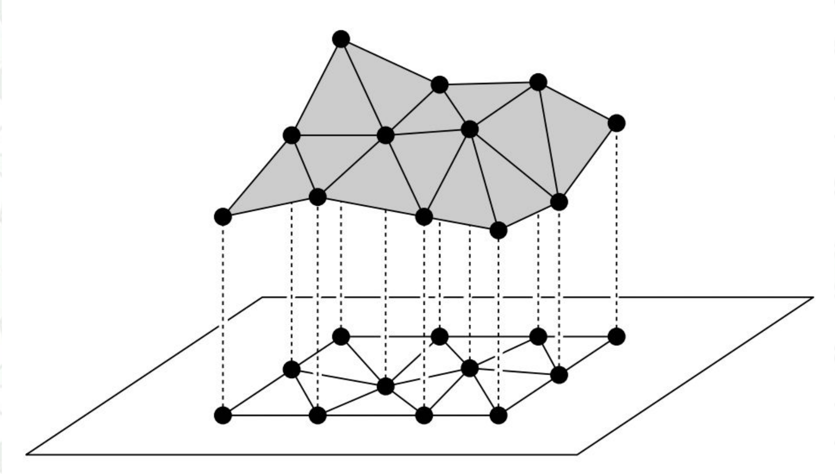
Motivation: Terrain Height Map



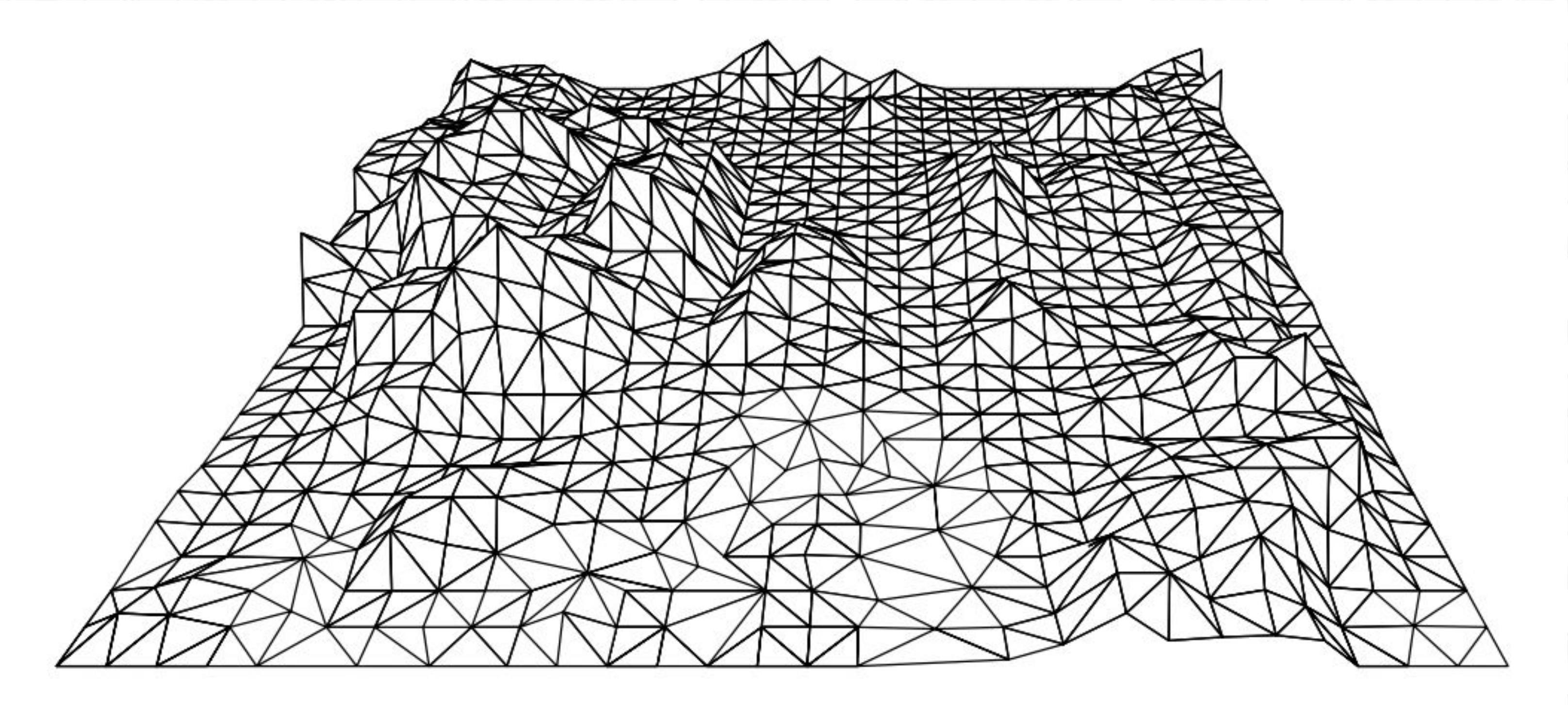
Nearest Neighbor



Bi-Linear Interpolation

Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 9

Motivation: Terrain Height Map



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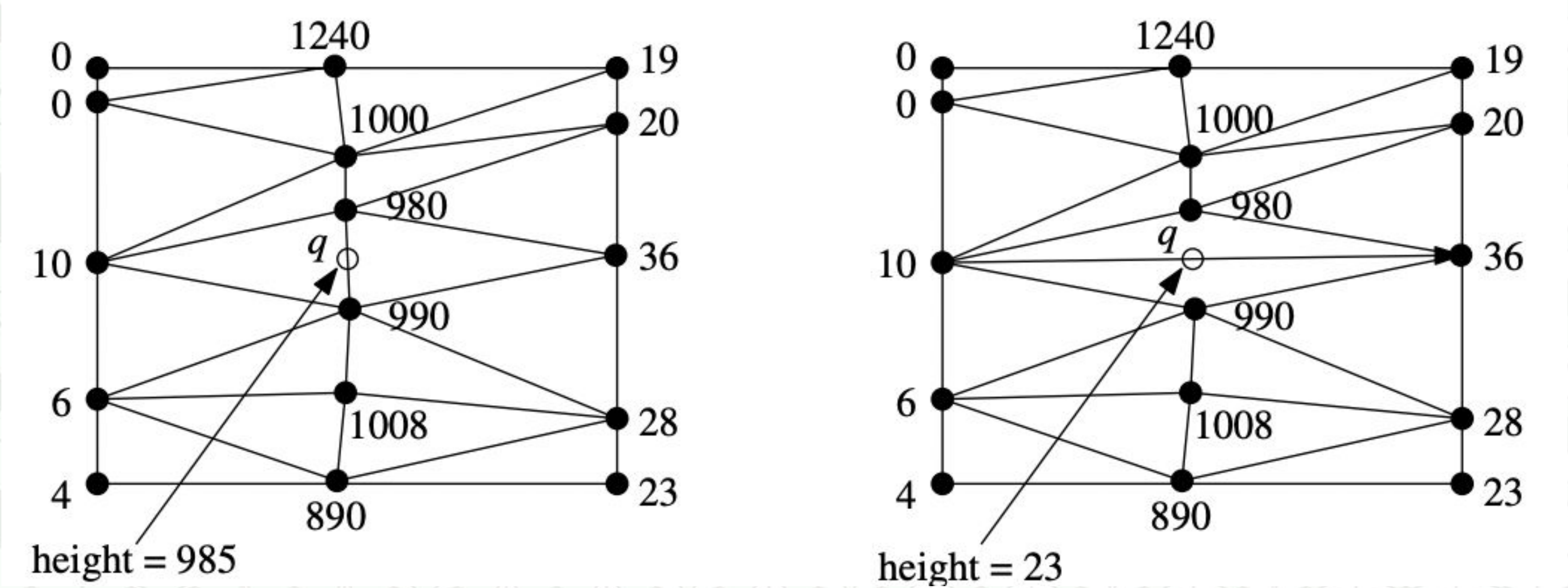
Not all Triangulations are the same!

this triangulation is better *this triangulation is worse*



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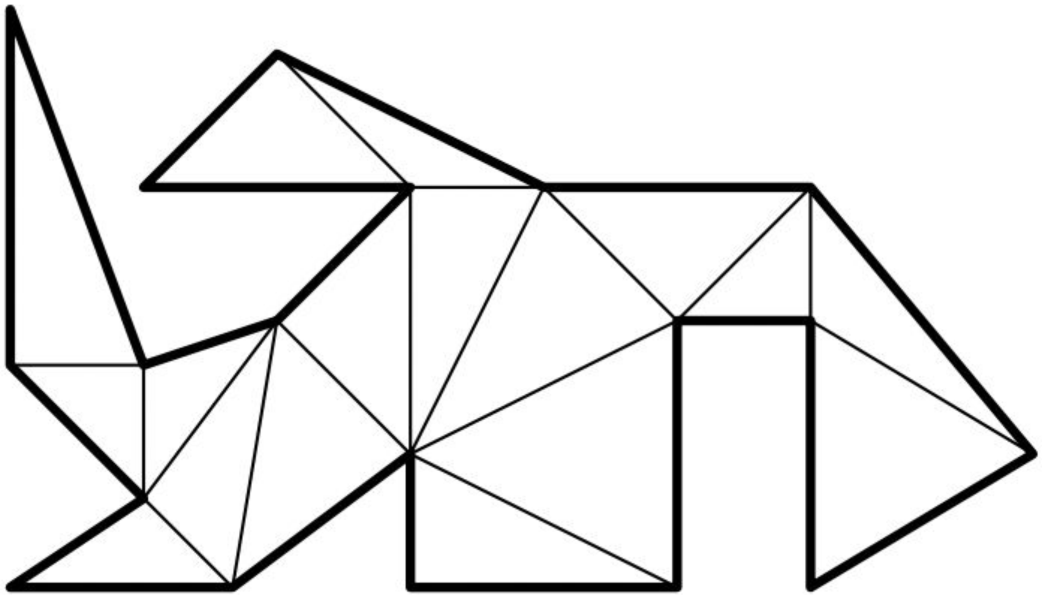
Polygon Triangulation (from Lecture 4)

- Boundary specified
  - Typically non-convex
- *No interior points*
  - *vertices are 3 colorable*
- Typically has many solutions
- For *n* input points
  - Each solution has n-2 triangles



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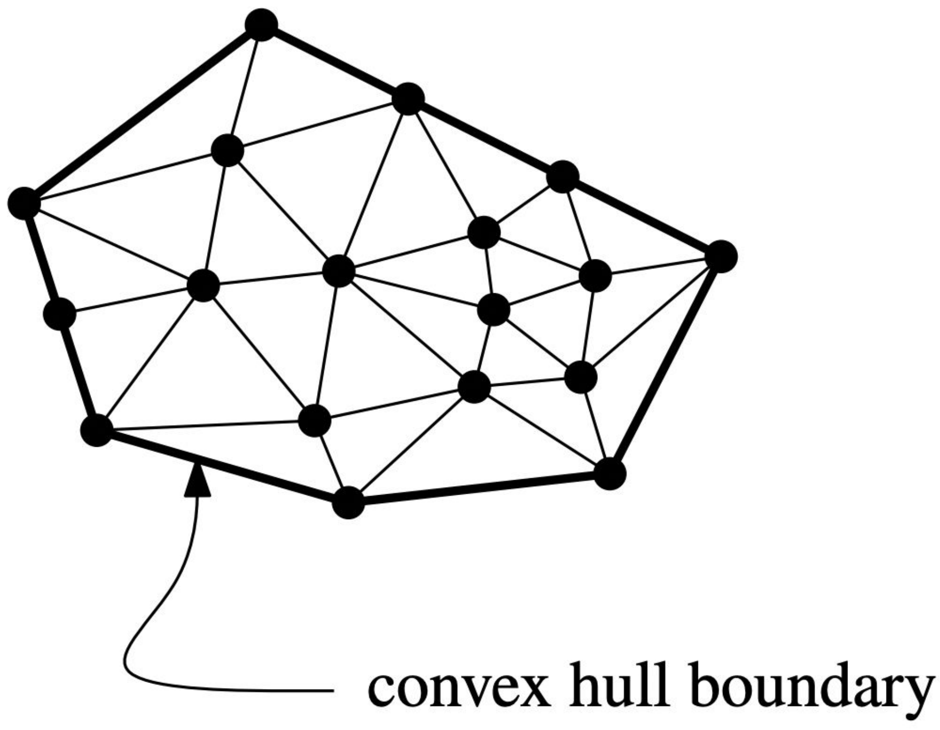
Point Set Triangulation

- A triangulation is a *Maximal Planar Subdivision* of a vertex set
- No edge connecting two vertices can be added without destroying planarity
- Every face will have 3 vertices
- For *n* input points, with *k* points on hull boundary
  - Each solution has $2n - 2 - k$ triangles



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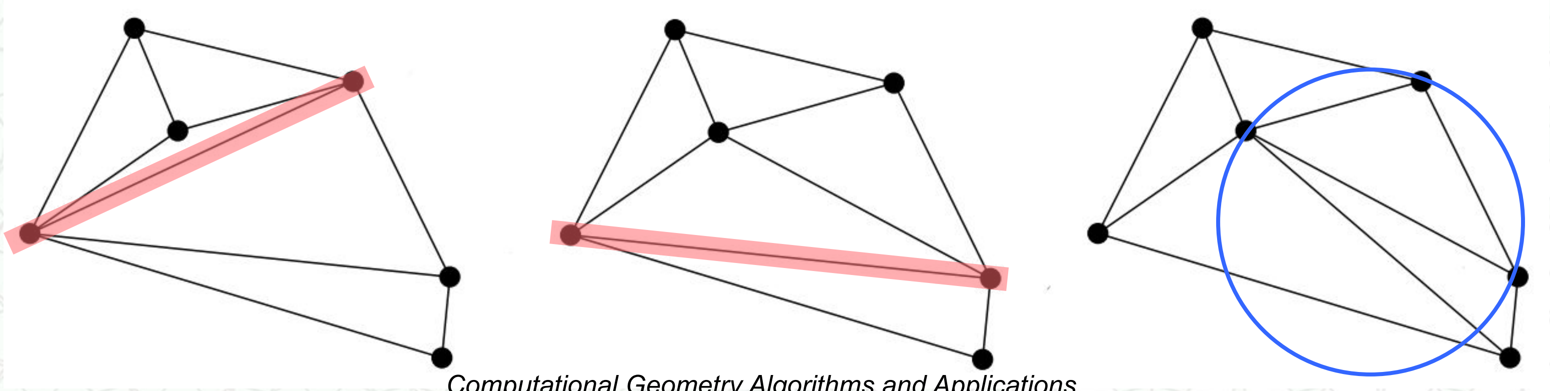
Last Time: Angle Optimal Triangulation Analysis

- Brute force (enumerate all triangles, construct circles, reject…)
- Construct any triangulation & Flip until all edges are legal



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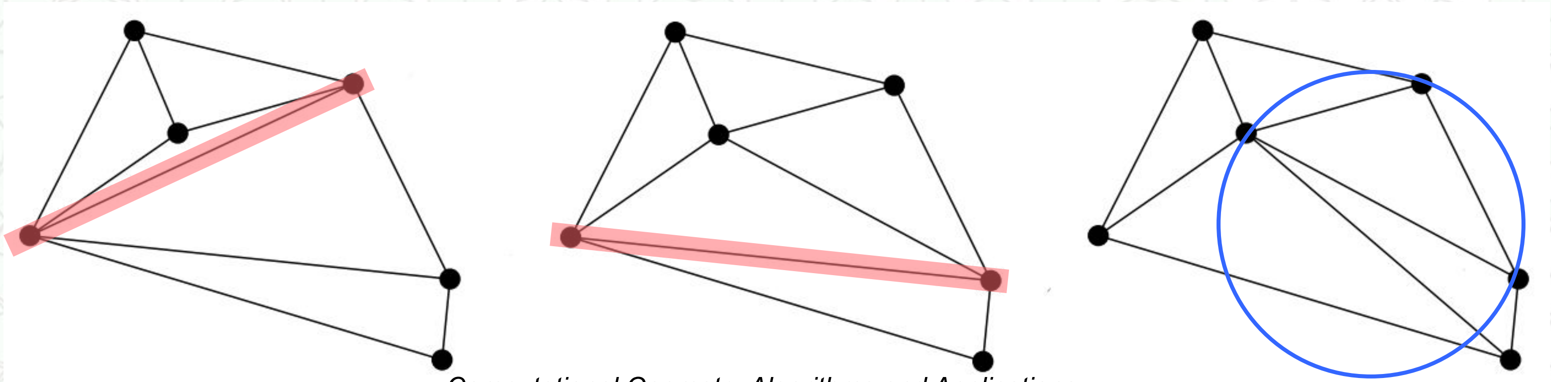
Last Time: Angle Optimal Triangulation Analysis

- Brute force (enumerate all triangles, construct circles, reject…)
  $\rightarrow O(n^3 * n) \; = \; O(n^4)$
- Construct any triangulation & Flip until all edges are legal
  $\rightarrow O(n * c^n)$

How do we create the initial triangulation?

Are there exponentially many different triangulations?



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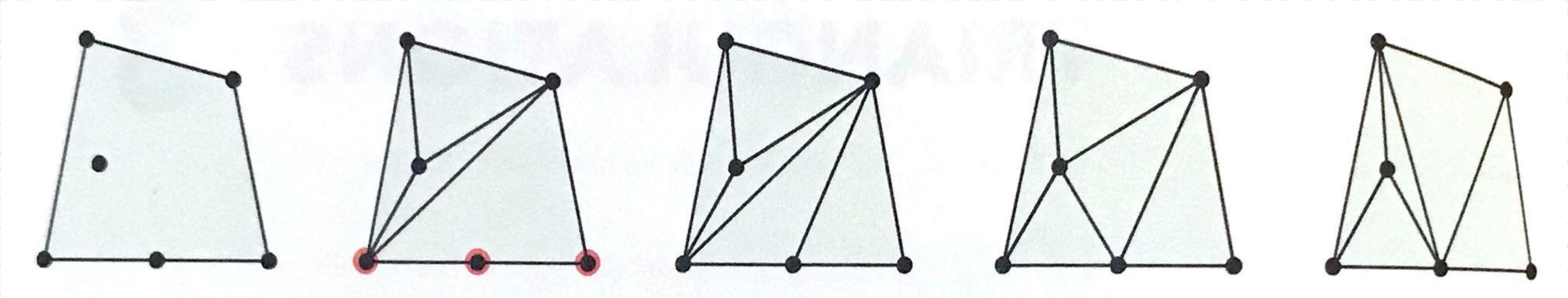
How many Different Triangulations?

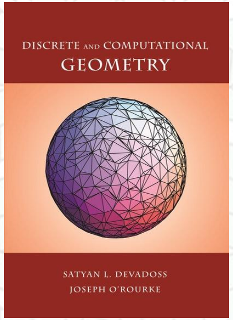


a point set &
its convex hull

a (non-maximal)
planar subdivision

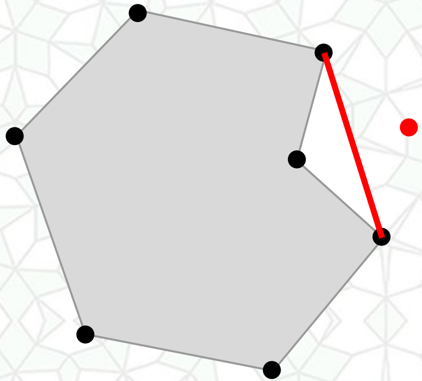
Several triangulations
of this point set

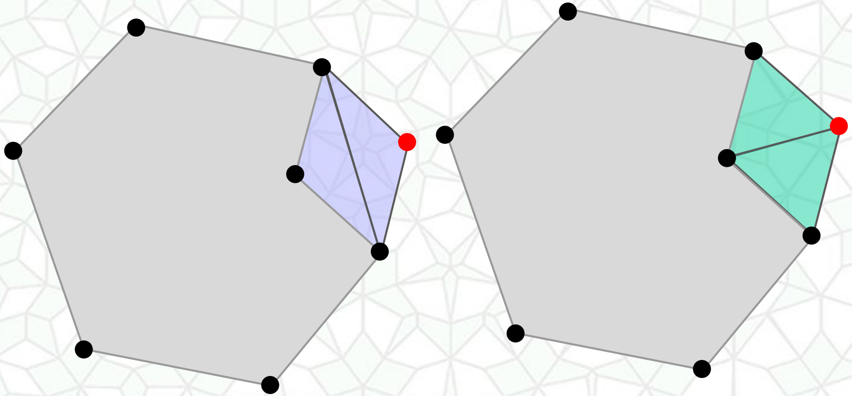
“Discrete and Computational Geometry”,
Devadoss & O’Rourke,
Princeton University Press 2011,
Chapter 3



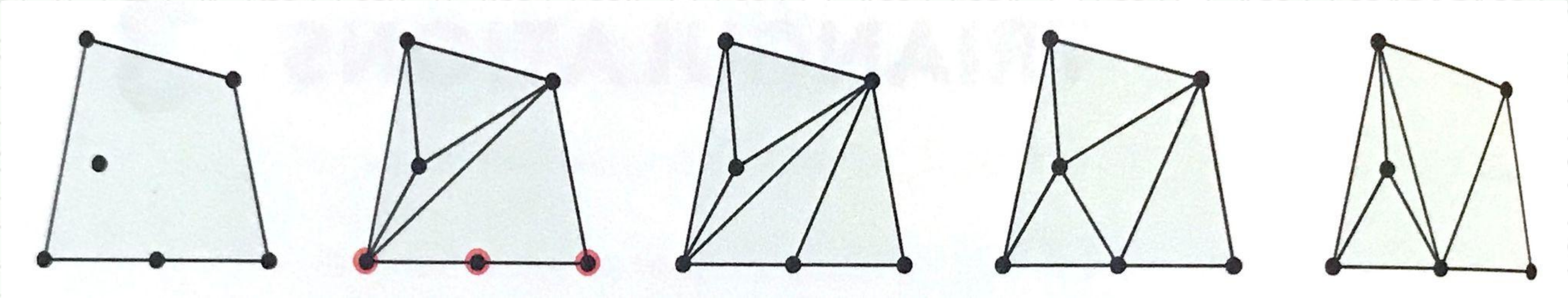
Are *Exponential* Triangulations Possible?

- Yes!
- Enumerate all *X* triangulations of a set of *n-1* points.
- Identify an edge on the boundary of the convex hull. Add one point just outside that edge.
- Each of the *X* triangulations can be used to produce 2 different triangulations.
- That's **at least** *2*X* triangulations of *n* points. → **at least** $O(2^n)$ triangulations of *n* points!





How many Different Triangulations?



a point set & its convex hull

a (non-maximal) planar subdivision

Several triangulations of this point set

"Discrete and Computational Geometry", Devadoss & O'Rourke, Princeton University Press 2011, Chapter 3

- Actually, counting the number of triangulations is hard!
- For *n* points, the number of triangulations is exponential
- Open Problem: Can we count the number of triangulations in *O(n)* time?

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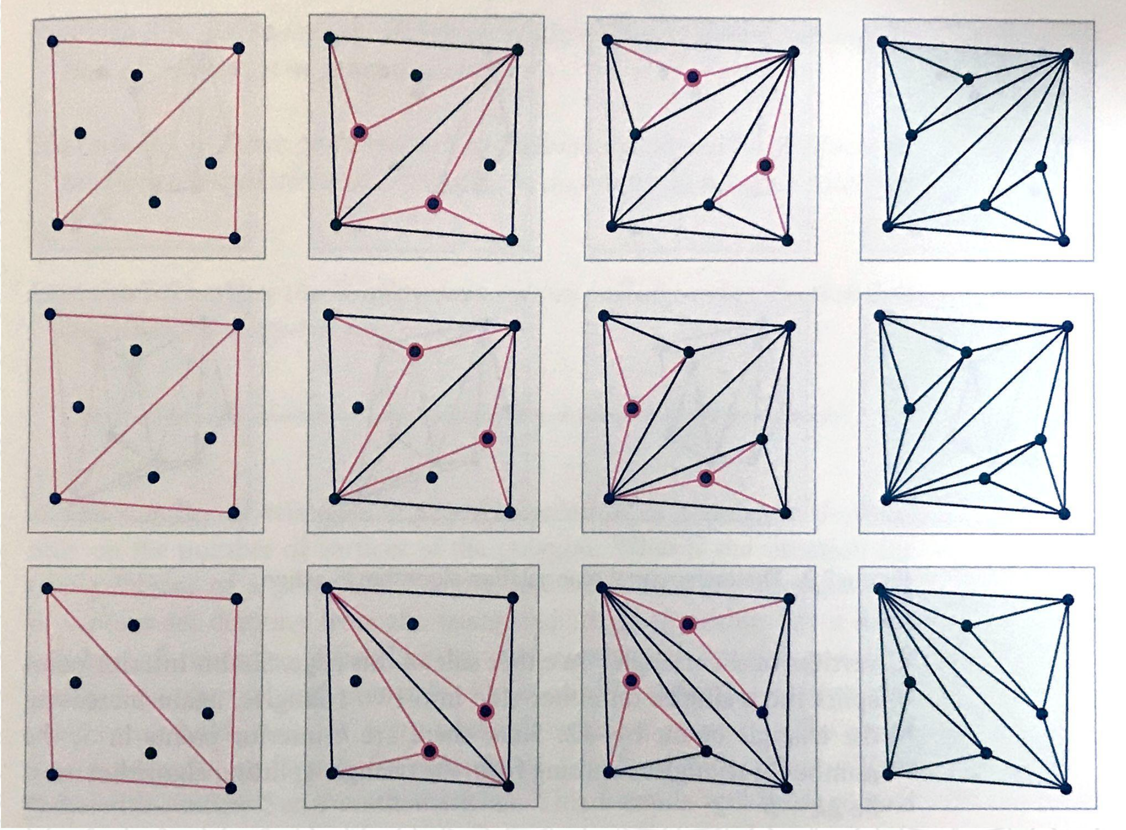
Construction by Point Insertion

"Discrete and Computational Geometry", Devadoss & O'Rourke, Princeton University Press 2011, Chapter 3

- Start with convex hull
  - Triangulate it
  - *k*-2 triangles
- For some ordering of the other points
  - Determine which triangle the point lies inside of
  - Replace that triangle with 3 triangles
  - (*n* - *k*) * 2 additional triangles
- 2**n* - *k* - 2 total triangles!



(partial enumeration of the triangulations constructed by point insertion)

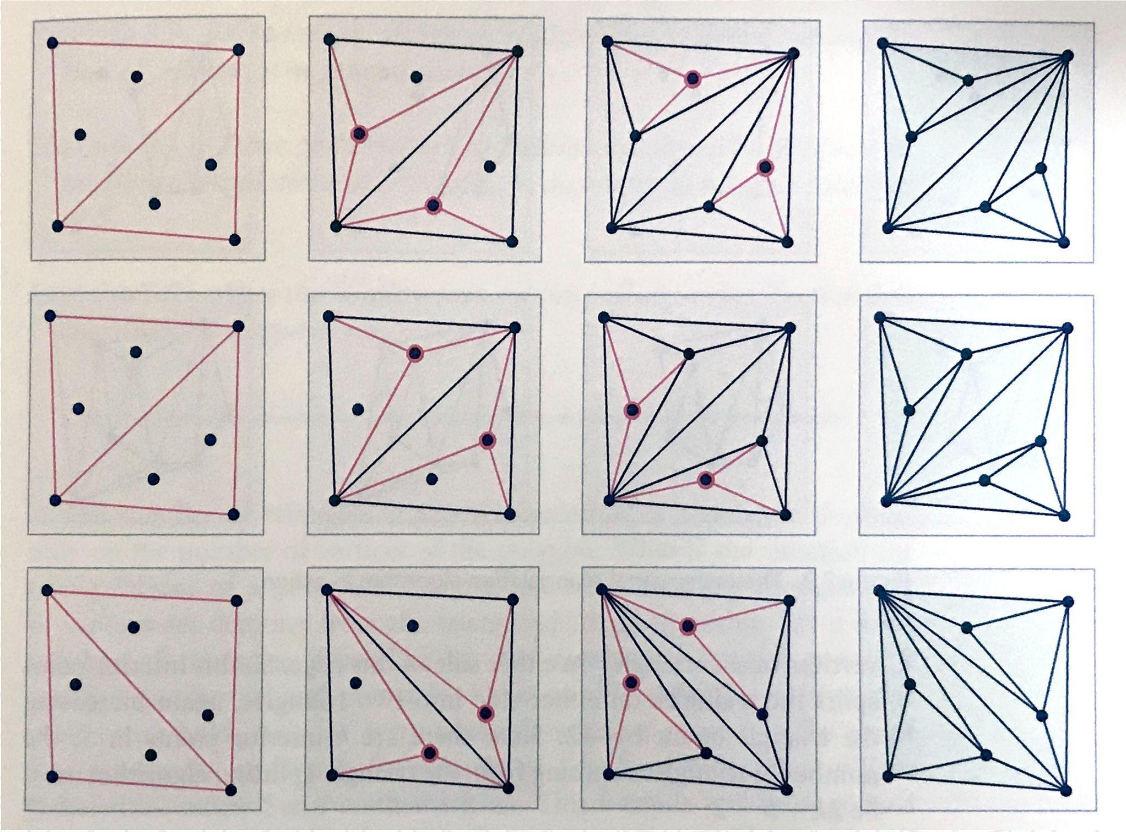
"Discrete and Computational Geometry", Devadoss & O'Rourke, Princeton University Press 2011, Chapter 3

Construction by Point Insertion

- Every solution has $2*n - k - 2$ total triangles!
- If we enumerate **every triangulation** of the hull and **every sequence** of point intersections:

  Is every triangulation unique?

  Do we generate every possible triangulation?



(partial enumeration of the triangulations constructed by point insertion)

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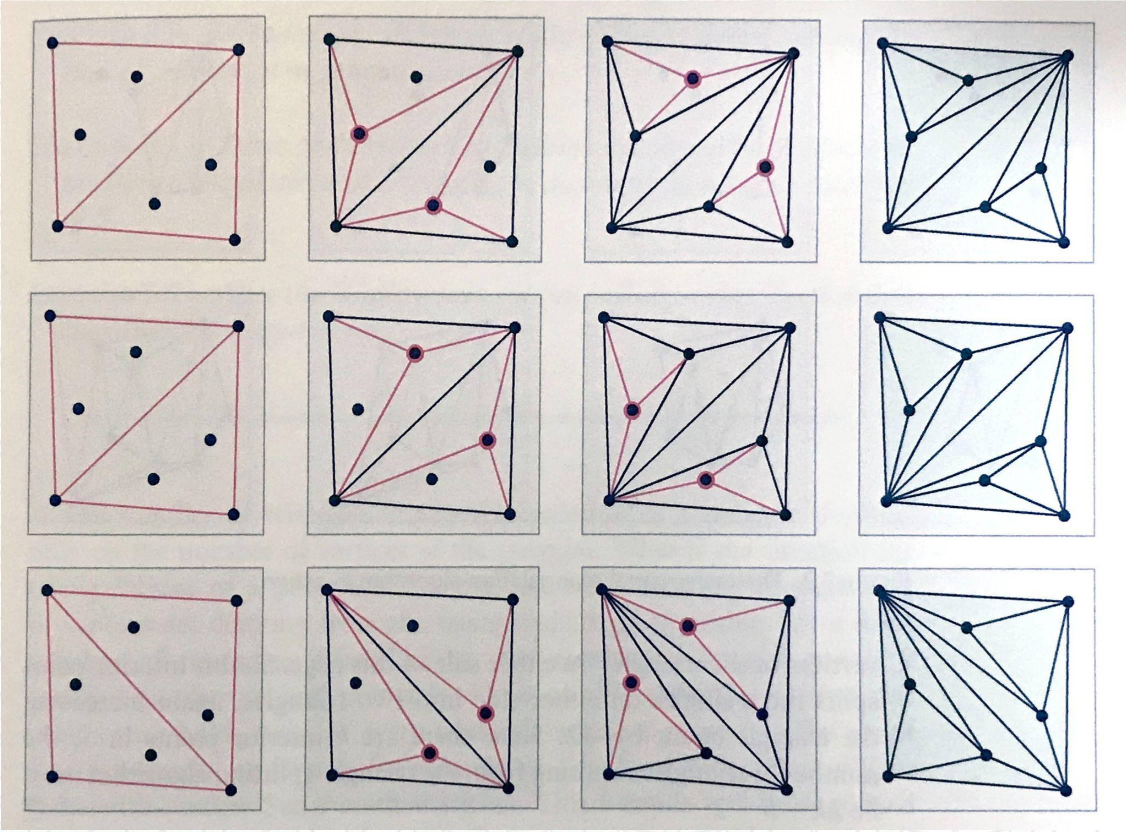
Construction by Point Insertion

- Every solution has $2*n - k - 2$ total triangles!

- If we enumerate **every triangulation** of the hull and **every sequence** of point intersections:

  Is every triangulation unique?
  No!
  Do we generate every possible triangulation?
  No! What are we missing?



(partial enumeration of the triangulations constructed by point insertion)

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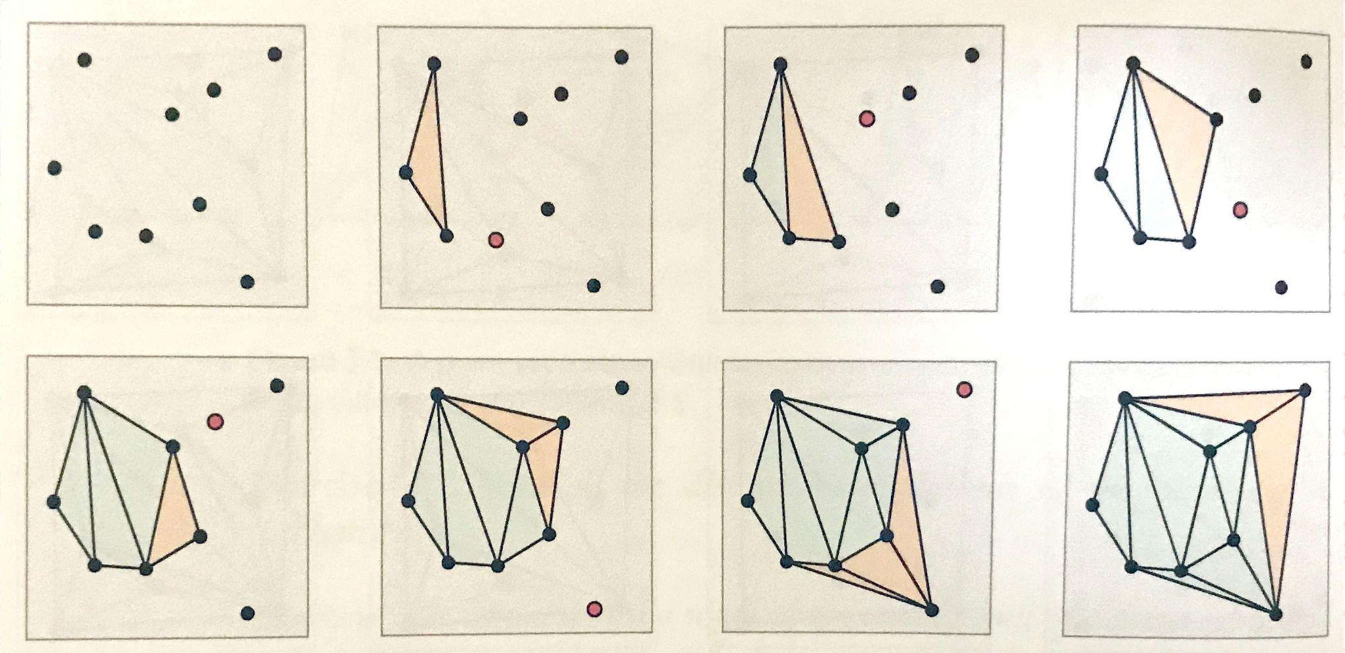
Construction by Line Sweep

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3

- Sort the input points by x
- Form a triangle with the 3 leftmost points



- Add every other point from left to right
  - Determine which points on the current hull are visible from the new point
  - Add a fan of triangles connecting the new point to the visible hull points

Construction by Line Sweep

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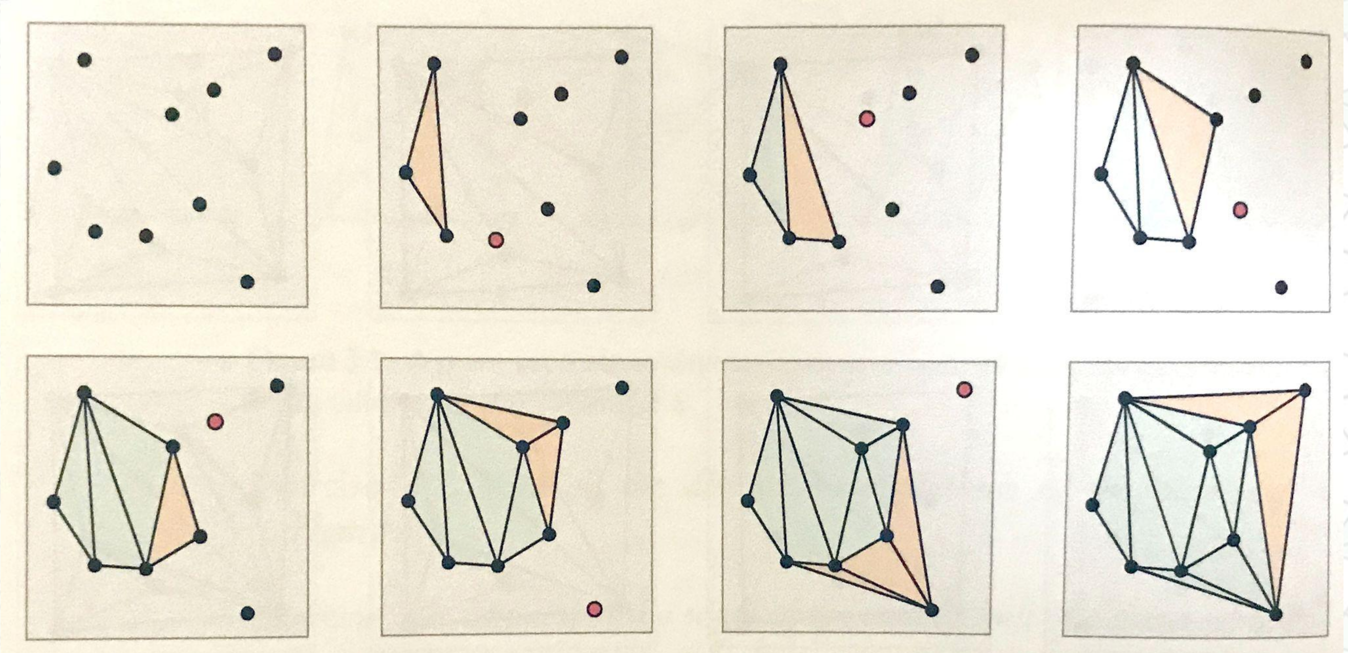
- If we enumerate **every sweep orientation** (rotate the coordinate system)...

- *Can we generate every triangulation?*



Construction by Line Sweep

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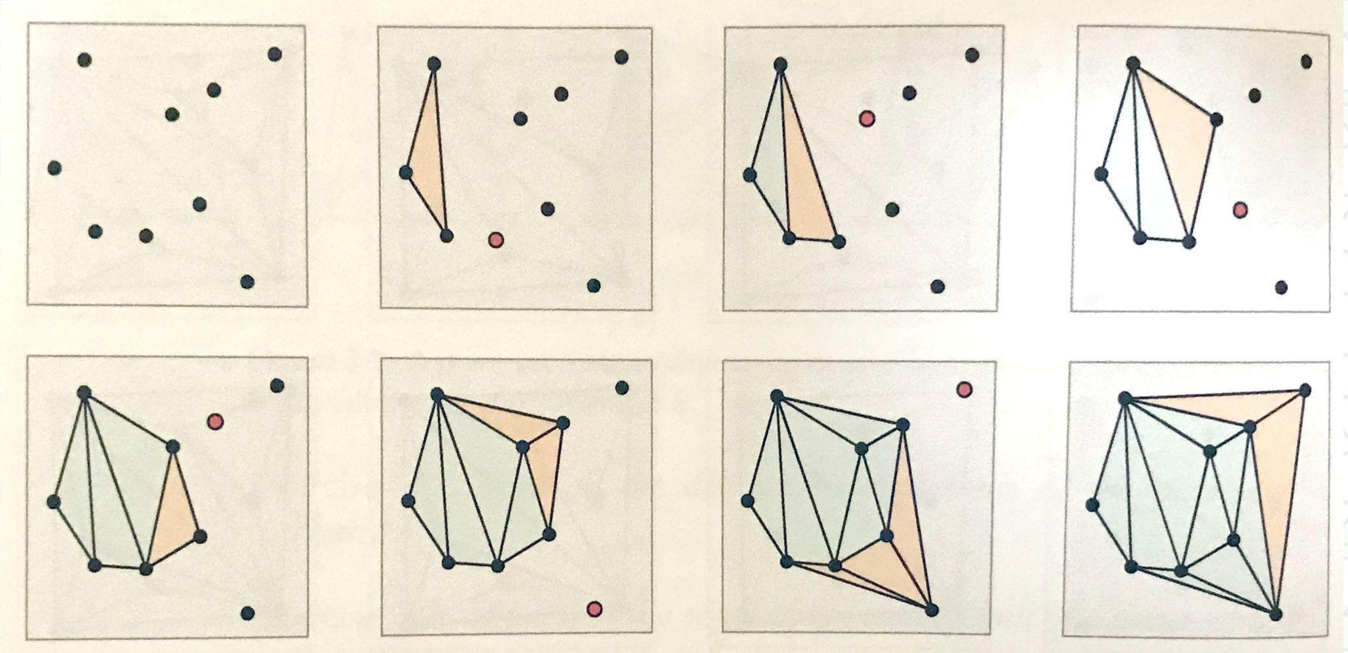
- If we enumerate **every sweep orientation** (rotate the coordinate system)...

- *Can we generate every triangulation?* *No…*



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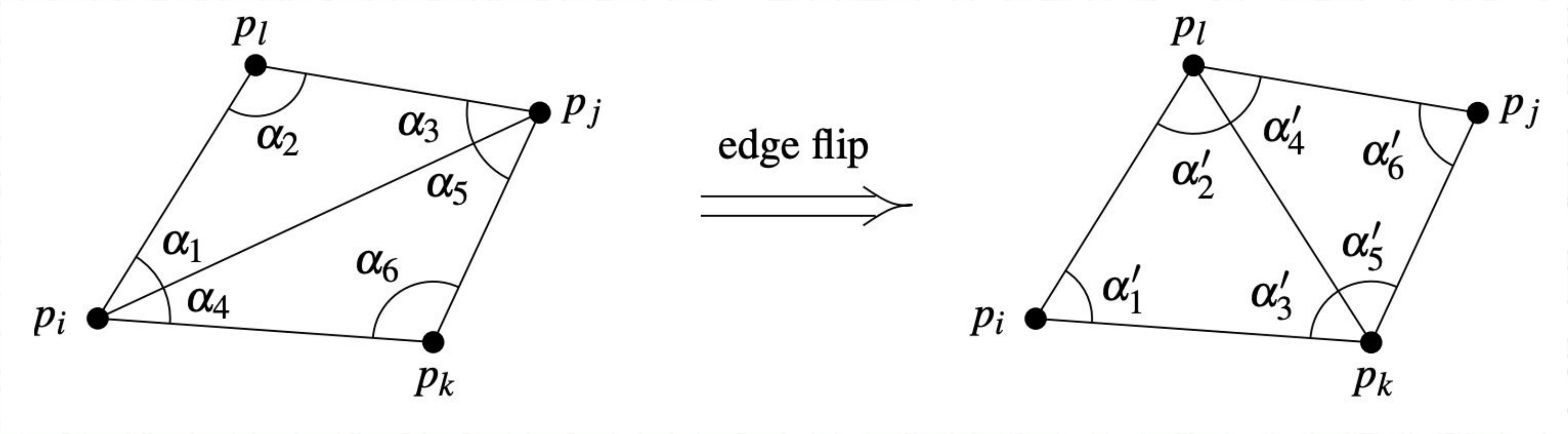
Triangle Swap (a.k.a. Flip the Edge)

- Replace any edge between two triangles with the edge connecting the other vertices of those two triangles (only possible if the combined area of the two triangles is convex)



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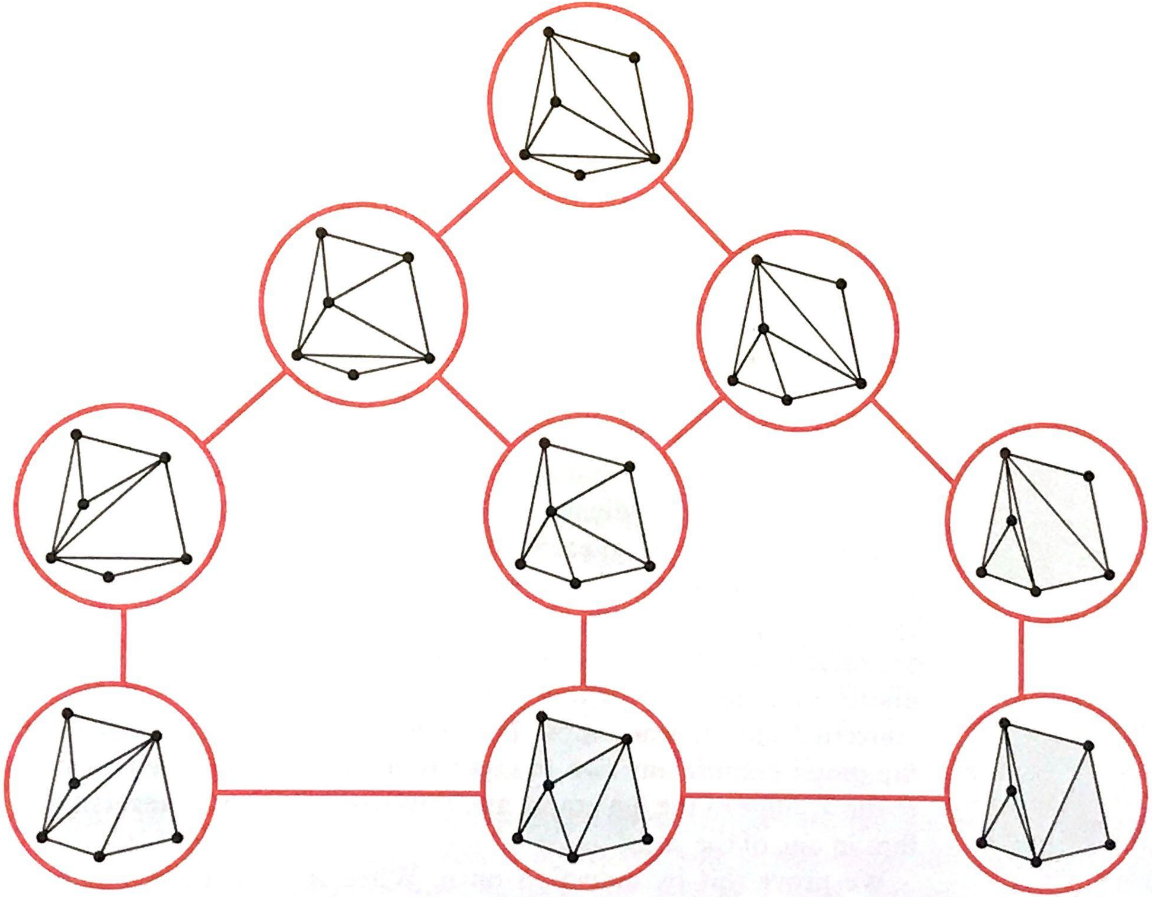
The Flip Graph

- If we did generate every triangulation…

- Let's organize the triangulations as nodes in a graph
- We'll put an edge between two nodes if flipping a single edge converts one triangulation into the other triangulation



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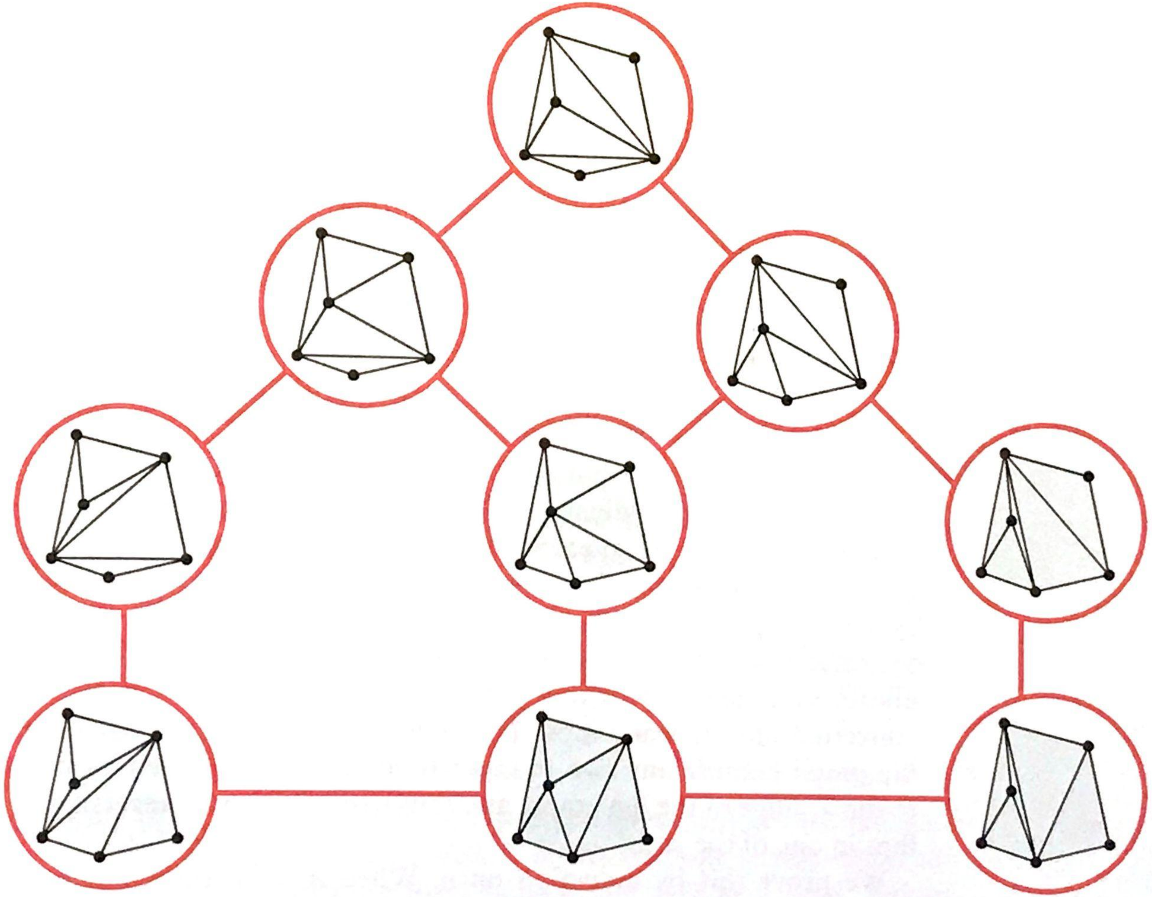
The Flip Graph

- Is this graph guaranteed to be connected?

- *Are we always able to find a sequence of edge flips that converts one triangulation into another triangulation?*



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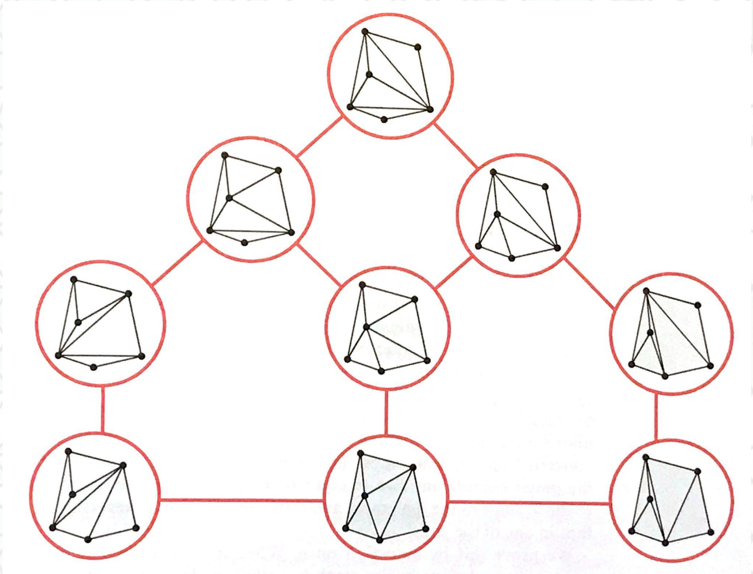
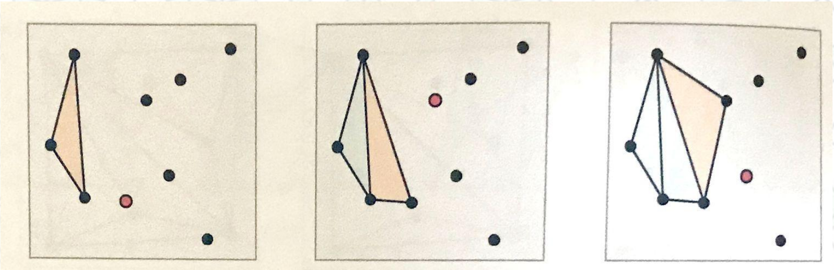
The Flip Graph *is Connected*

- Let's show that every triangulation can be converted by edge flips to the triangulation that results from the *x* axis sweep line construction.

- This will prove the flip graph is connected.





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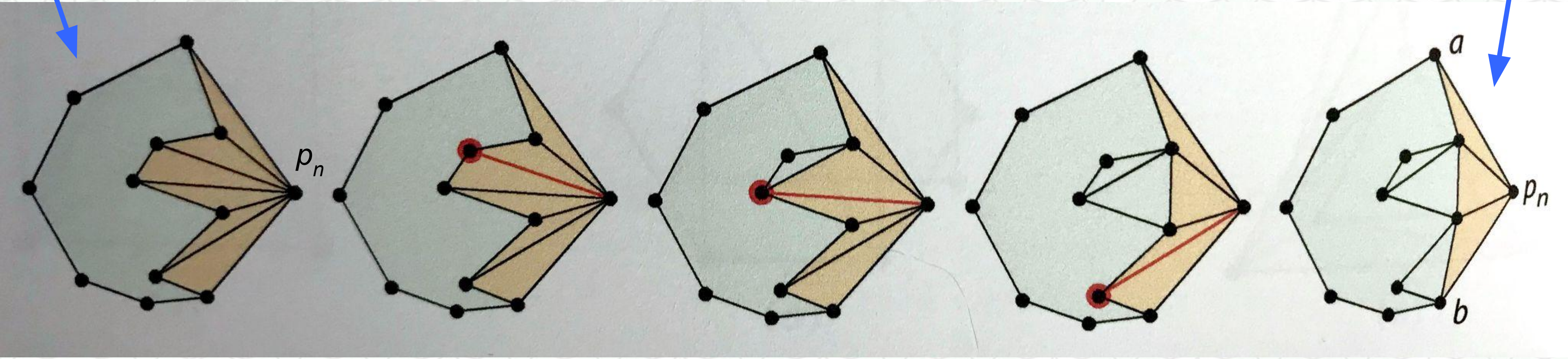
Proof by Induction & Construction in Reverse

- Given any target triangulation, let's deconstruct the triangulation by removing one vertex at a time, from right to left
- **Identify all triangles that touch the current rightmost vertex, p_n**

Our target triangulation

Triangulation constructed by Line Sweep



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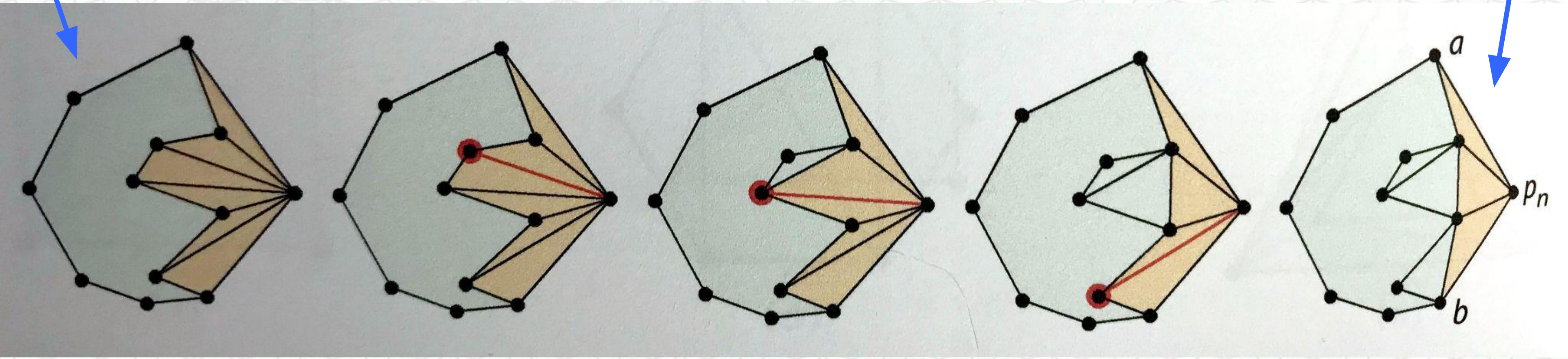
Proof by Induction & Construction in Reverse

- **Identify a vertex touching p_n that is not on the hull of the Line Sweep triangulation without p_n**
- *Flip that edge (if that quadrilateral is not convex, find one that is!)*

Our target triangulation

Triangulation constructed by Line Sweep



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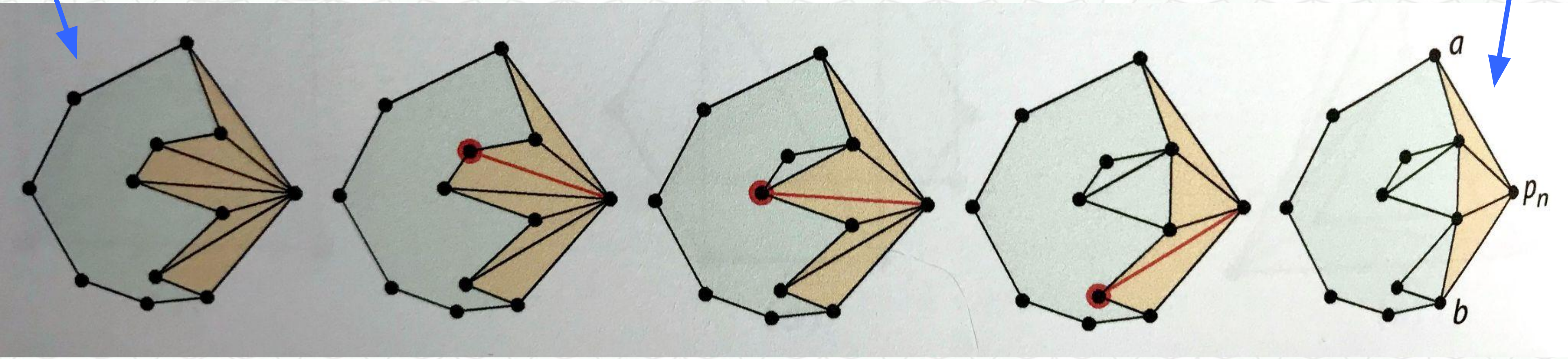
Proof by Induction & Construction in Reverse

- **… and continue to deconstruct the Line Sweep triangulation from right to left, editing the target triangulation to match as needed.**
- Therefore, the flip graph is connected!

Our target triangulation

Triangulation constructed by Line Sweep



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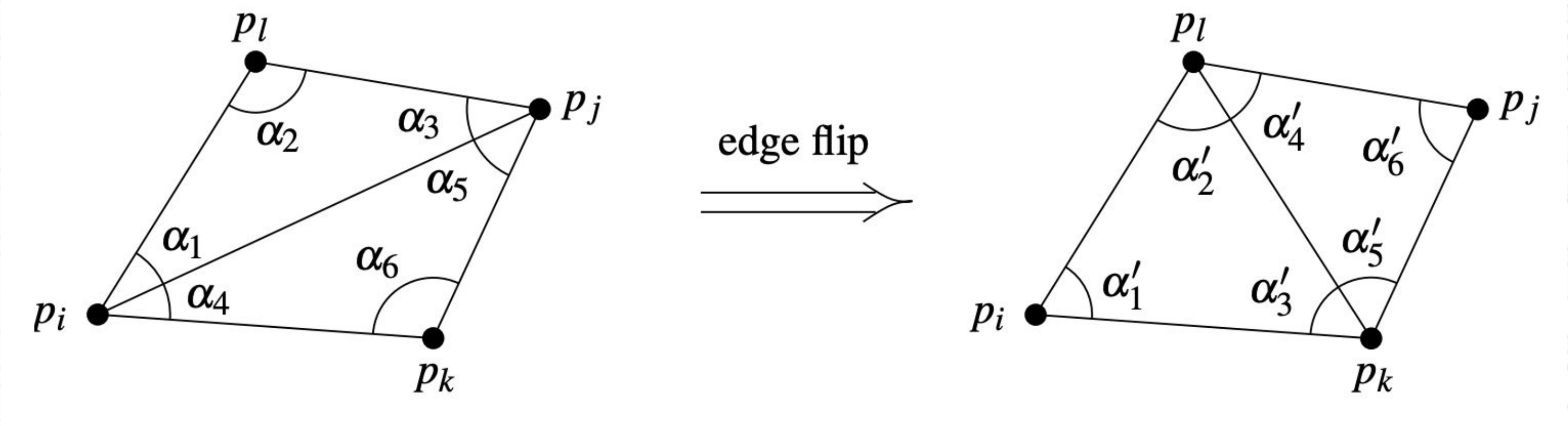
Definition: Angle-Optimal Triangulation

- We want to maximize the smallest angle
- Consider replacing each edge between two triangles with the edge connecting the other vertices of those two triangles (only possible if the combined area of the two triangles is convex)



- Edge $p_i p_j$ is said to be *illegal* if: $$\min_{1 \leqslant i \leqslant 6} \alpha_i < \min_{1 \leqslant i \leqslant 6} \alpha'_i$$

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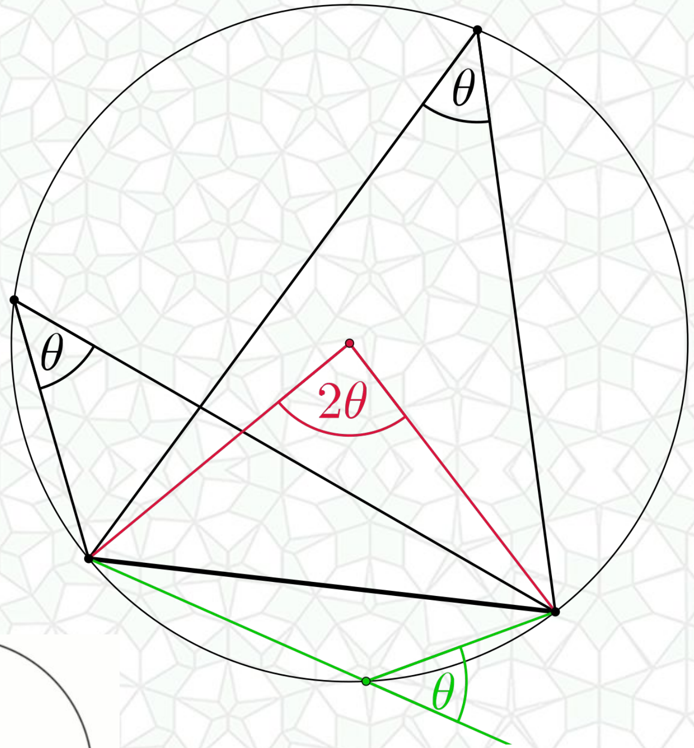
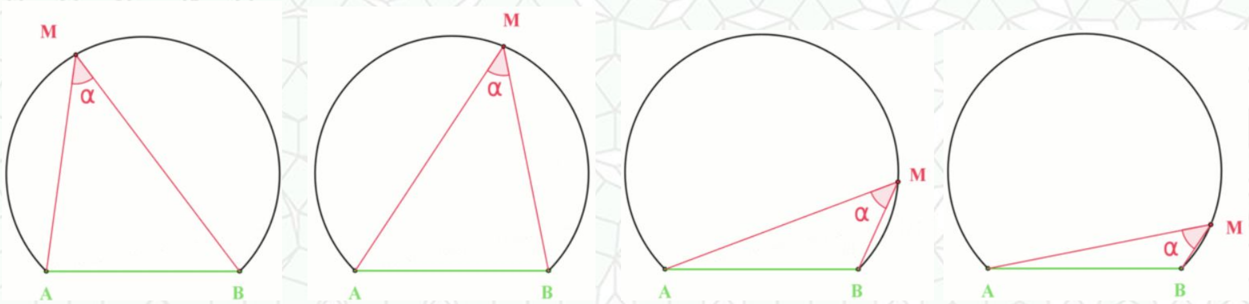
Inscribed Angle Theorem

The inscribed angle θ is half of the central angle 2θ that subtends the same arc on the circle. The angle θ does not change as its vertex is moved around on the circle.





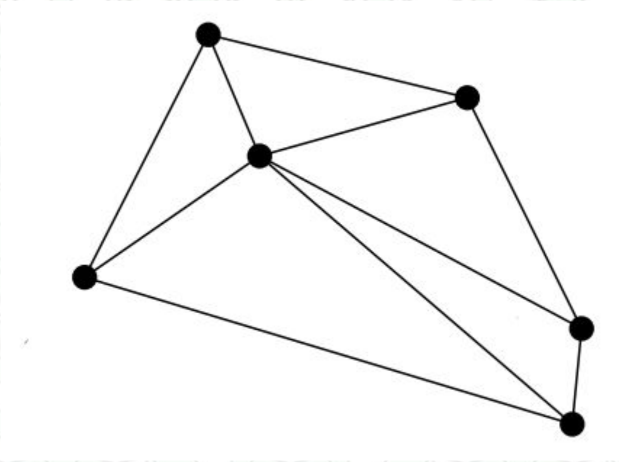
https://en.wikipedia.org/wiki/Inscribed_angle#Theorem

Constructing an Angle-Optimal Triangulation

- Brute Force

- Try all combinations of 3 vertices
- Construct the circumscribed circle
- If no other vertex is inside of that circle, keep it
- Only works if no more than 3 vertices are on the circle

- Analysis?



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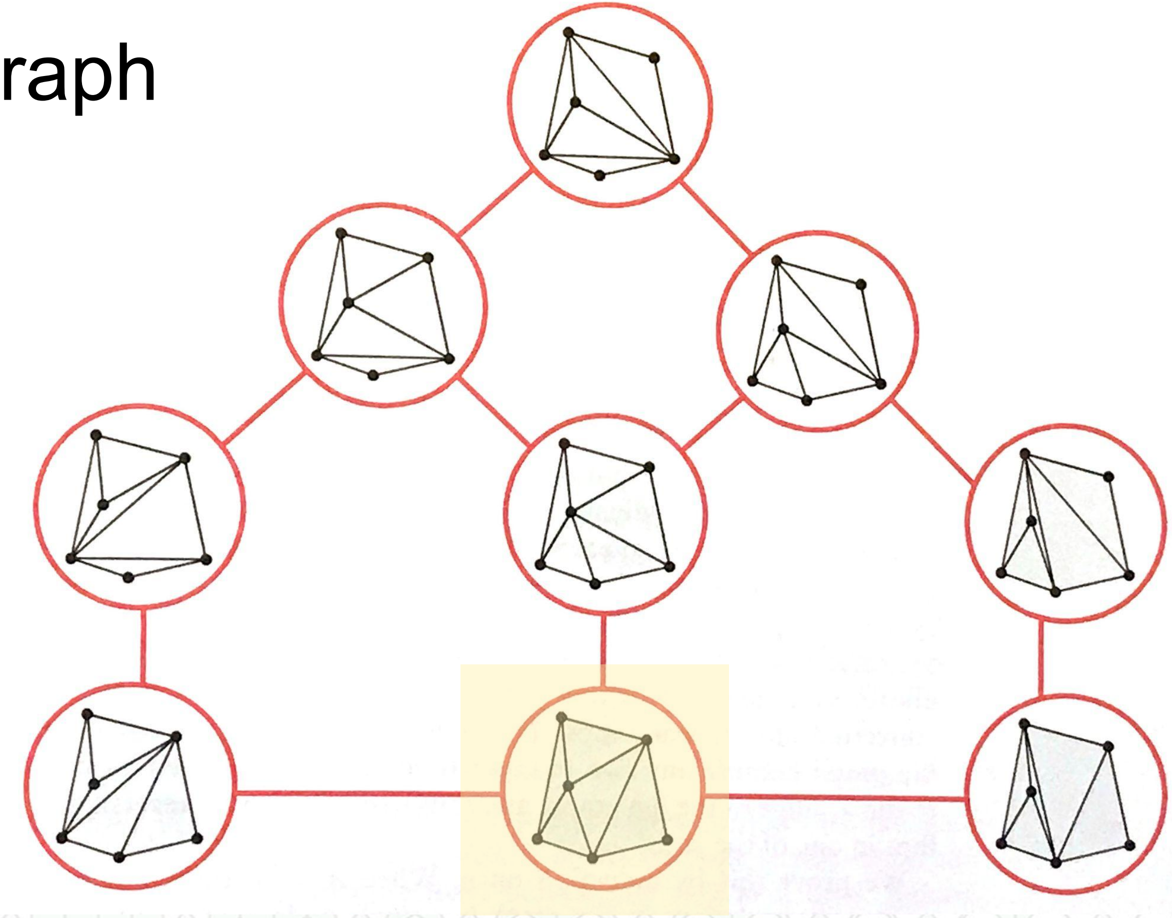
Walking the Flip Graph

- The Delaunay Triangulation is the Angle-Optimal Triangulation



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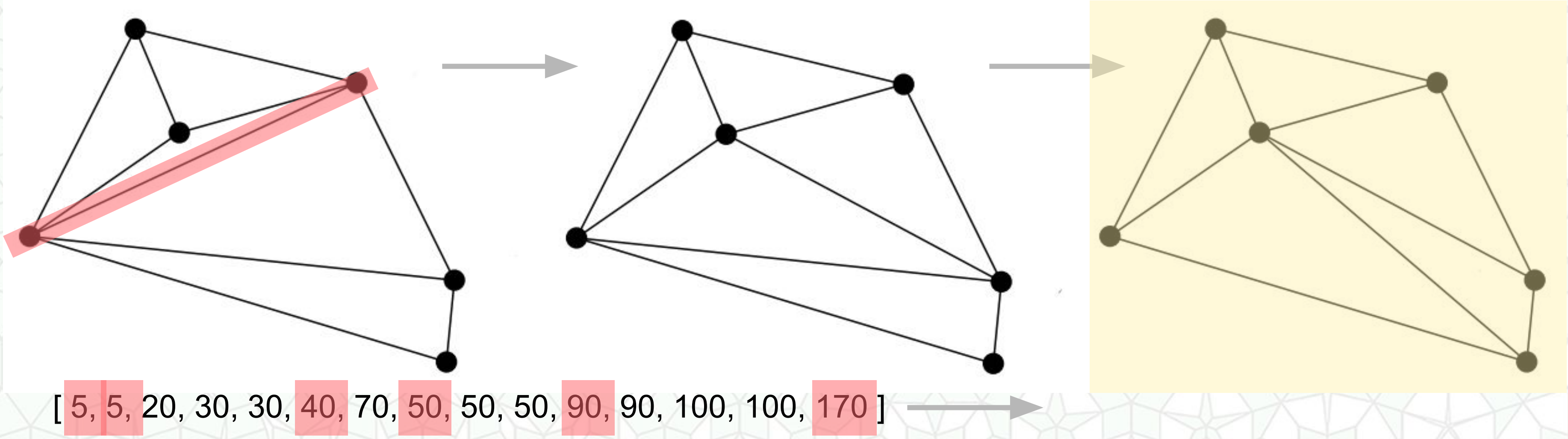
Guaranteed to Terminate? Yes!

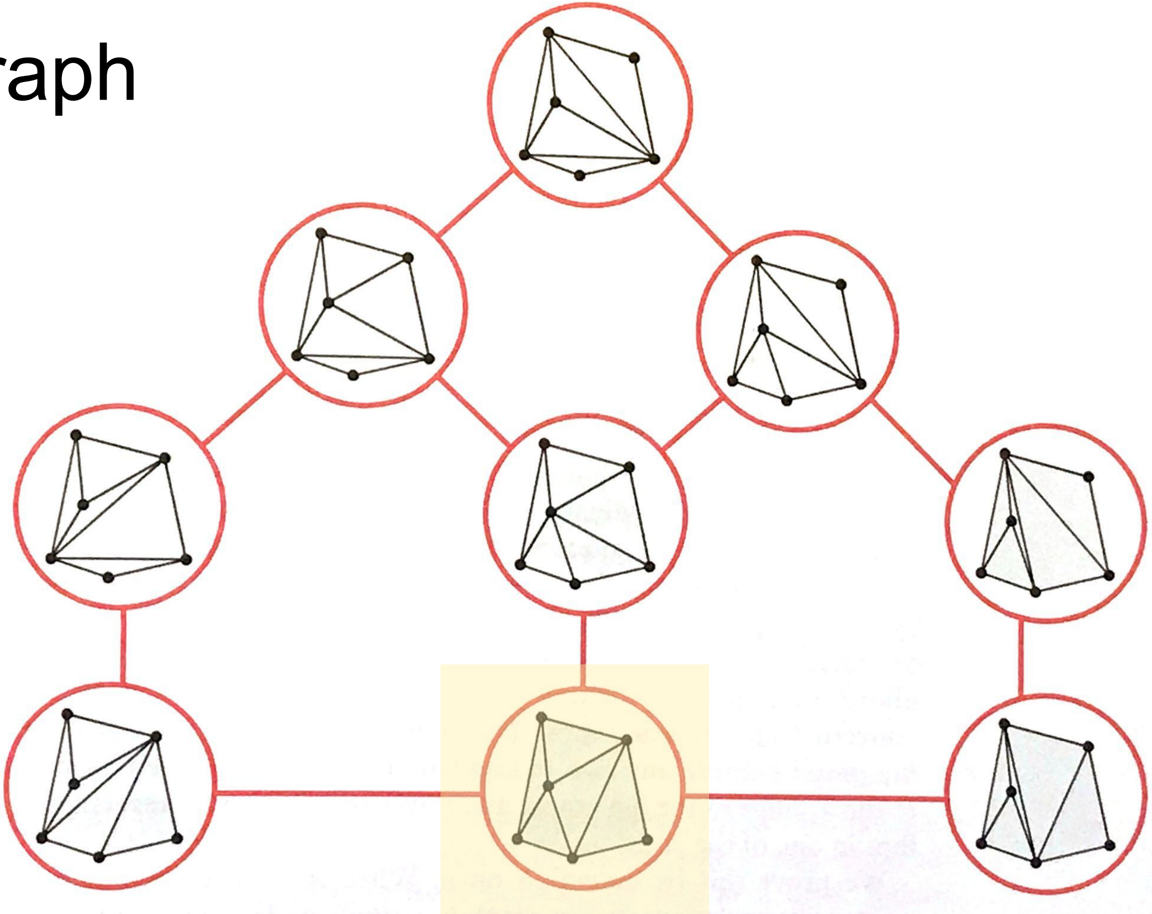
- Create a sorted vector of all of the angles of every triangle
  vector length = 3 * # of triangles
- Each edge flip replaces one of the smaller angles
- New sorted vector representation is the same up to that angle..
  (it comes lexicographically after the previous vector representation)



Walking the Flip Graph

- The Delaunay Triangulation is the Angle-Optimal Triangulation

- *How many flips are necessary to reach the Delaunay Triangulation?*



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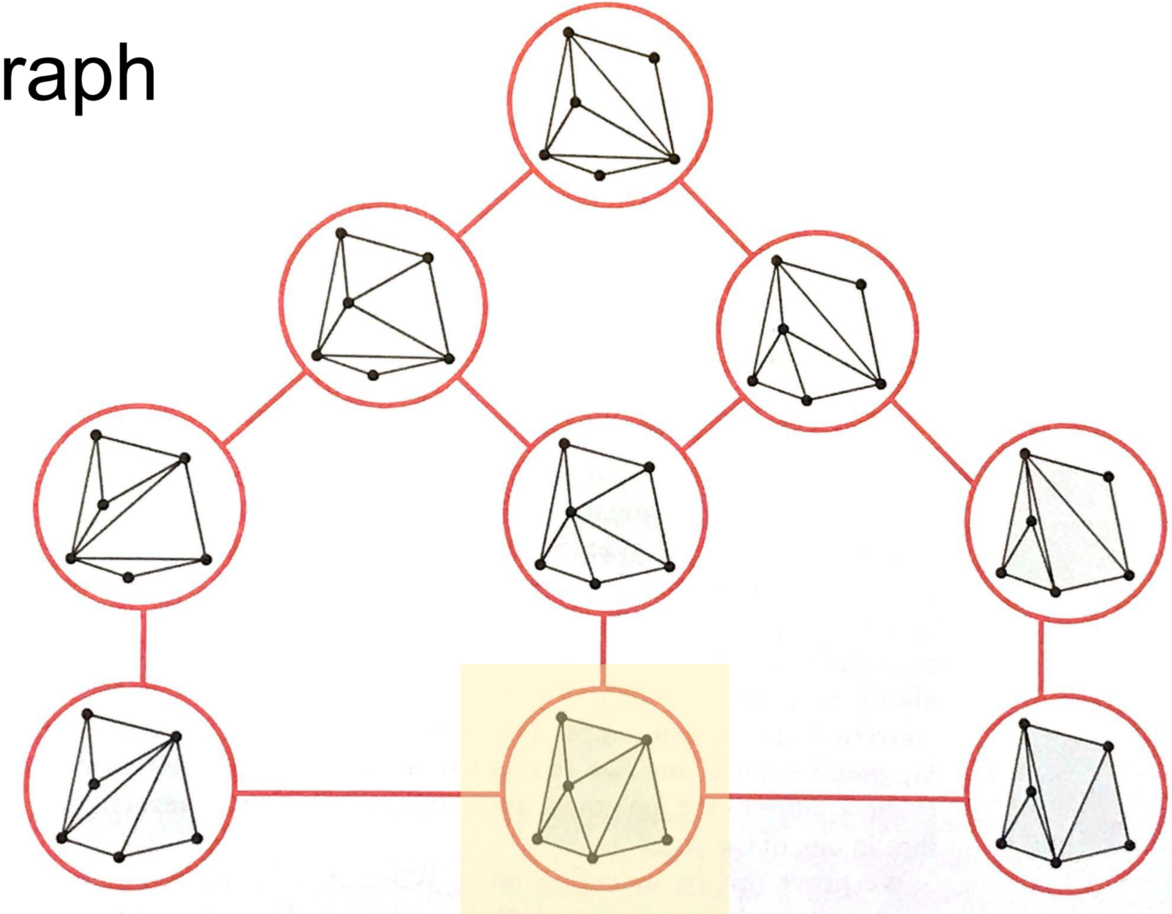
Walking the Flip Graph

- There are exponential triangulations / nodes in the flip graph…
- *Could we end up visiting every/most of these nodes in our walk??*



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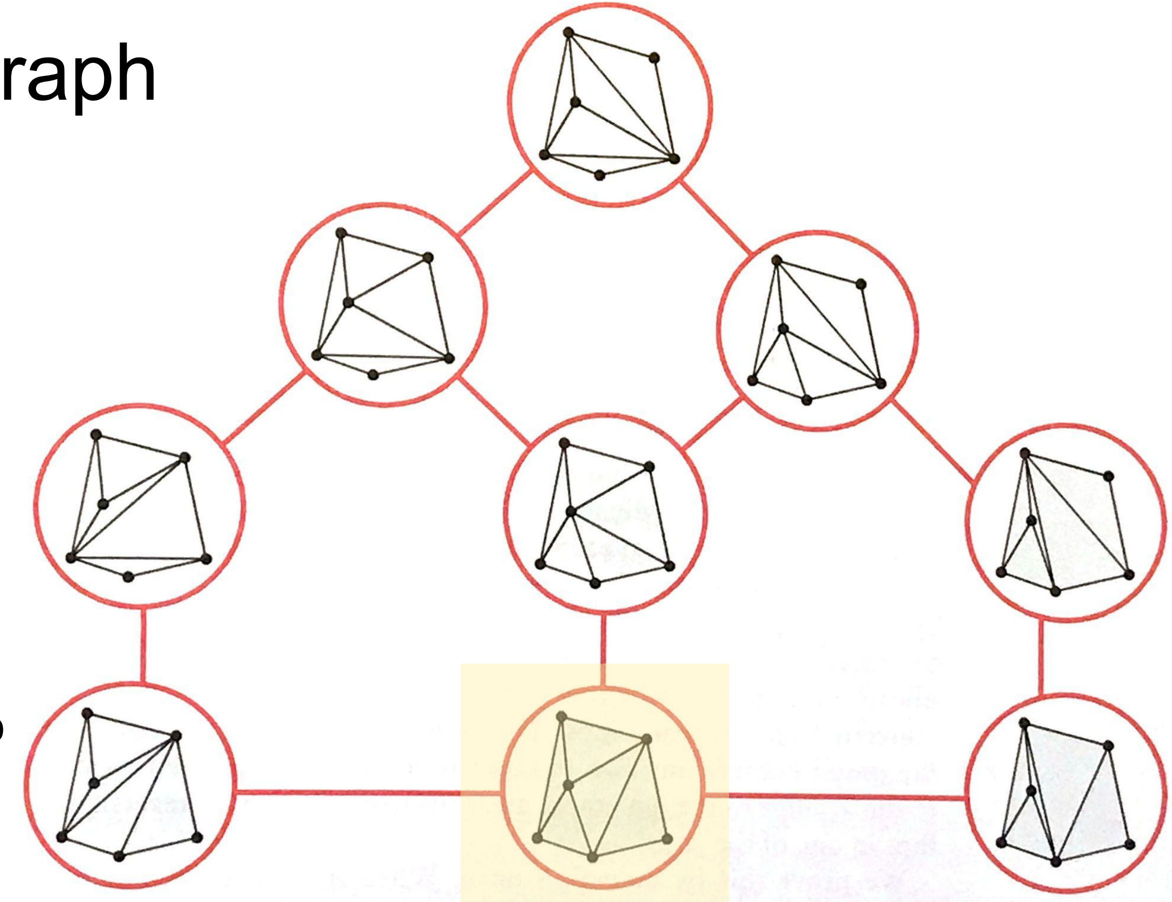
Walking the Flip Graph

- There are exponential triangulations / nodes in the flip graph…
- *Could we end up visiting every/most of these nodes in our walk??*
  - Fortunately, no..
- What is the diameter (longest path between two nodes) in the flip graph?
  - *At most (n-2) * (n-3)*

See book for proof…



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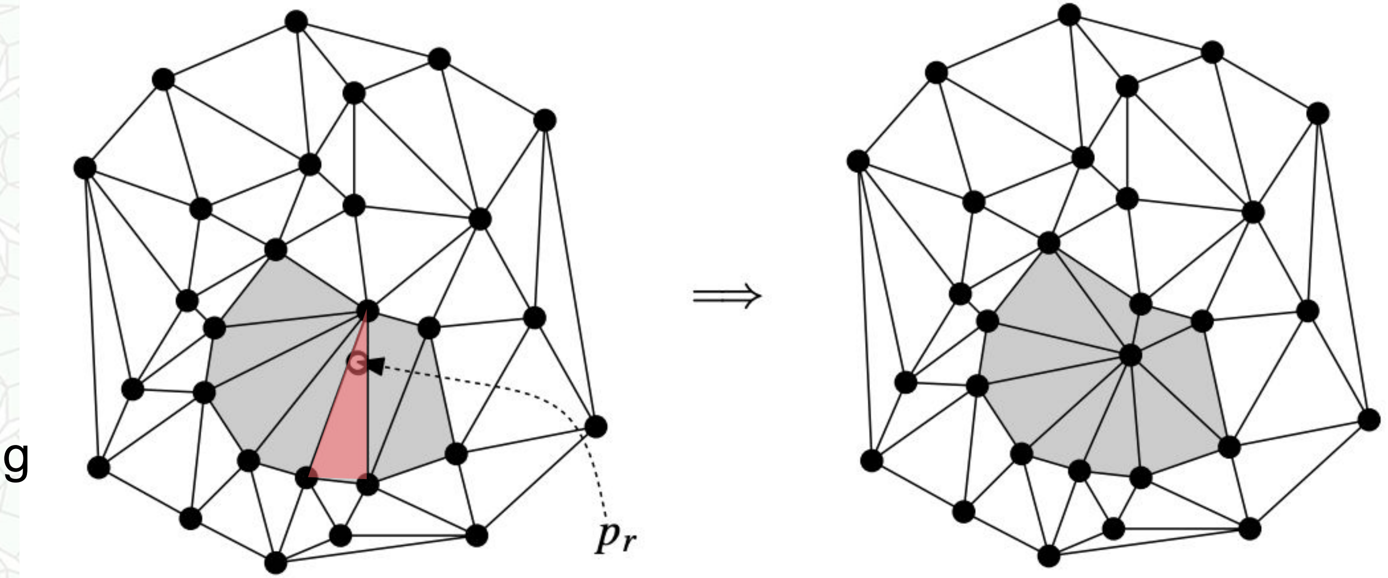
Randomized Incremental Construction of Delaunay Triangulation

- Randomize order of points and insert one at a time
- Identify which triangle contains p_r
- Split into 3 smaller triangles
- Flip neighboring edges as necessary

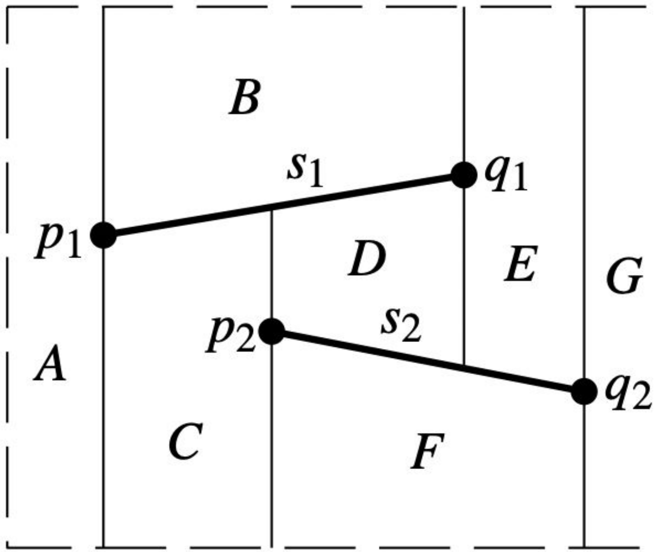


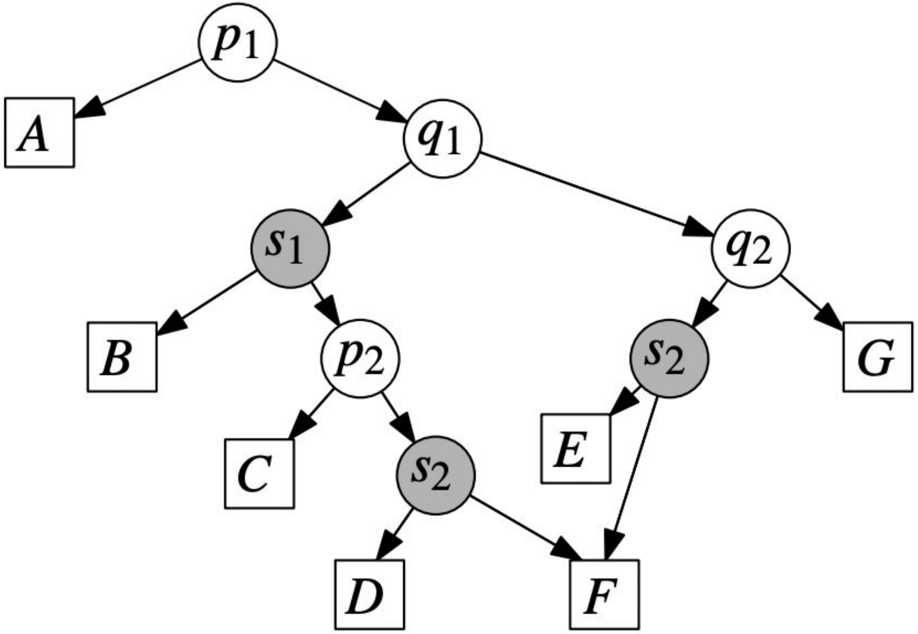
Hopefully the footprint of impact is small!

Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 9

Lecture 9: Point Location by Directed Acyclic Graph

- Intermediate notes are vertices (vertical lines) and line segments
- The leaves are the trapezoidal regions (map back to original polygons)





Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 6

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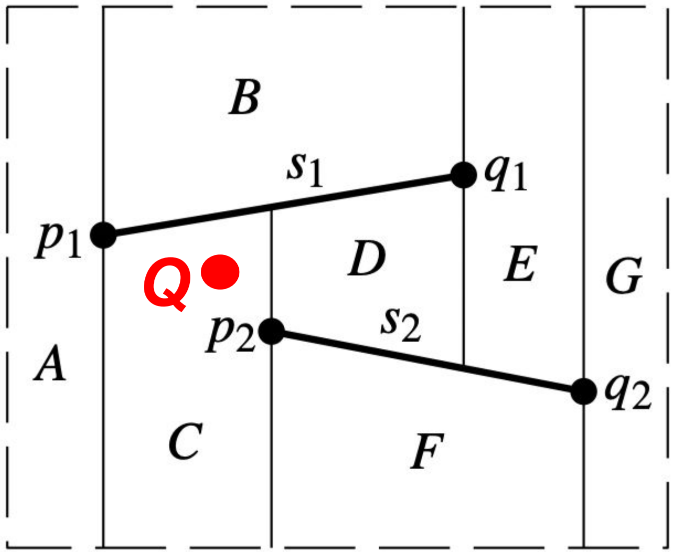
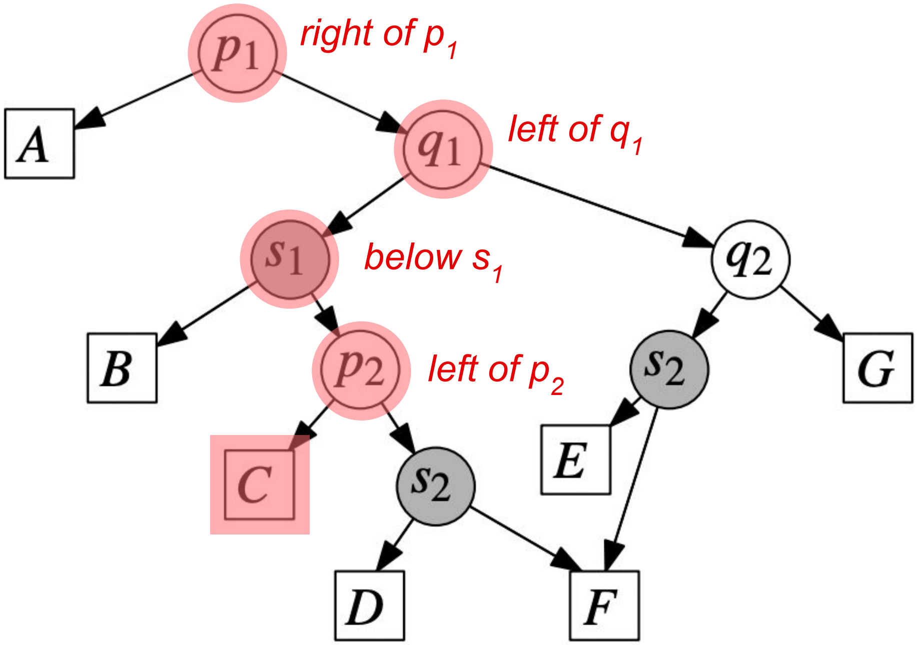
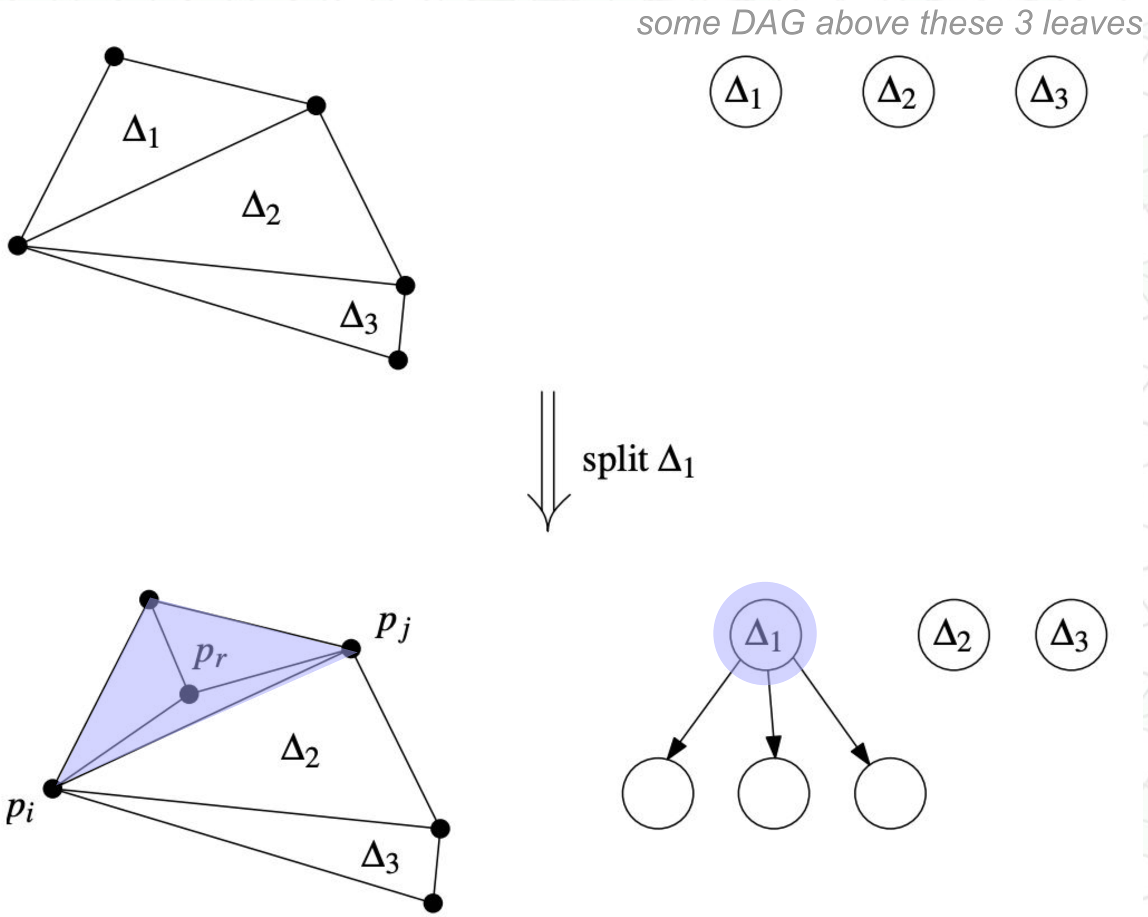




Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 6

- Similarly…
  we'll construct a directed acyclic graph (DAG) of triangles
- The leaves will be the final triangulation
- We can use this to identify which triangle contains p_r
- And then split this triangle into 3 smaller triangles

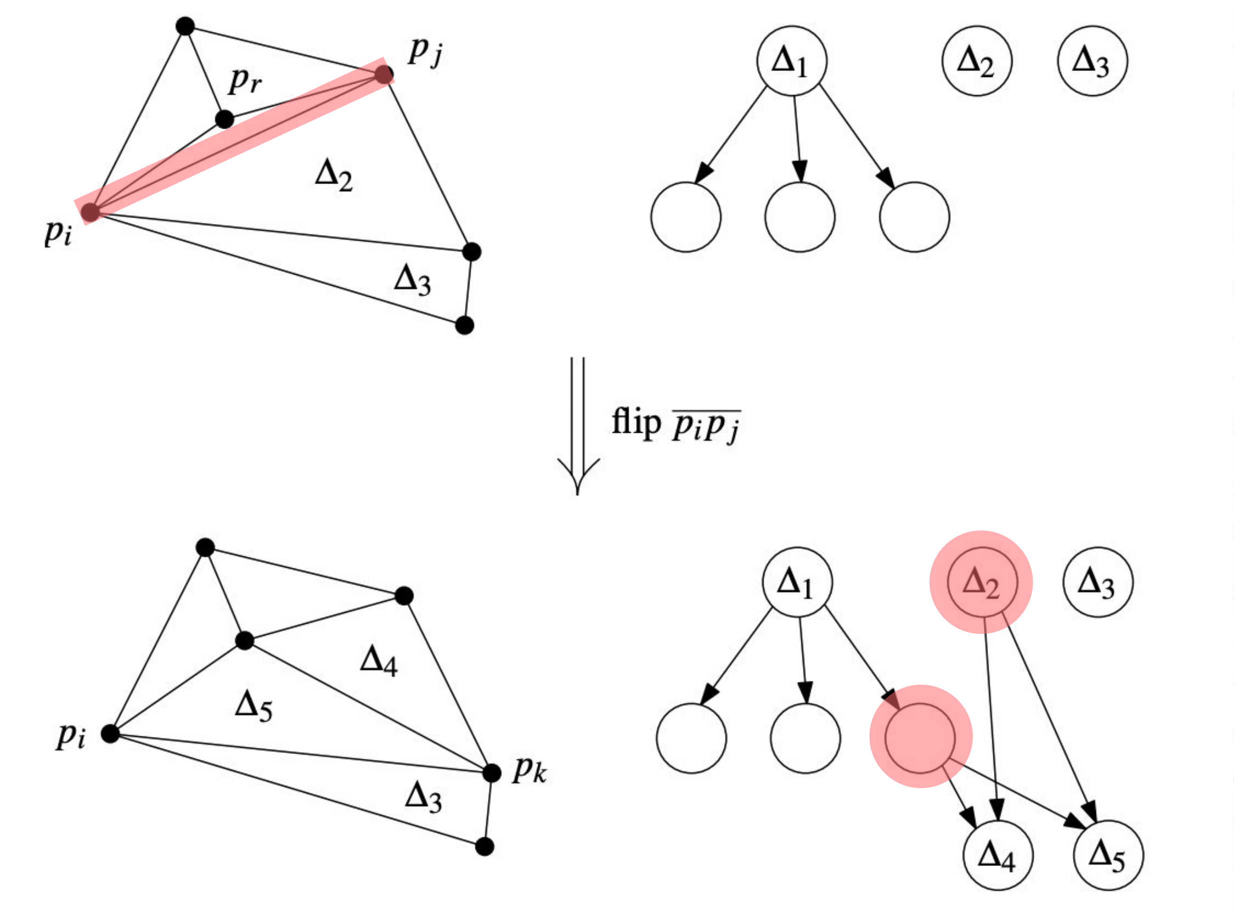


Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 9

- Check the "legality" of the edges of the new triangles

- Flip edges if necessary
- Add new triangles to DAG

- & recurse



Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 9

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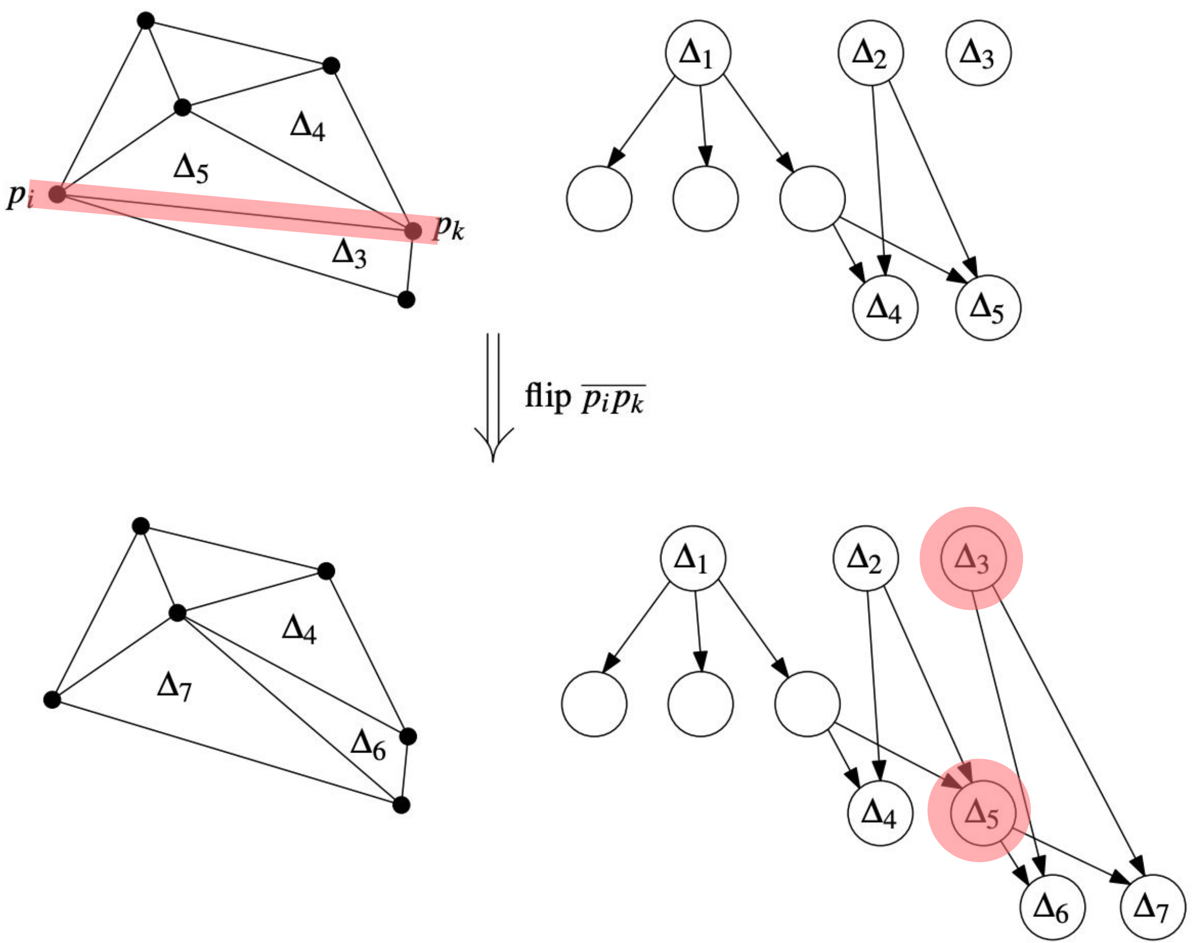


Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 9

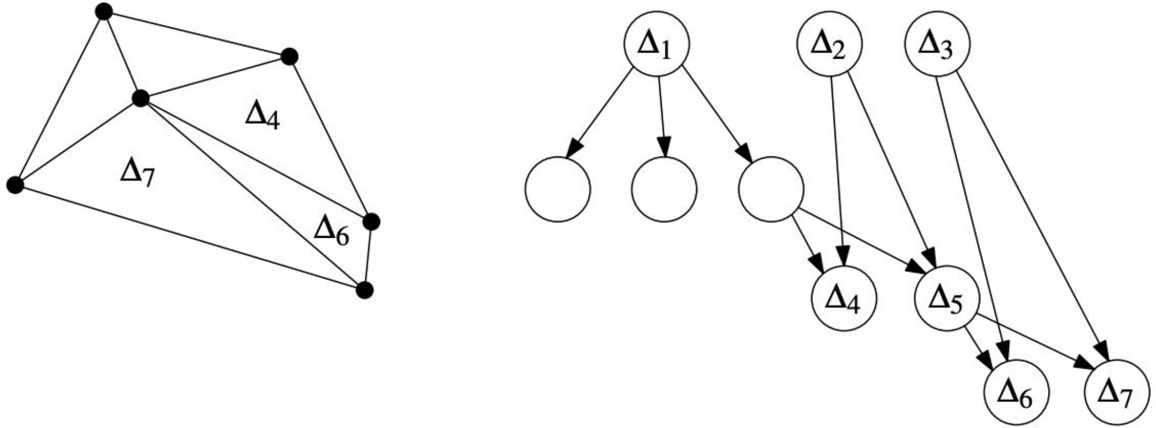
Randomized Incremental Construction of Delaunay Triangulation Analysis

- For *n* points, inserted one at a time
- Point location in DAG

- Split triangle

- Check edge legality
  Do edge flips
  & Recurse

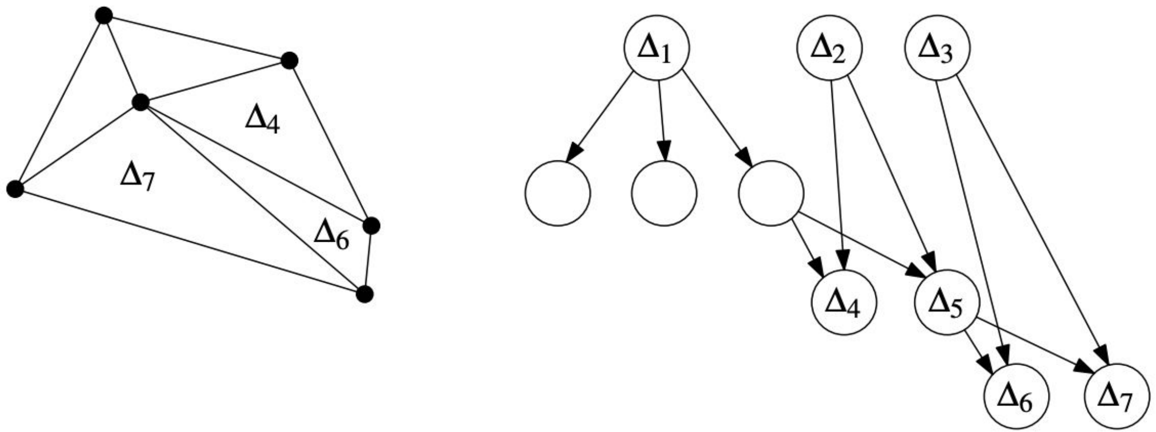
- Overall



Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 9

Randomized Incremental Construction of Delaunay Triangulation Analysis

- For *n* points, inserted one at a time
- Point location in DAG
  → *O(log n)*
- Split triangle
  → *O(1)*
- Check edge legality
  Do edge flips
  & Recurse
  → *O(1) expected*
  See book for proof…
- Overall
  → *O(n log n)*



Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 9

Outline for Today

- Final Project: Brainstorming & Feedback & Partner Matching
- Last Time / Motivation: Terrain Height Maps
- Polygon Triangulation vs. Point Set Triangulation
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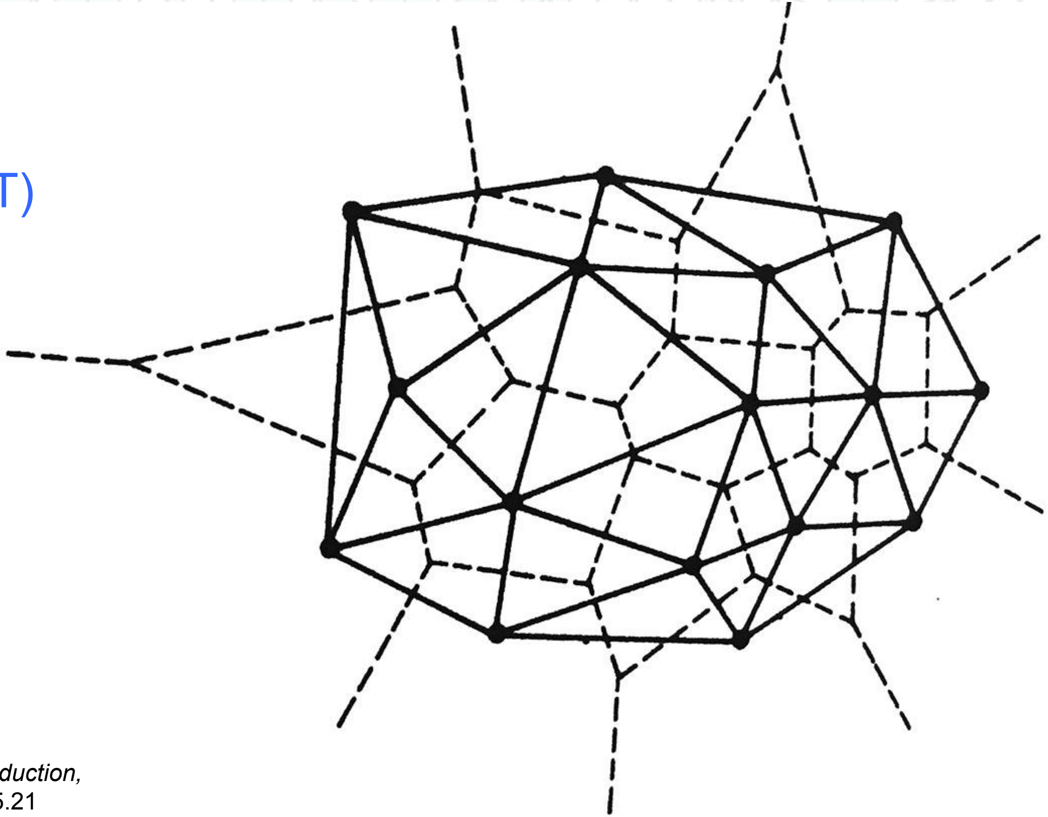
Dual: Voronoi Diagram & Delaunay Triangulation

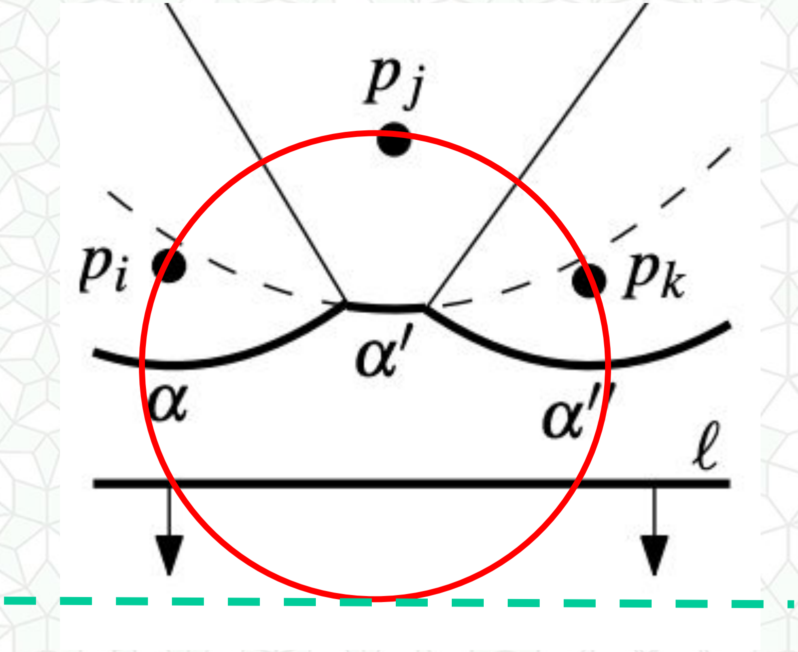
- The Voronoi Diagram (VD) *is the dual of the* Delaunay Triangulation (DT)
- Every Voronoi Site is a face in Voronoi Diagram and a vertex in the DT
- Every Voronoi Edge is an edge in the DT
- Every Voronoi Vertex is a triangle in the DT



Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.21

Lecture 10: Voronoi Sweep Line Algorithm

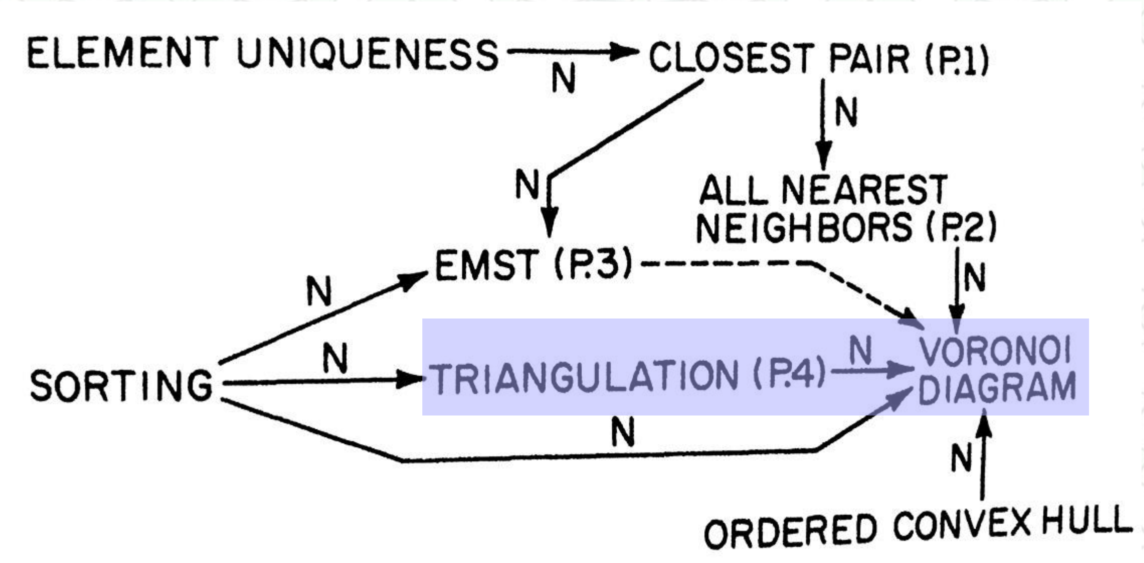
- For *n* Voronoi sites
- New Arc Events: Sort Voronoi sites vertically → *O(n log n)*
- Keep a horizontal sorted ordering of the parabolic arcs on the current beachline. 2*n* arcs maximum
- (Potential) Arc Absorption Events: For each triple of neighboring arcs α, α', α'' on the beachline, compute the **circle**, and **tangent sweep line** → *O(n)* Voronoi vertices
- Move sweep line to the next event…
- Overall: → *O(n log n)*



Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 7

Lecture 12: Problems that Reduce to Voronoi Diagram

- We can compute the Voronoi Diagram of *n* points in *O(n log n)* time and *O(n)* space.
- These other problems can be computed in *O(n)* additional time if given the Voronoi Diagram.



Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.30

- *Therefore they are also O(n log n) time and O(n) space.*

Delaunay Construction Analysis Summary

- Brute force (enumerate all triangles, construct circles, reject…)
- Construct any triangulation & Flip until all edges are legal
- Randomized Incremental Construction
- By duality, reduce to problem of Constructing the Voronoi Diagram

Delaunay Construction Analysis Summary

- Brute force (enumerate all triangles, construct circles, reject…)

  → $O(n^3 * n) = O(n^4)$

- Construct any triangulation & Flip until all edges are legal

  → $O(n^2)$

- Randomized Incremental Construction

  → $O(n \log n)$

- By duality, reduce to problem of Constructing the Voronoi Diagram

  → $O(n \log n)$

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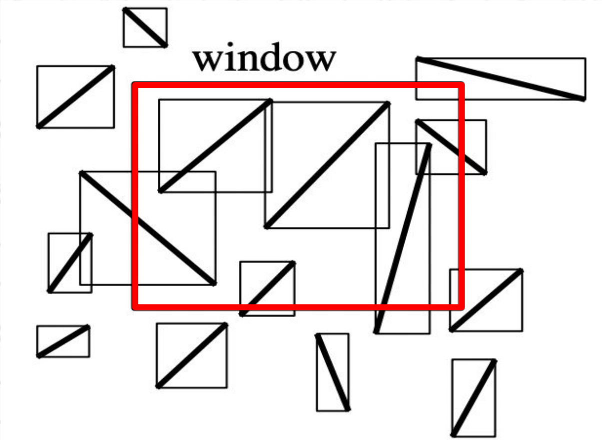

Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 10

How should we handle segments that overlap splits in a standard BST??