CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/F23/

Lecture 16: Windowing, Interval & Segment Trees

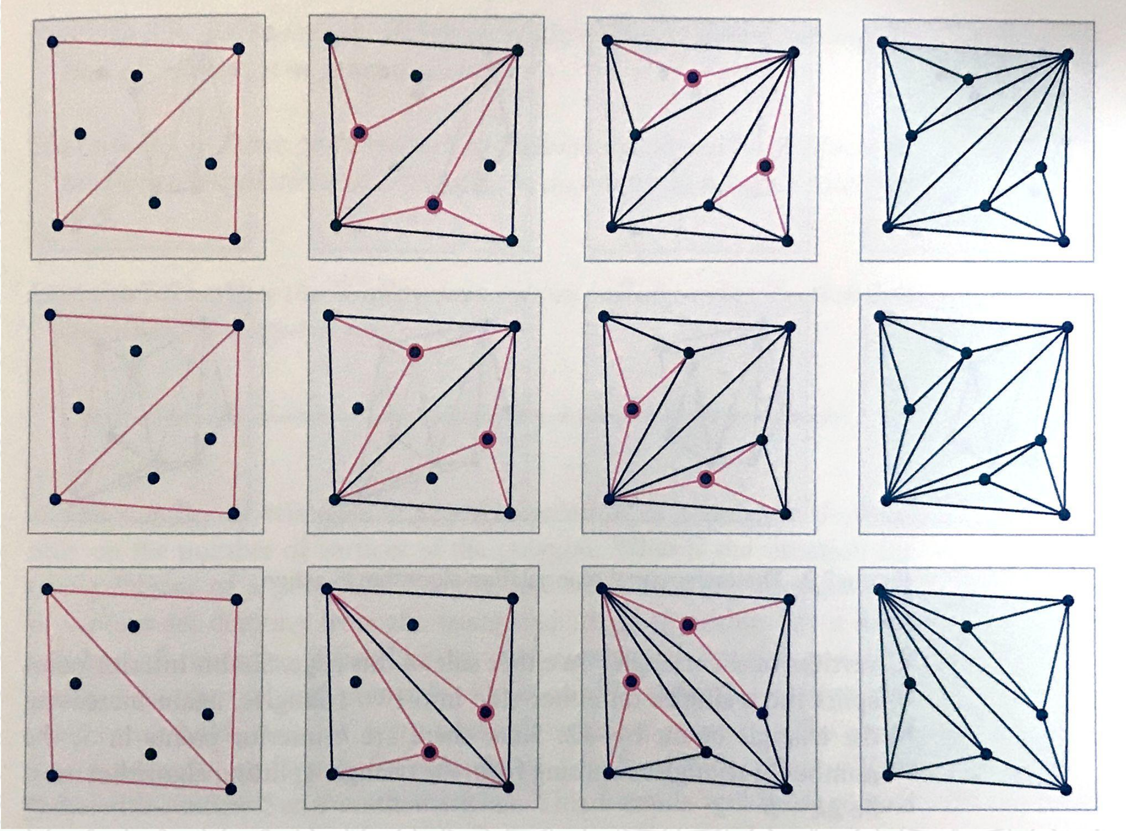
Outline for Today

- Review from Last Time: Delaunay Triangulations
- Motivation: Cartography Windowing & Data Selection
- Lecture 8 Review: Points in k-D trees and Range Trees
- 1D Interval Tree
- 1D Interval Tree Analysis
- 2D Interval Tree + Range Tree
- 2D Interval Tree + Range Tree Analysis
- 2D Axis Aligned Segment Query
- Segment Tree for general 2D Segment Query
- Segment Tree Analysis
- Next Time: Quad Trees

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3

Construction by Point Insertion

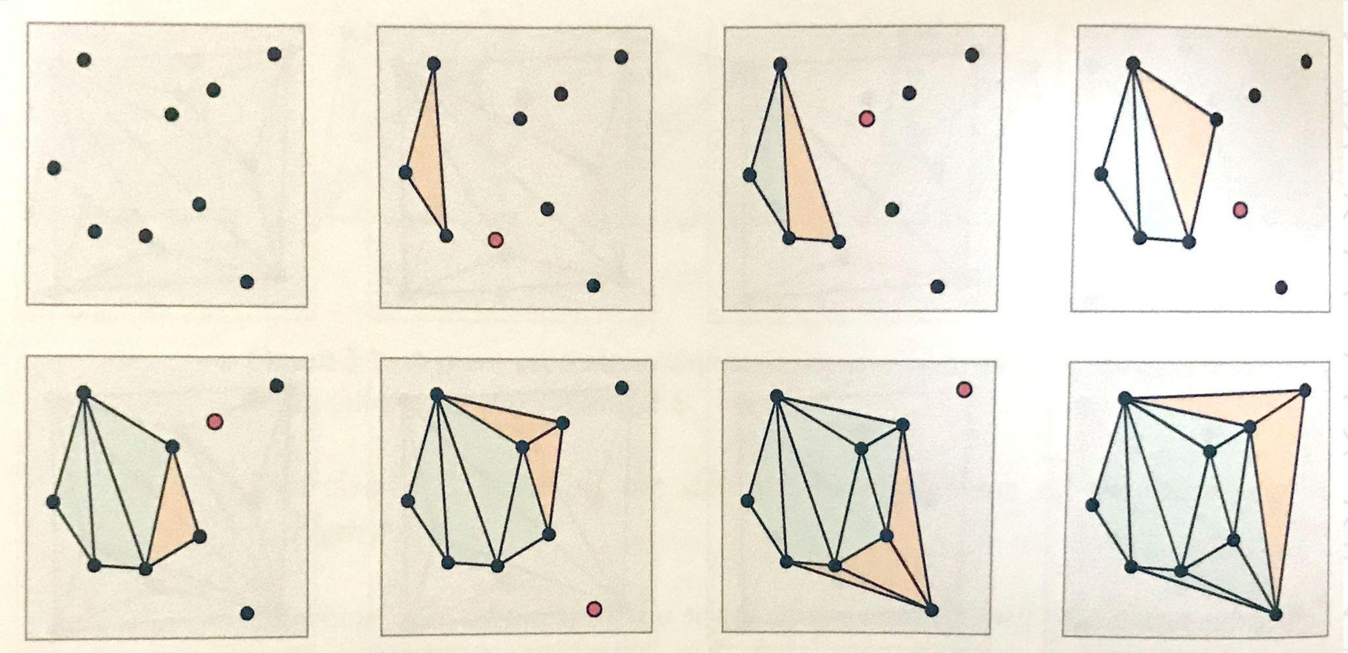
- Start with convex hull
  - Triangulate it
  - *k*-2 triangles
- For some ordering of the other points
  - Determine which triangle the point lies inside of
  - Replace that triangle with 3 triangles
  - (*n* - *k*) * 2 additional triangles
- 2**n* - *k* - 2 total triangles!



Construction by Line Sweep

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3

- Sort the input points by x
- Form a triangle with the 3 leftmost points

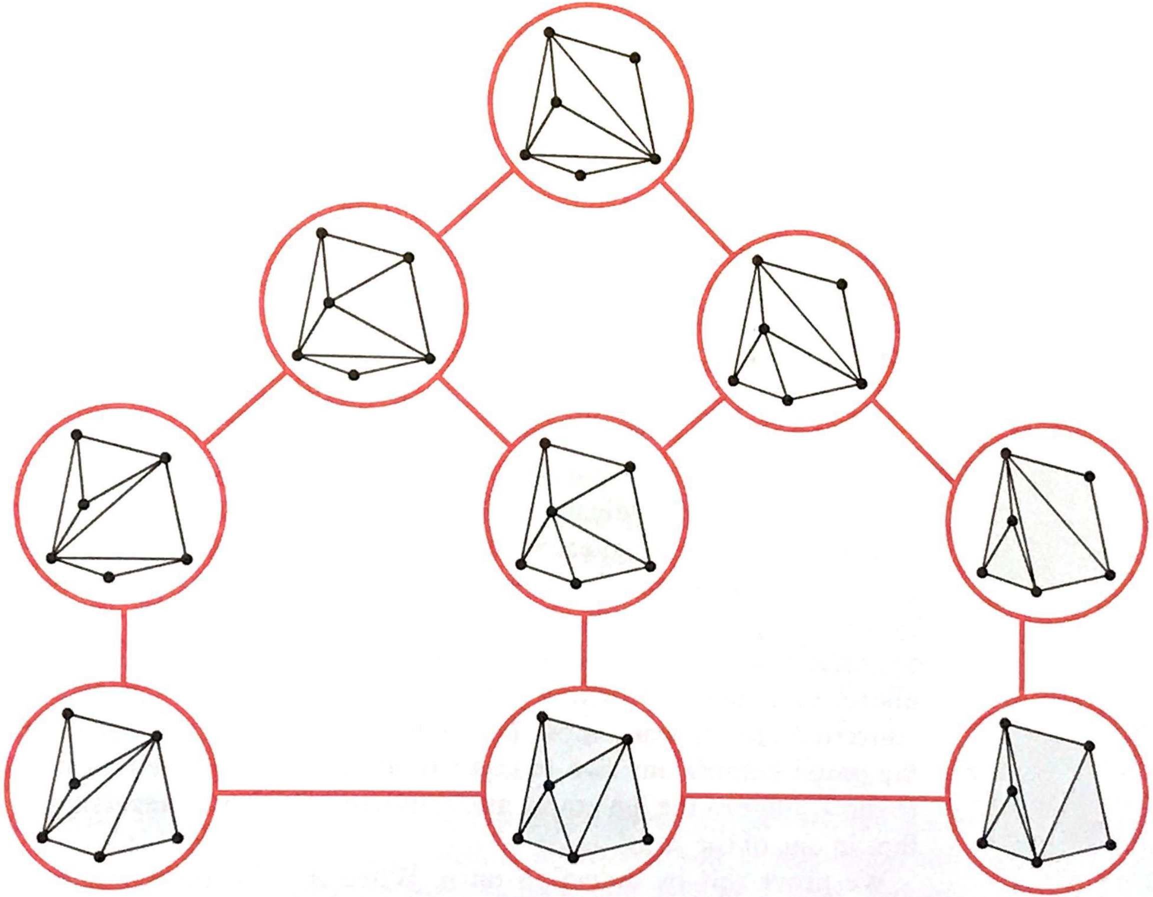


- Add every other point from left to right
  - Determine which points on the current hull are visible from the new point
  - Add a fan of triangles connecting the new point to the visible hull points

The Flip Graph

- If we did generate every triangulation…

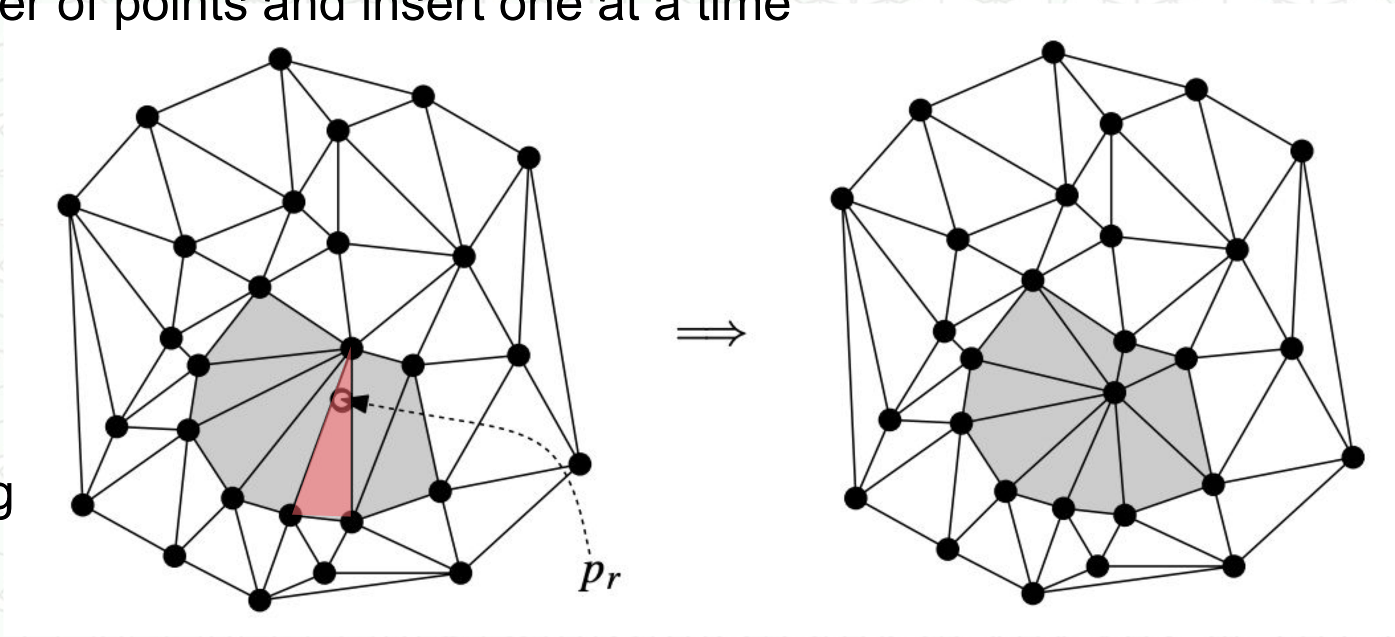
- Let’s organize the triangulations as nodes in a graph
- We’ll put an edge between two nodes if flipping a single edge converts one triangulation into the other triangulation



“Discrete and Computational Geometry”, Devadoss & O’Rourke,
Princeton University Press 2011, Chapter 3

Randomized Incremental Construction of Delaunay Triangulation

- Randomize order of points and insert one at a time
- Identify which triangle contains p_r
- Split into 3 smaller triangles
- Flip neighboring edges as necessary



Hopefully the footprint of impact is small!

Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 9

Delaunay Construction Analysis Summary

- Brute force (enumerate all triangles, construct circles, reject…)

  → $O(n^3 * n) = O(n^4)$

- Construct any triangulation & Flip until all edges are legal

  → $O(n^2)$

- Randomized Incremental Construction

  → $O(n \log n)$

- By duality, reduce to problem of Constructing the Voronoi Diagram

  → $O(n \log n)$

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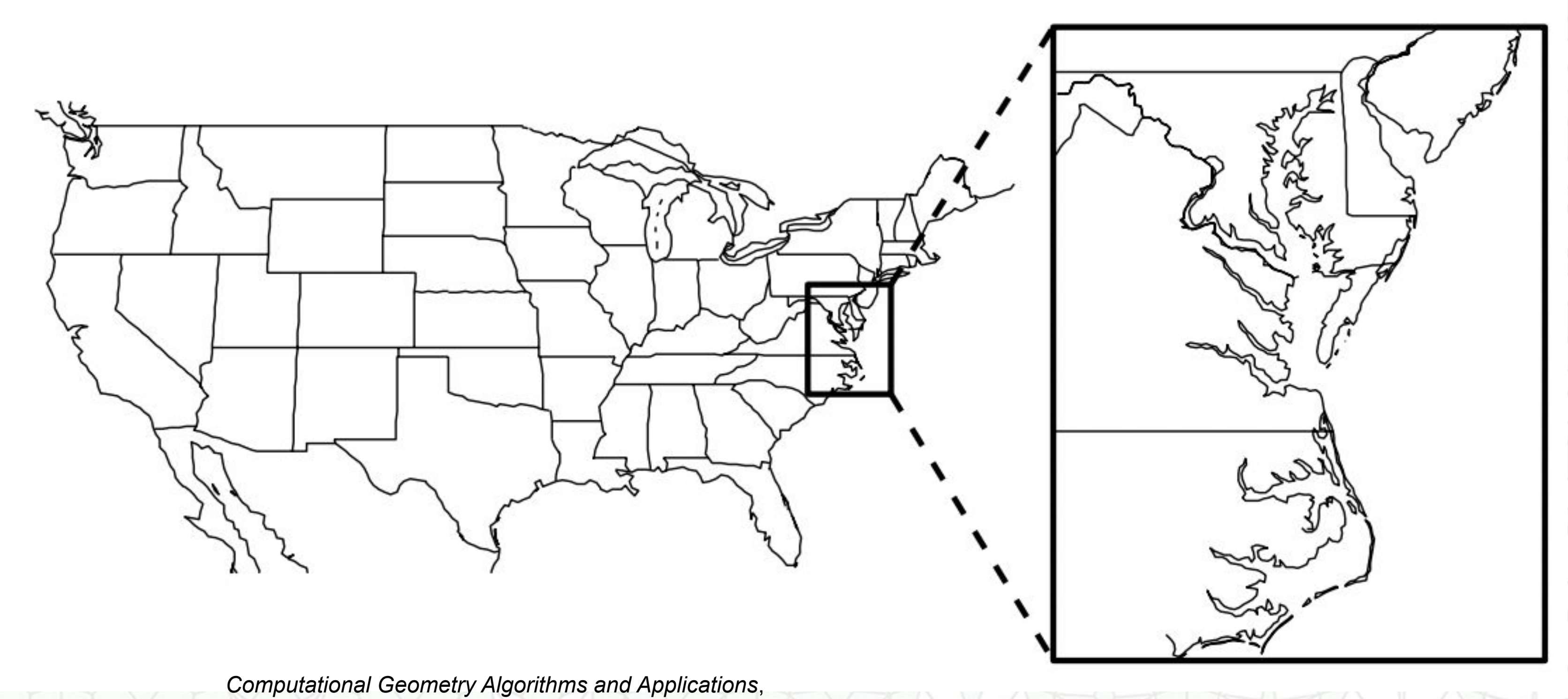
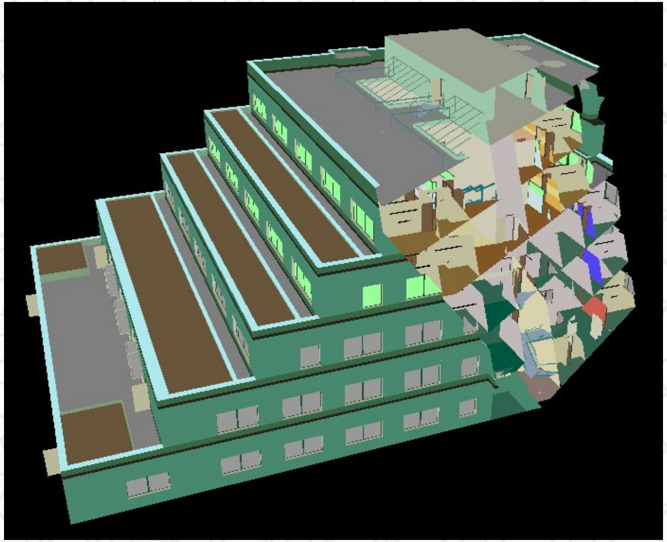
Motivation: Cartography (Map-Making)

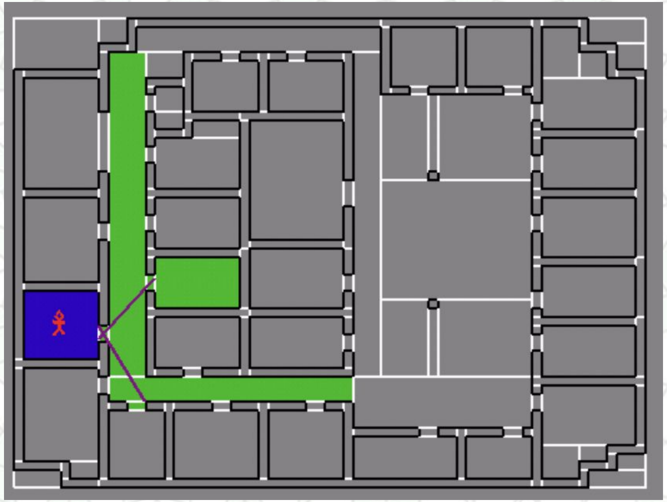
- Select a small rectangular region to display in a window at larger scale



Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 10

Motivation: Visibility

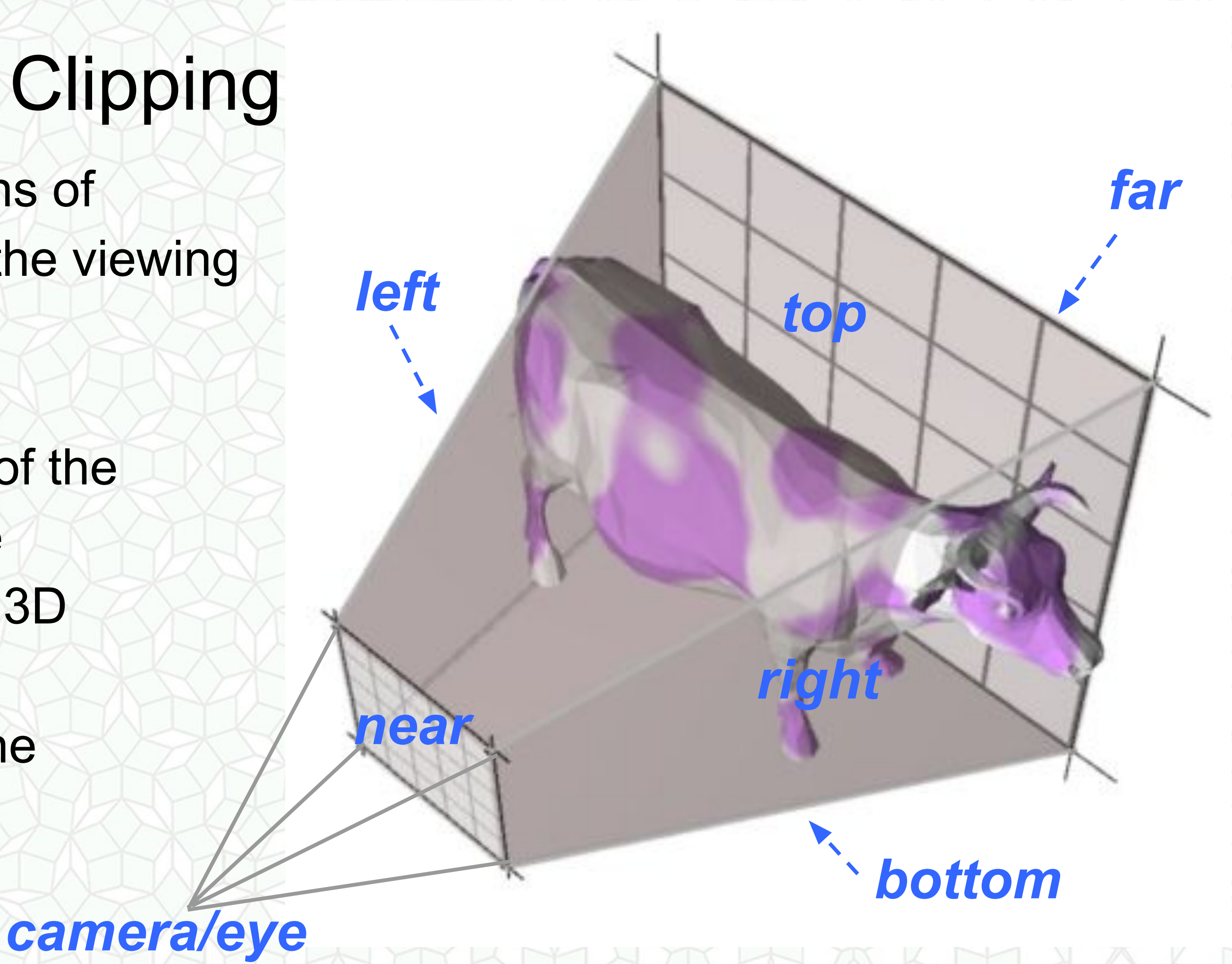




Seth Teller, PhD thesis, 1992,
Berkeley Soda Hall walkthrough

Graphics: 3D Clipping

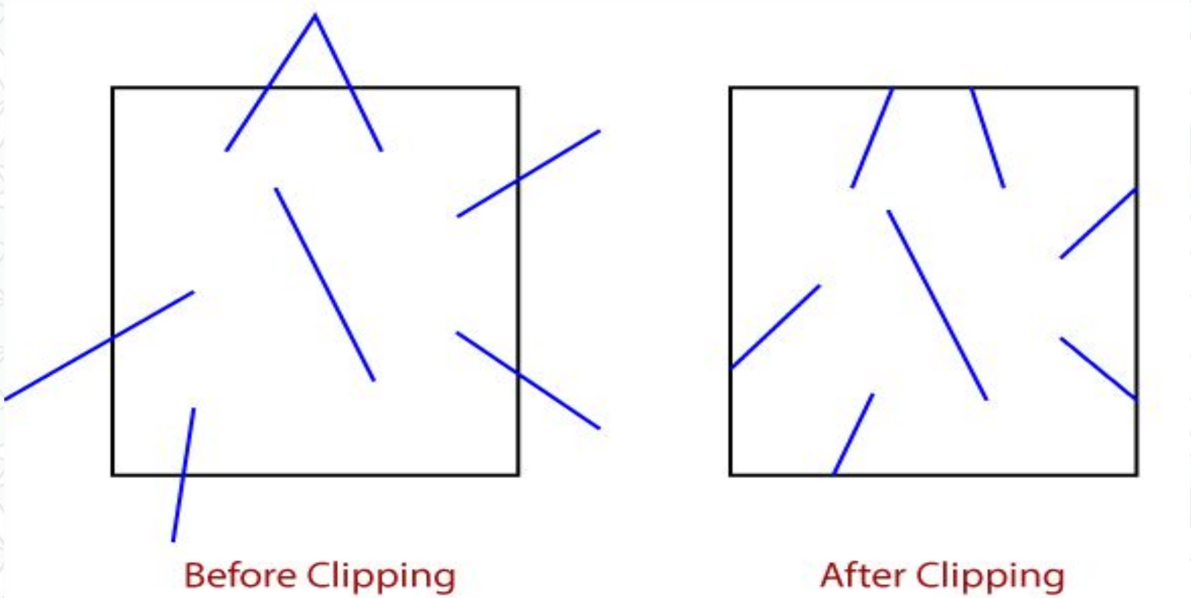
- Eliminate portions of objects outside the viewing frustum
- View Frustum
  - boundaries of the image plane projected in 3D
  - a near & far clipping plane



Graphics: 2D Clipping

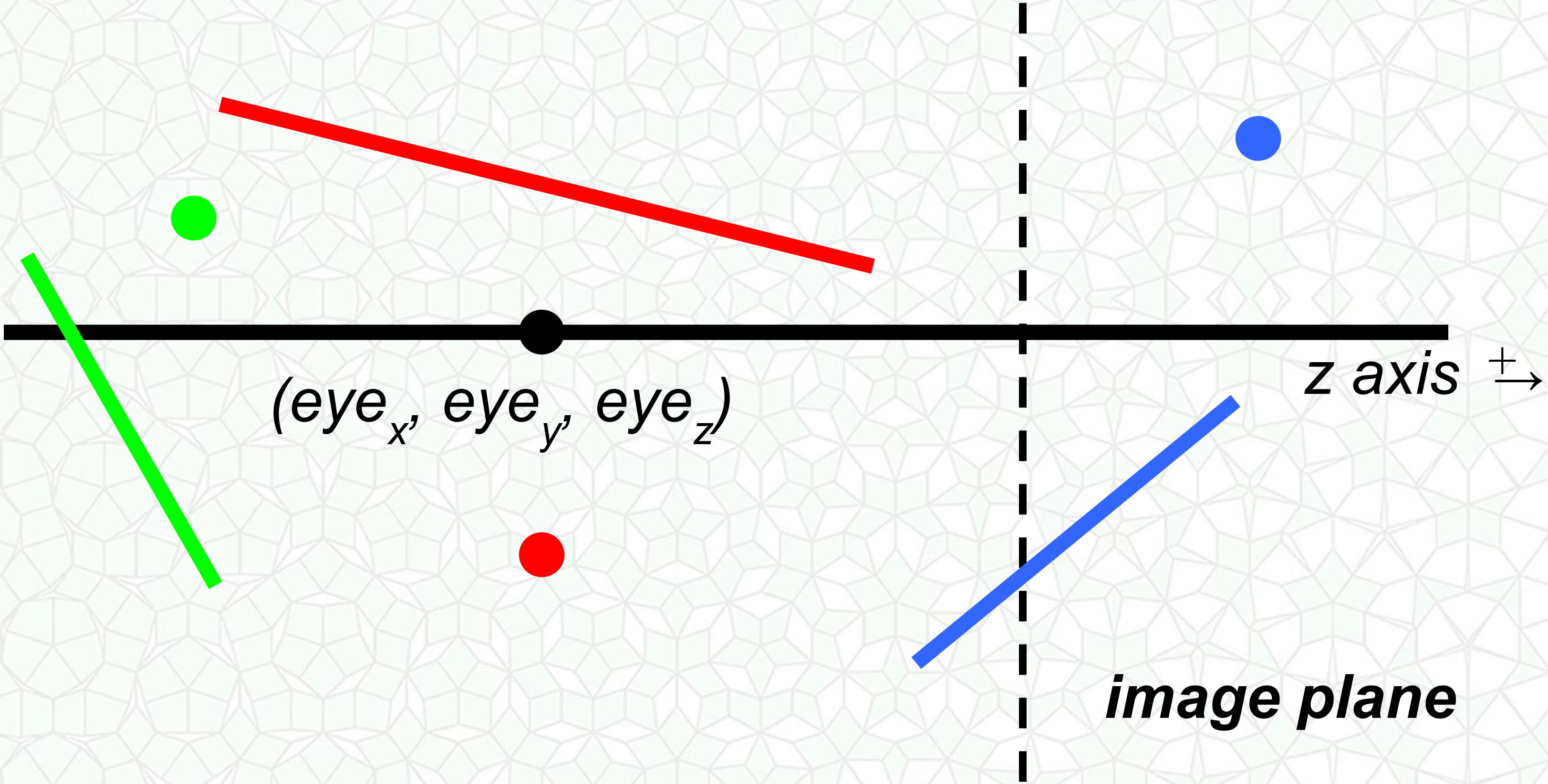
Why do it?

- Reduce amount of geometry going through graphics pipeline
- Prevent rendering bugs from overflow, wraparound, things behind the camera, etc.

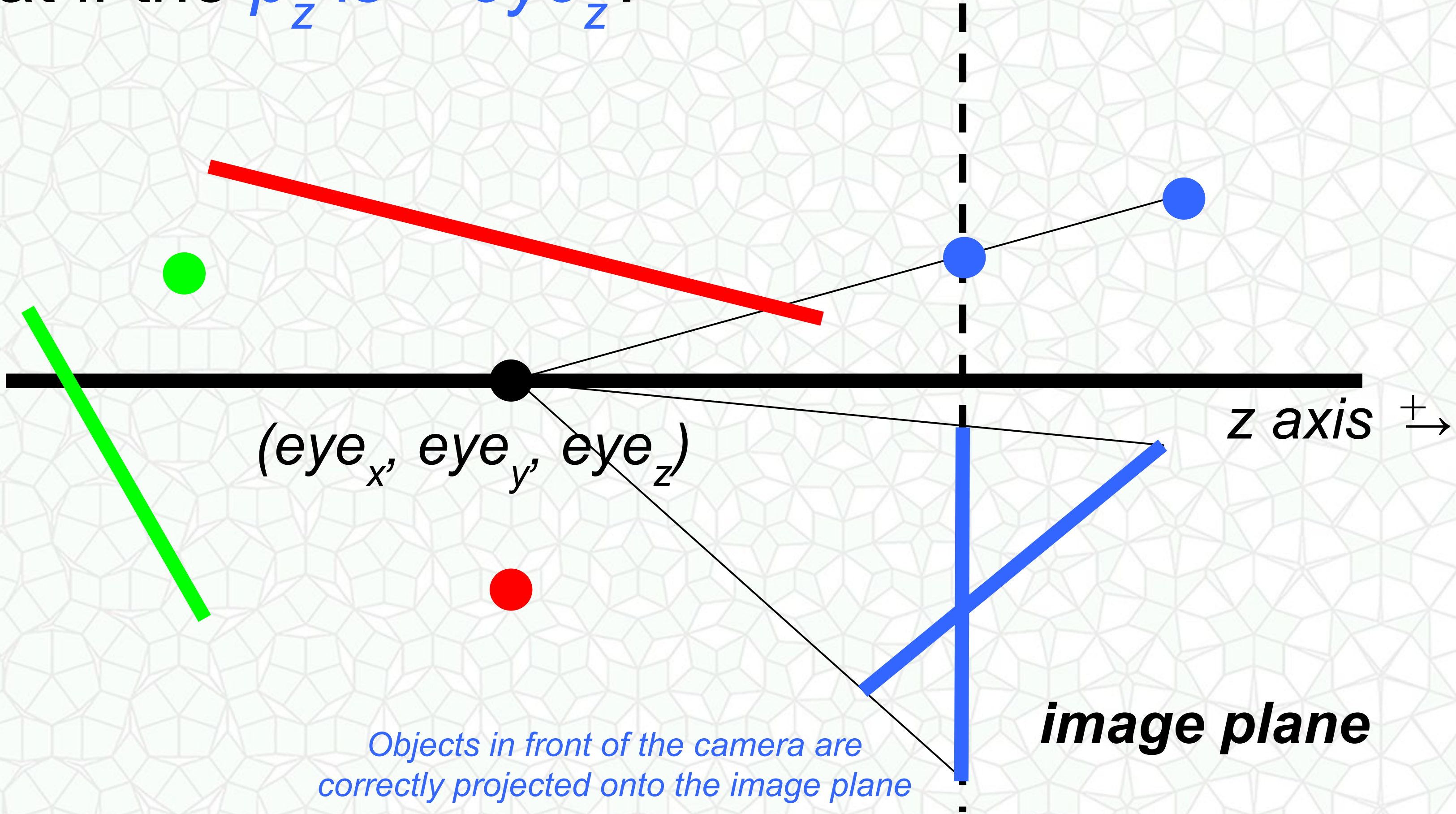


https://www.tutorialandexample.com/clipping-in-computer-graphics

Perspective Projection onto Image Plane

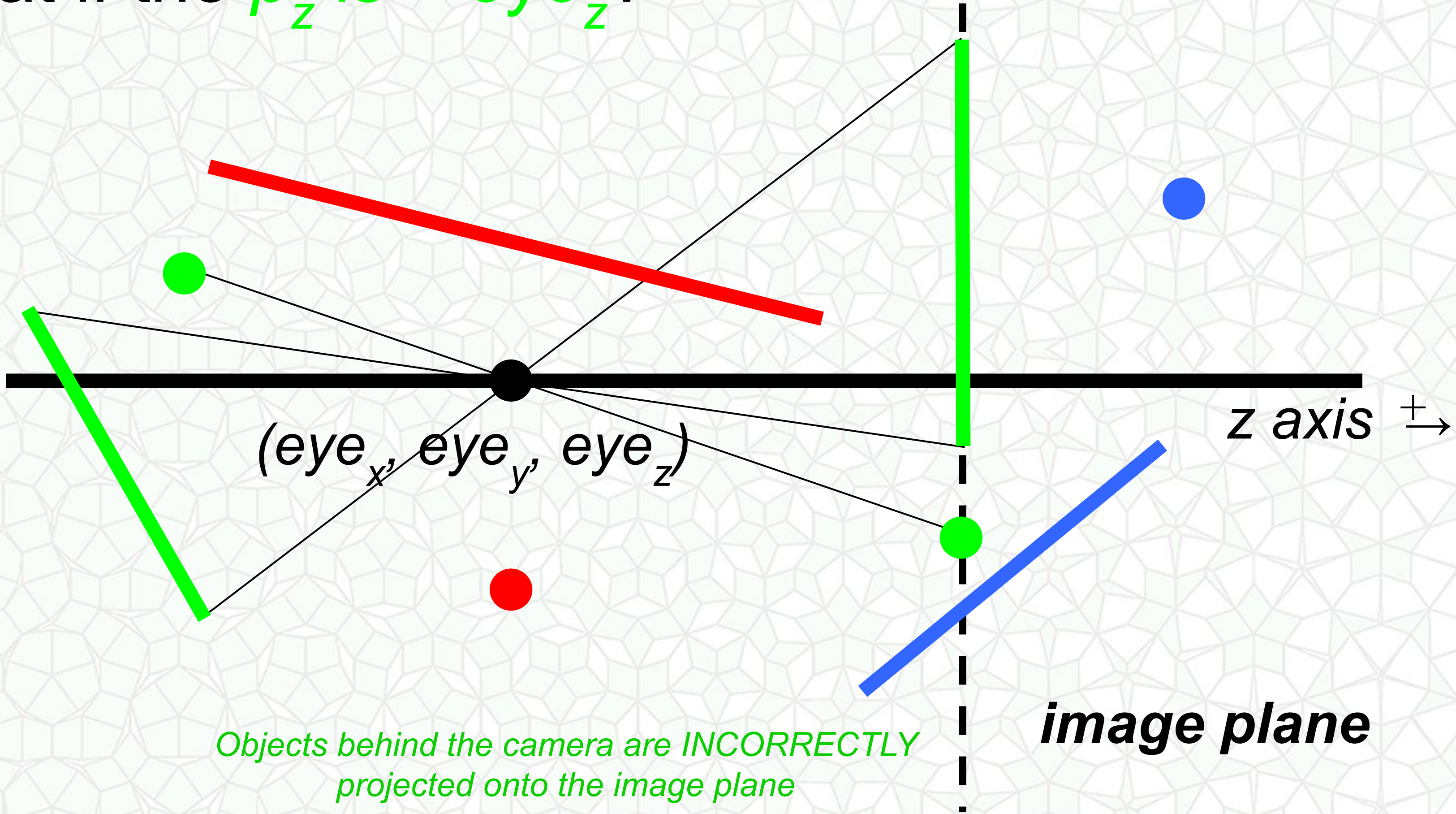


What if the p_z is > eye_z?



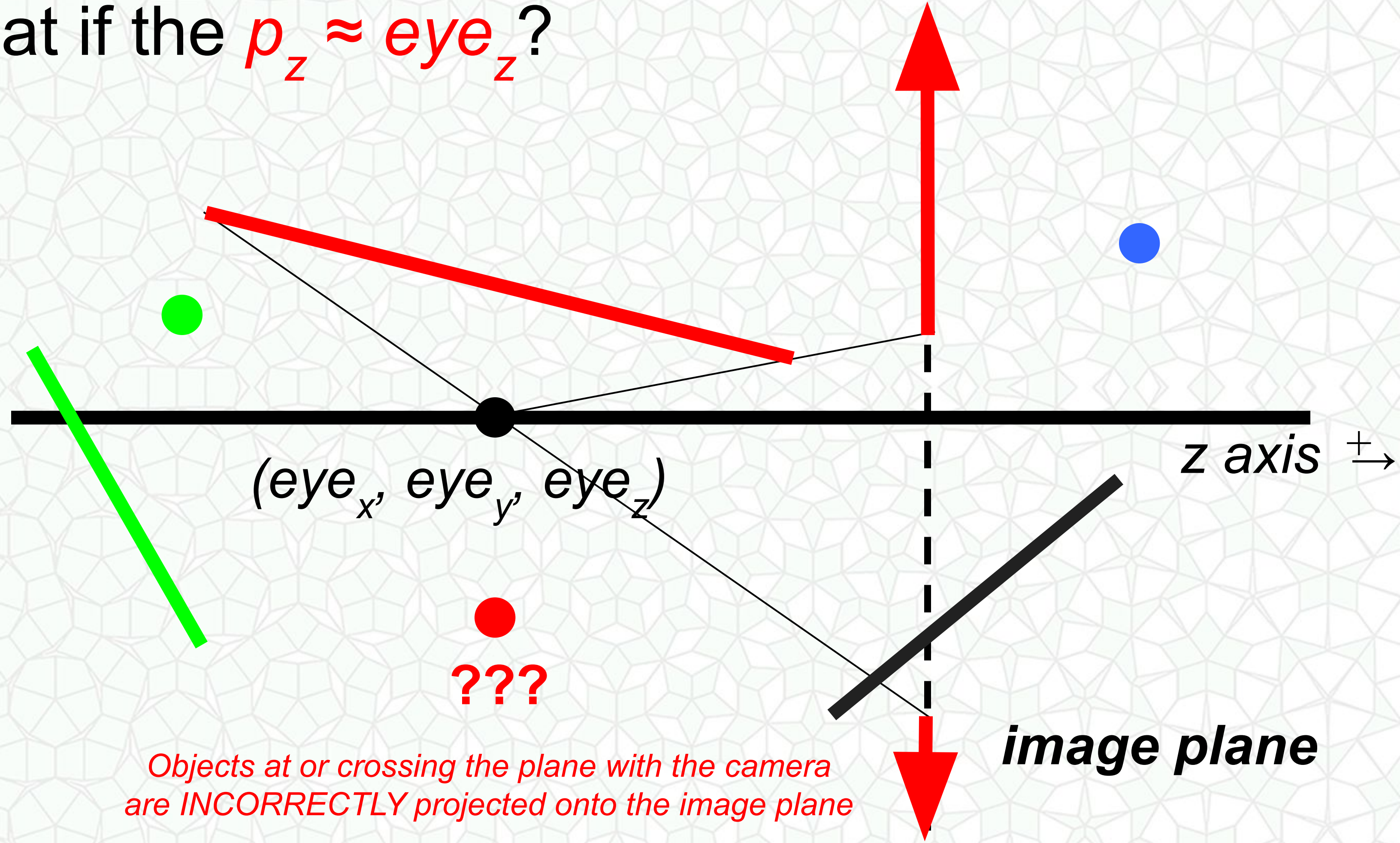
Objects in front of the camera are correctly projected onto the image plane

What if the p_z is < eye_z?



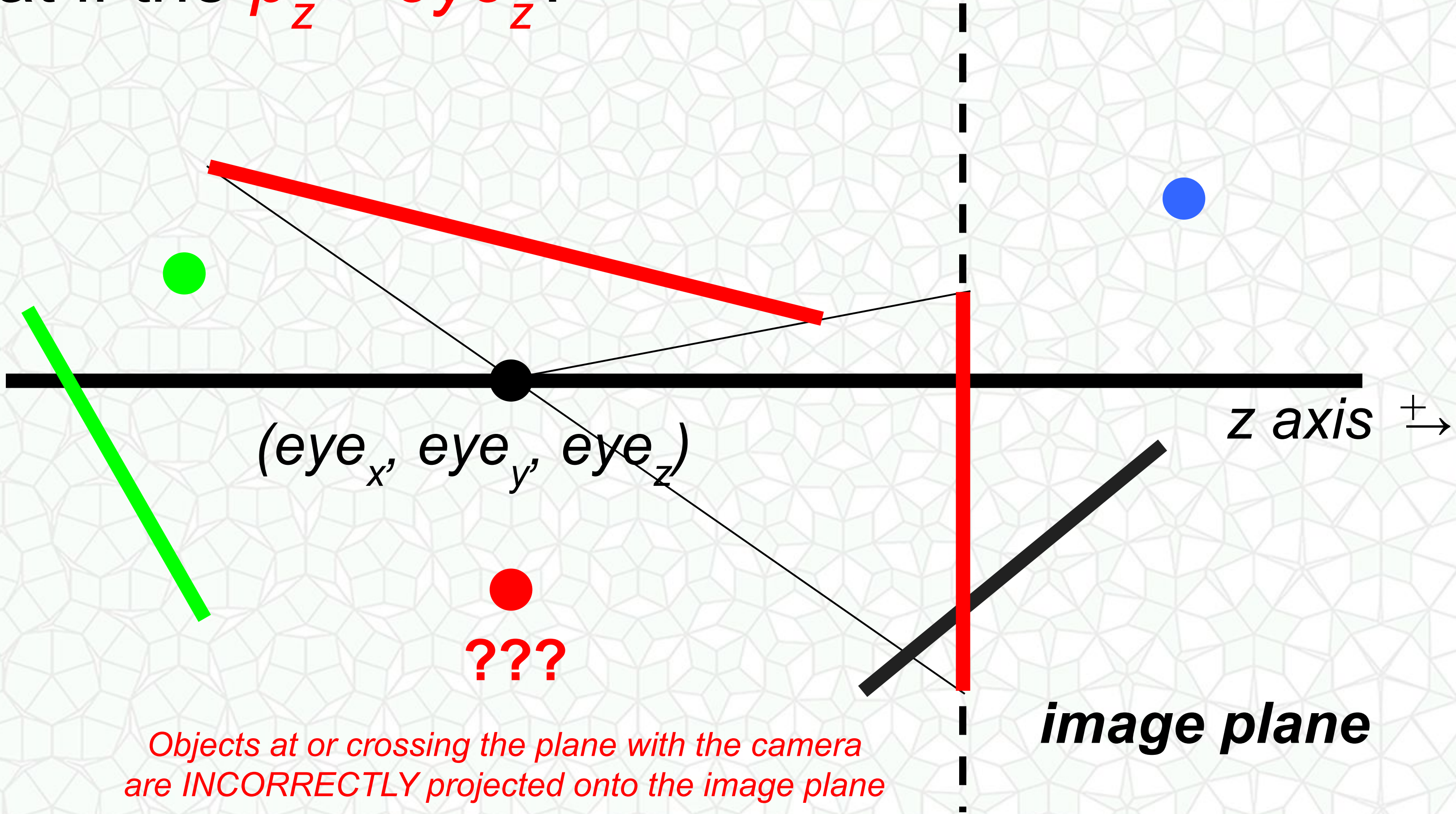
Objects behind the camera are INCORRECTLY projected onto the image plane

What if the $p_z \approx eye_z$?



Objects at or crossing the plane with the camera are INCORRECTLY projected onto the image plane

What if the $p_z \approx eye_z$?



Objects at or crossing the plane with the camera are INCORRECTLY projected onto the image plane

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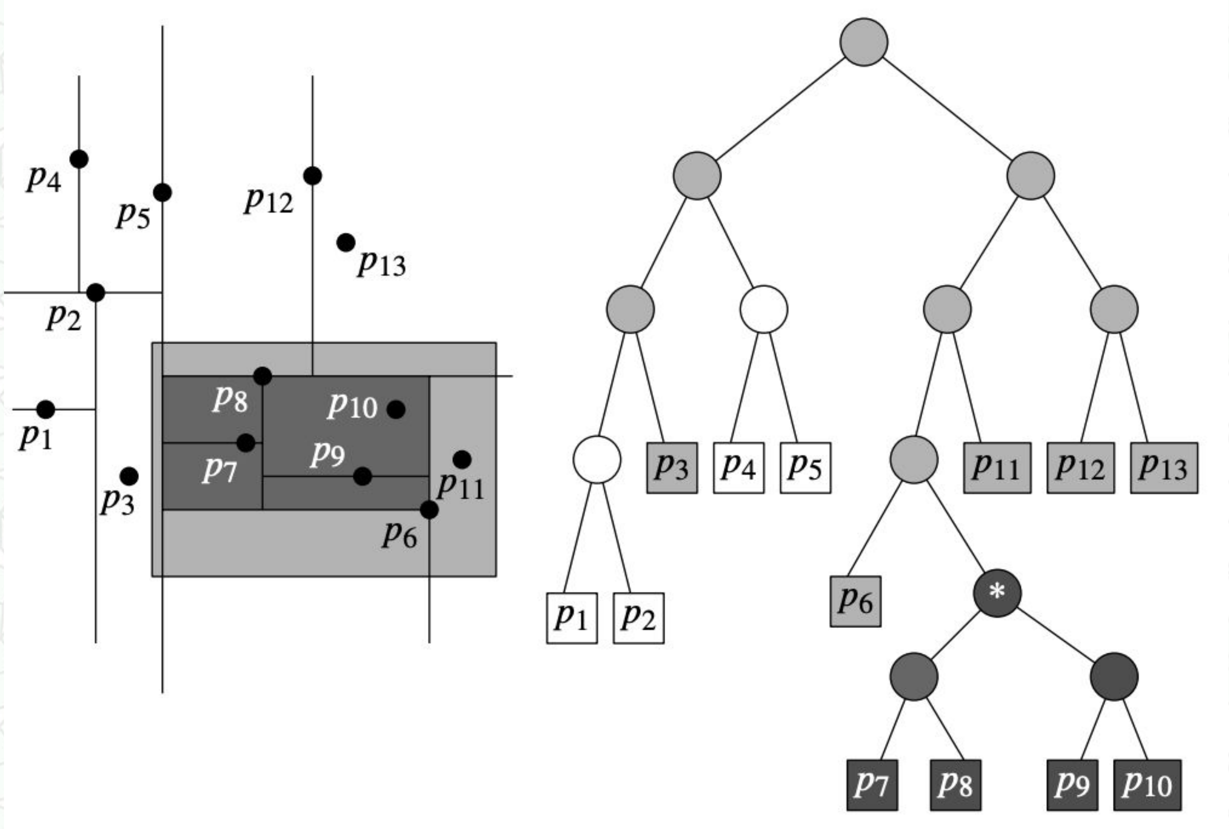
Review from Lecture 8: 2D k-d Tree

- Used to store points
- Alternate splitting horizontally & vertically
- If data is available for preprocess, the structure is easy to balance
- Point data is only stored at the leaves



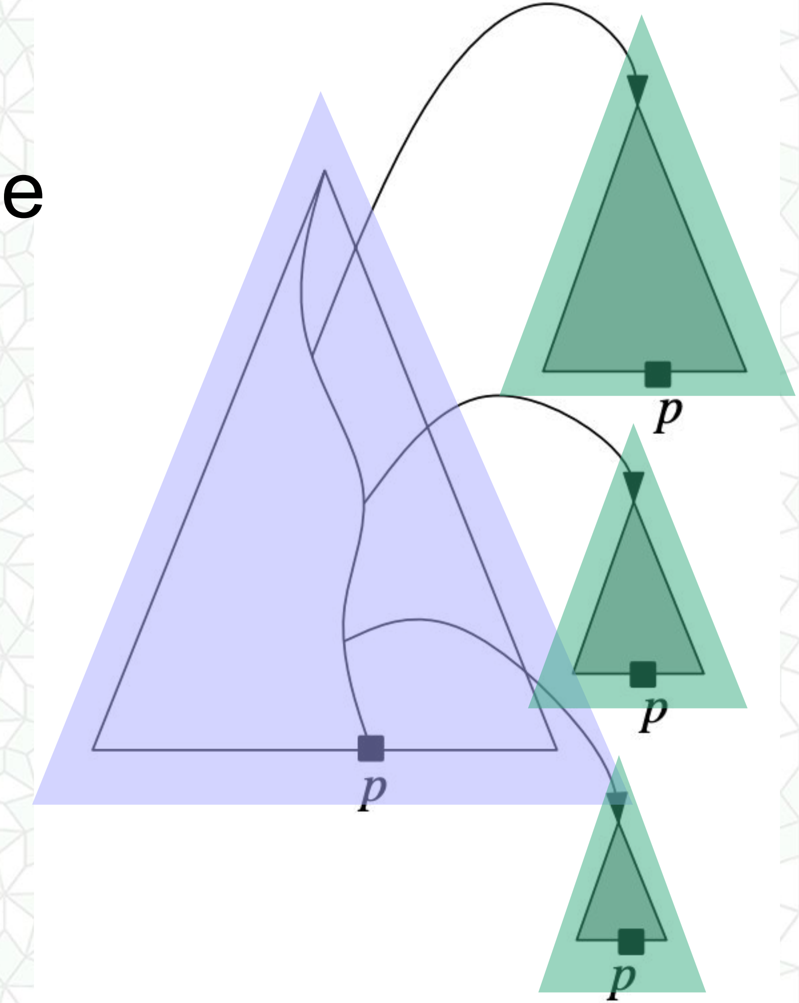
Review from Lecture 8: How to Query 2D Range Tree

- Search through level 1 (blue) tree for all intermediate nodes that fit completely inside the query's *x* range

  For each matched intermediate blue node

  - Search through the corresponding level 2 (green) trees for all nodes and leaves that fit completely inside the query's y range

    Return all matching data!



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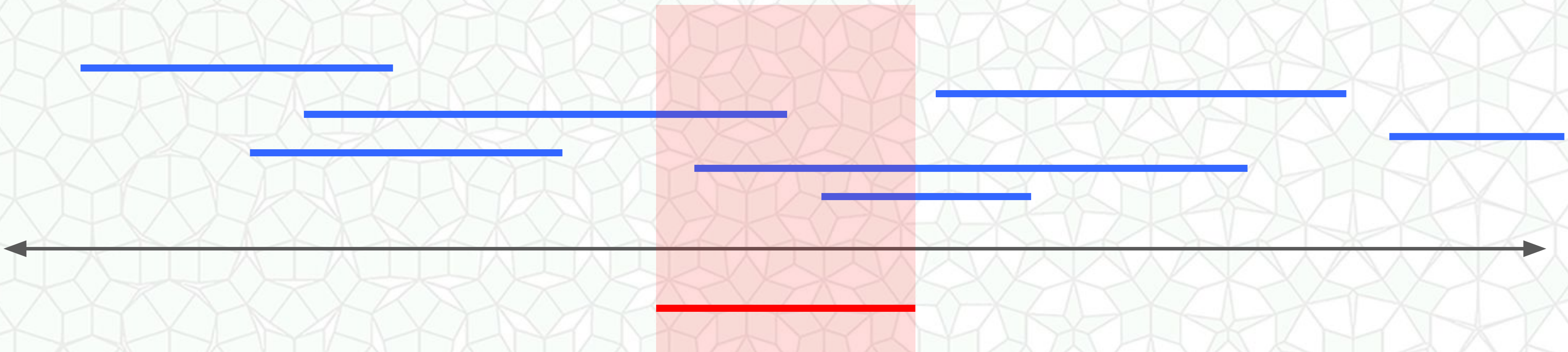
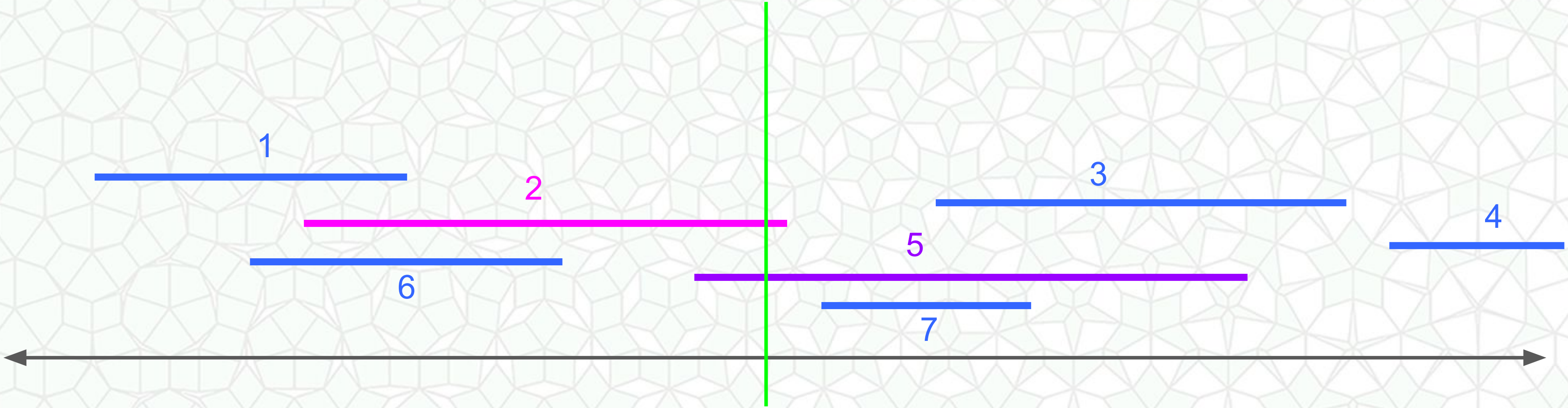
What about Segments? Let’s Tackle 1D First…

- Input: A collection of *n* line segments on the x-axis
- For a query interval, return all line segments that overlap the query interval



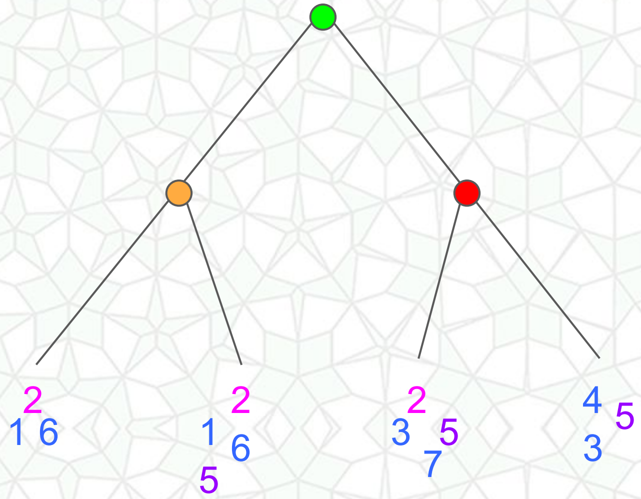
Traditional Binary Search Tree

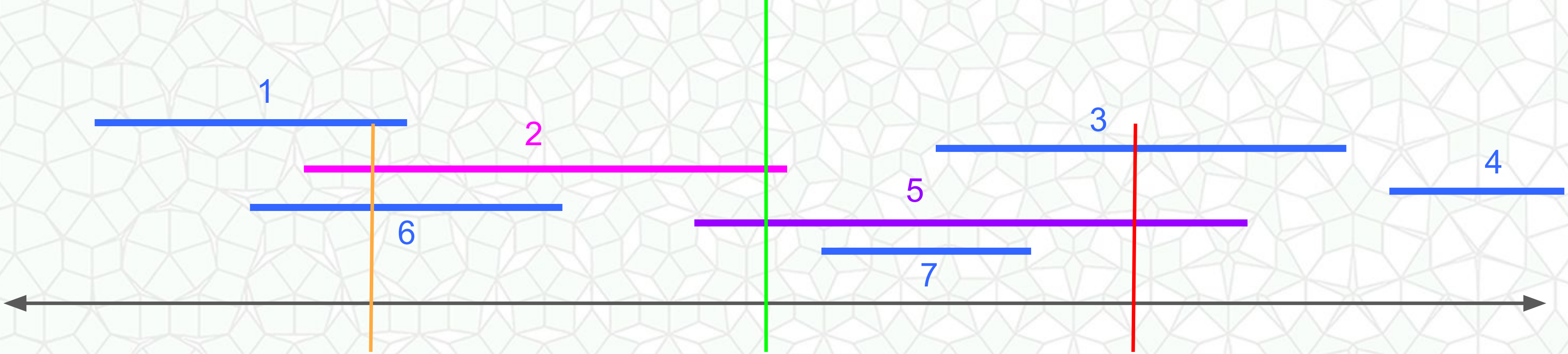
- Select split point near middle of data
- What about segments that overlap the split?



Traditional Binary Search Tree

- Select split point near middle of data
- What about segments that overlap the split?
- Should we store them on both sides?
  - Uses extra memory
  - Have to remove duplicates
  - We may lose our O(log n) performance!

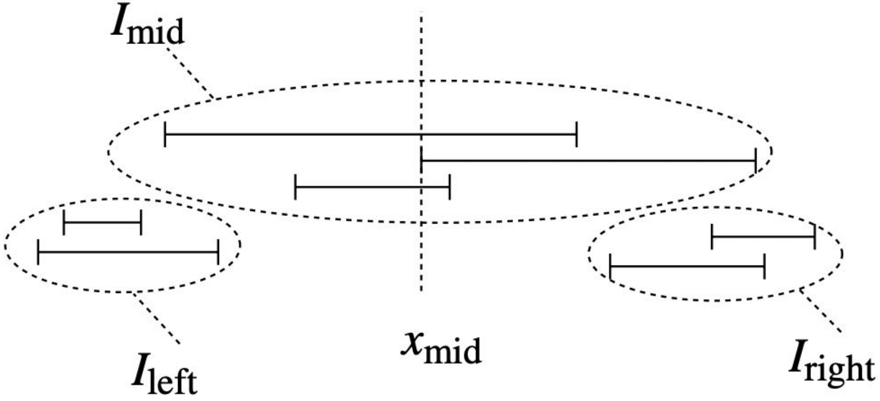


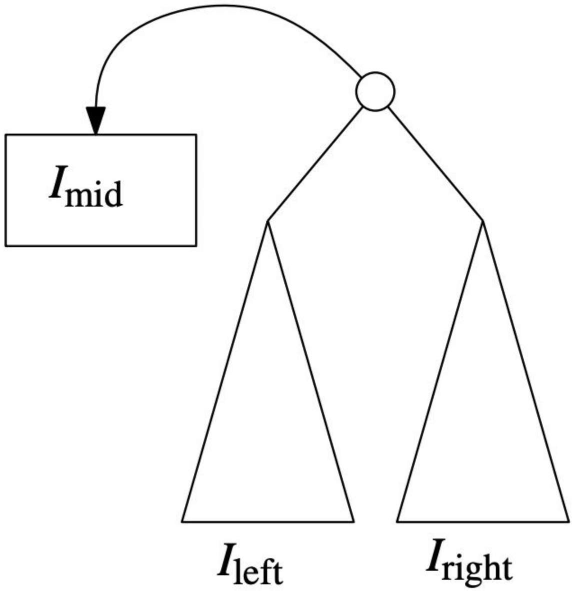


Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 10

Interval Tree

- Chose a split point and make 3 groups:
  - I_{mid} = Segments that overlap the split
  - I_{left} = Segments completely to the left
  - I_{right} = Segments completely to the right





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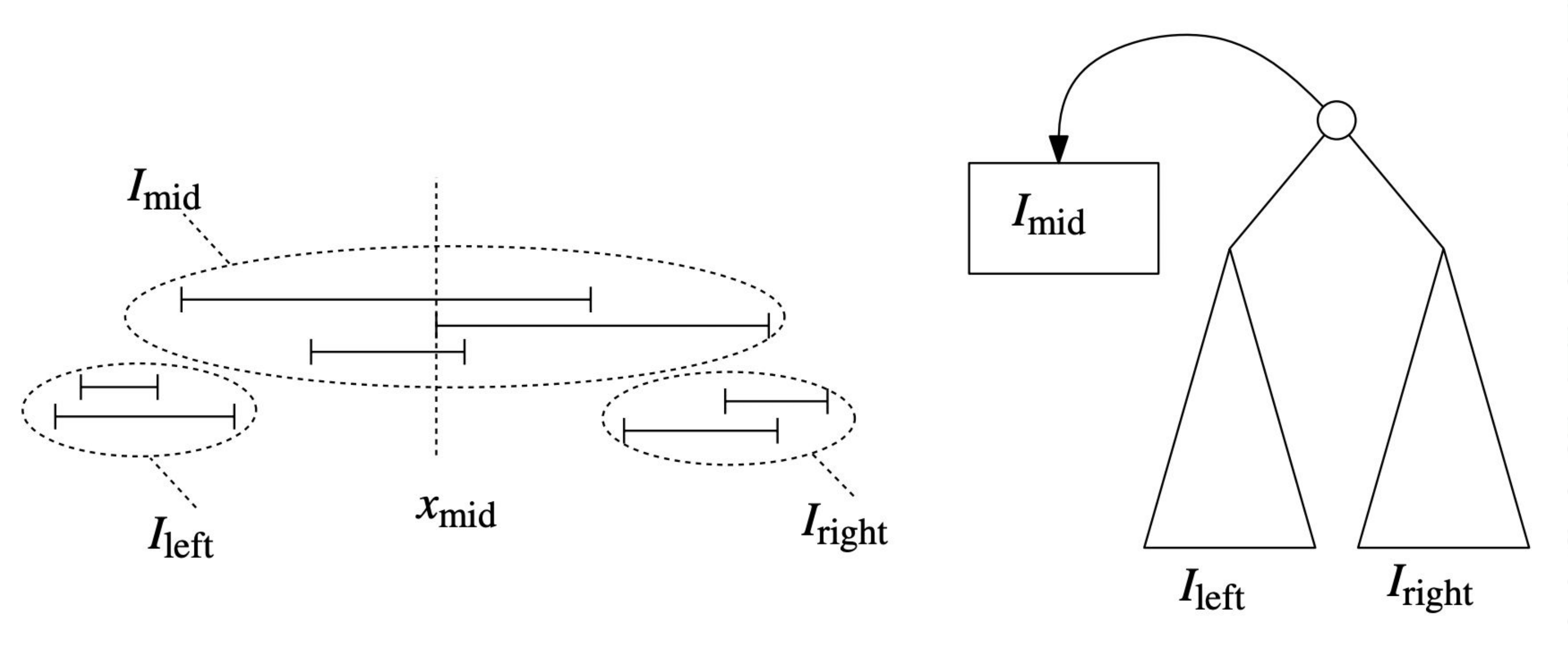
Interval Tree

- Recurse down the tree only with items that DO NOT overlap the split point.

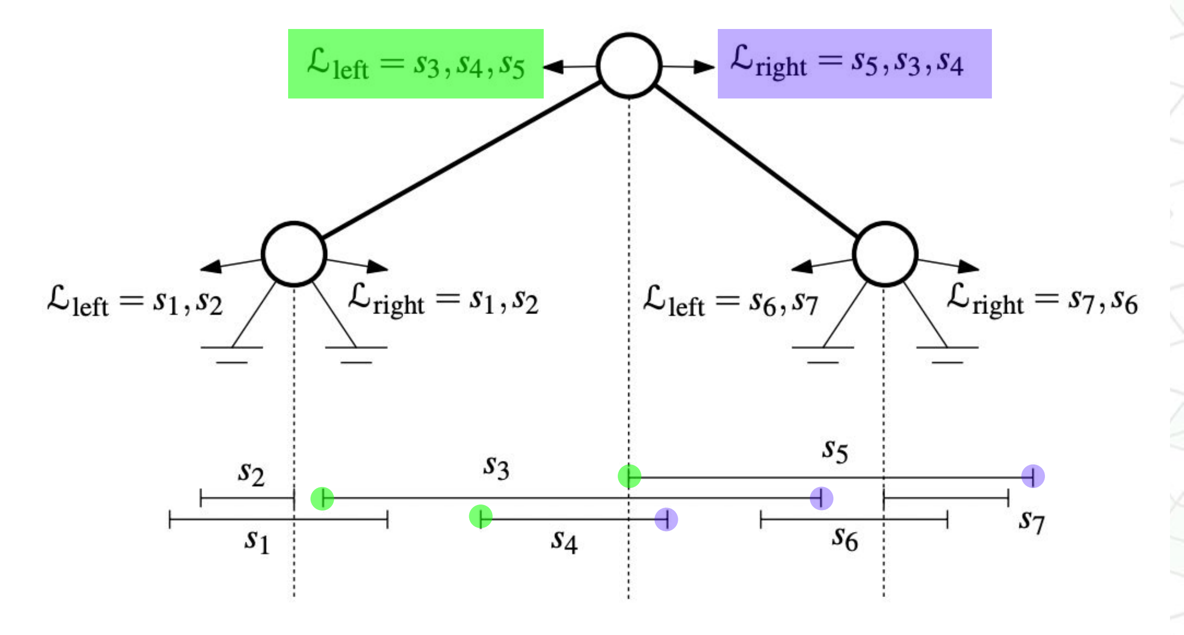


Interval Tree

- Items in I_{mid} group will stay at the current node

- Each node stores two two sorted lists:

- L_{left} = I_{mid} sorted by left endpoint (increasing)
- L_{right} = I_{mid} sorted by right endpoint (decreasing)



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$\mathcal{L}_{\text{right}} = s_5, s_3, s_4$
$\mathcal{L}_{\text{left}} = s_1, s_2$
$\mathcal{L}_{\text{right}} = s_1, s_2$
$\mathcal{L}_{\text{left}} = s_6, s_7$
$\mathcal{L}_{\text{right}} = s_7, s_6$
s_2
s_3
s_5
s_1
s_4
s_6
s_7

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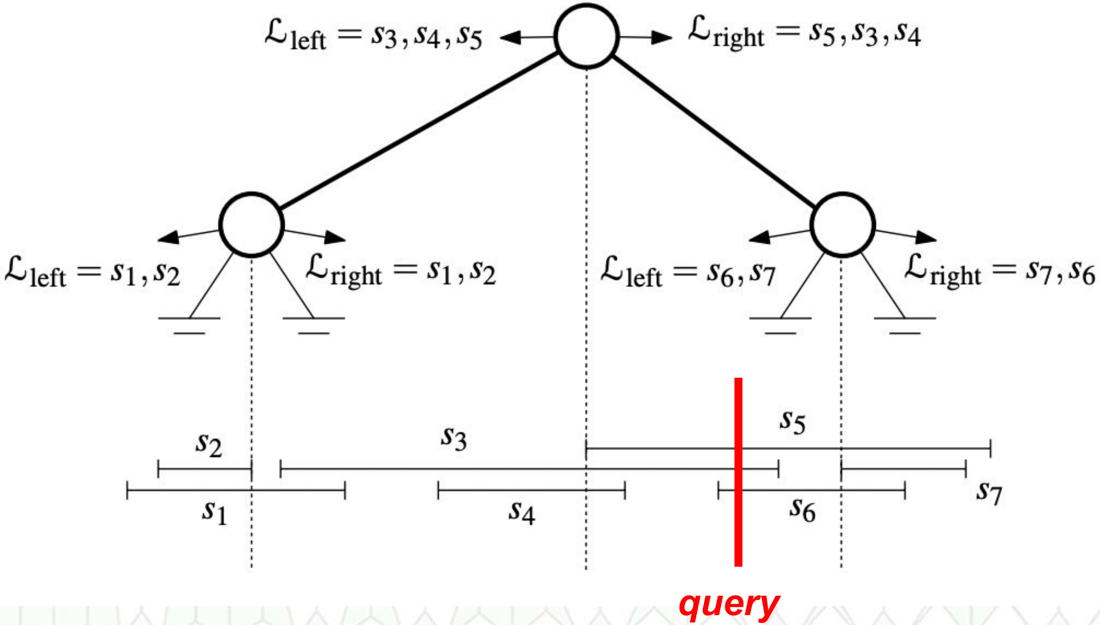
Interval Tree

For a specific *query*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s_5
s_2
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s_7
s_1
s_4
s_6
query

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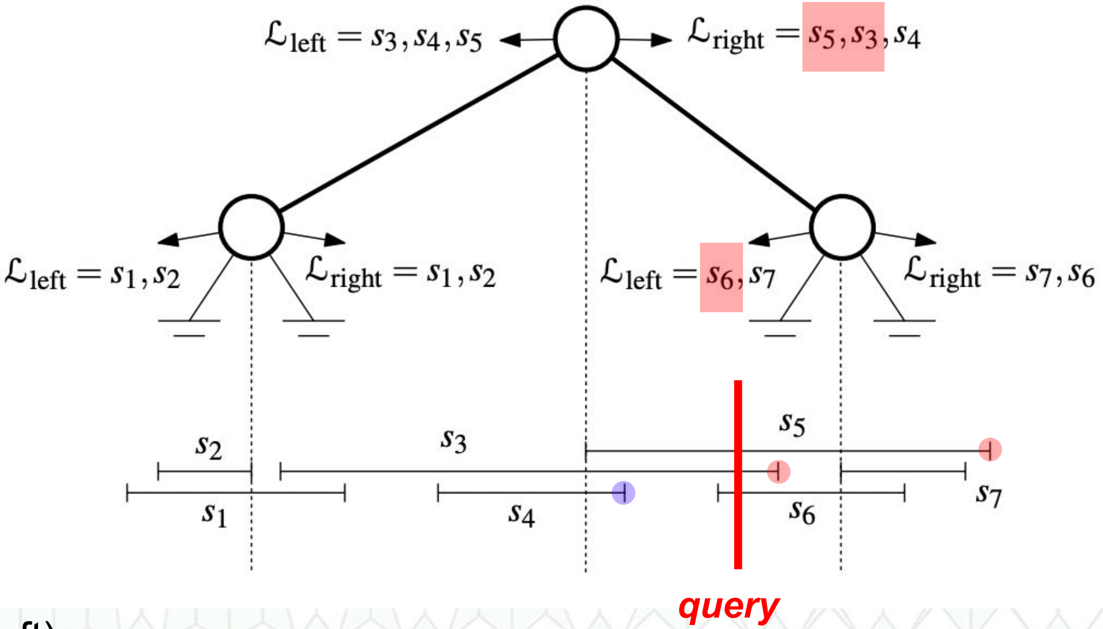
Interval Tree

For a specific *query*

- Determine if the query is to the right (or left) of the current node
- Walk through the L_{right} list (or L_{left} list) which is sorted by right (or left) endpoint
  - Return segments with endpoint further away from the query
  - *Stop when you find a segment that doesn't match the query*
- And recurse down the right (or left)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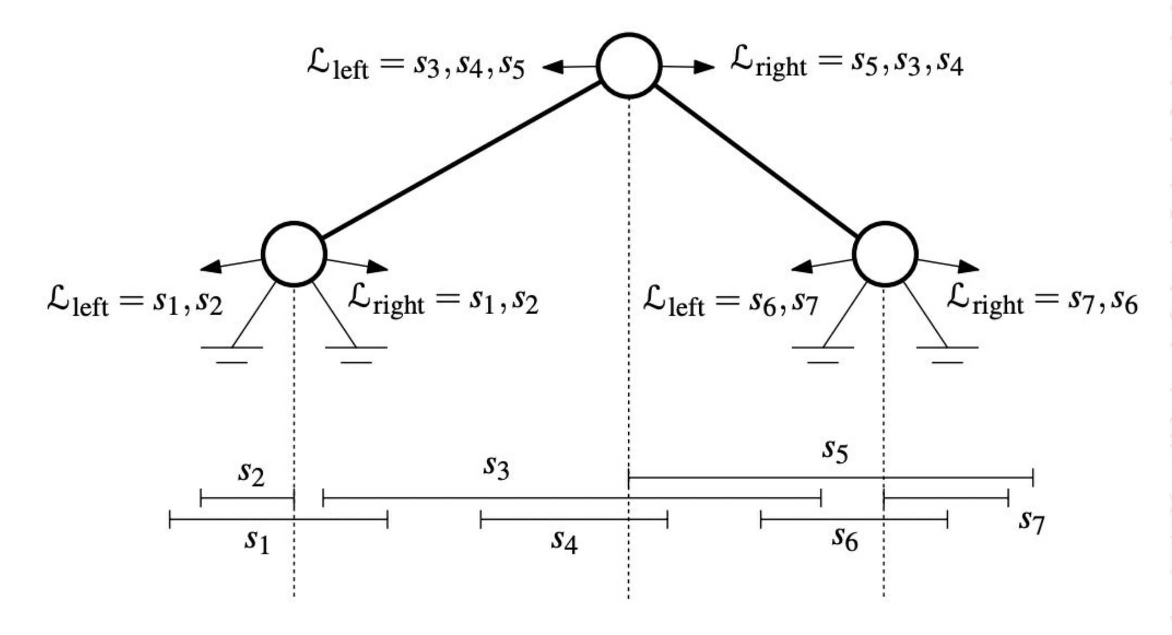
1D Interval Tree Analysis

- For *n* input segments and a query that will return *k* items

- Memory Usage:

- Construction Time:

- Query Time:



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s_6

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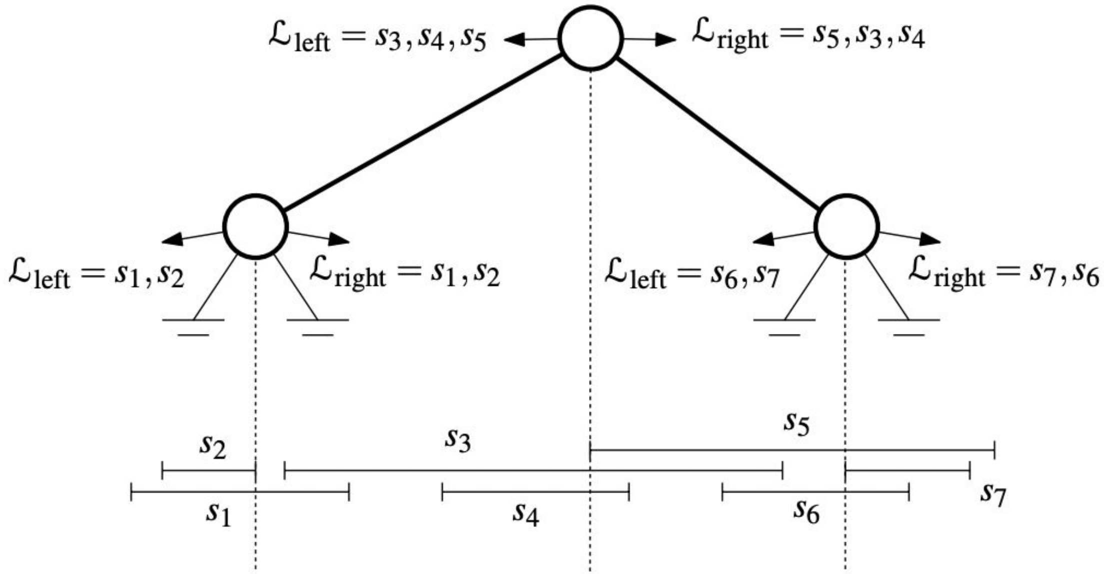
1D Interval Tree Analysis

- For *n* input segments and a query that will return *k* items

- Memory Usage:
  → *O(n)*

- Construction Time:
  → *O(n log n)*

- Query Time:
  → *O(log n + k)*



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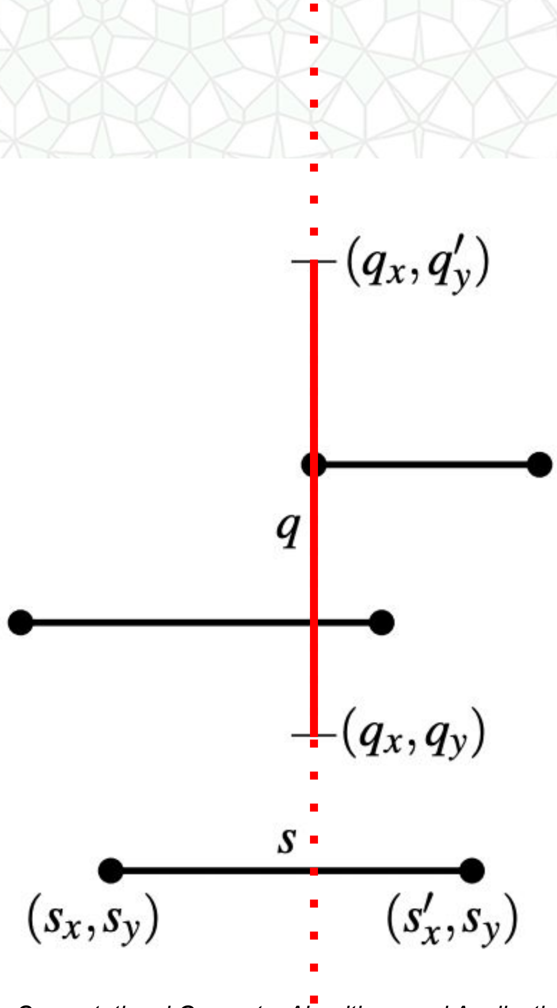
How do we Extend to 2D?

- First consider only horizontal input line segments
- And instead of a query line, we'll have a ***query line segment***



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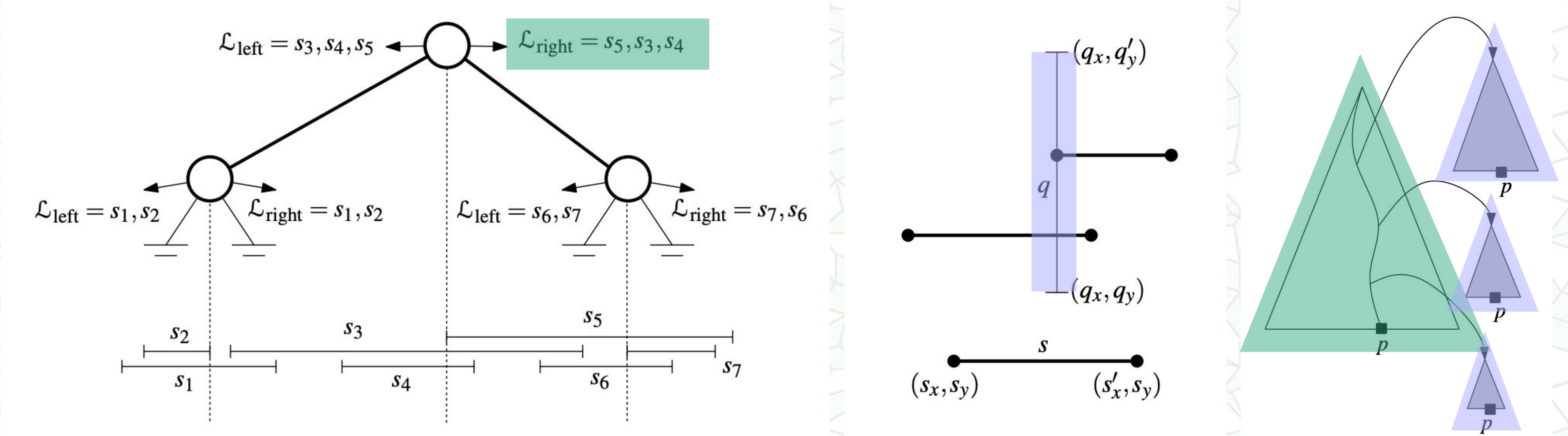
How do we Extend to 2D?

- We’ll replace the sorted lists of the interval tree with a 2D range query (first by x then by *y*) of the 2D endpoints (Lecture 8)
- This will now require additional memory…



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s_1
s_2
s_3
s_4
s_5
s_6
s_7
(q_x, q'_y)
q
(q_x, q_y)
s
(s_x, s_y)
(s'_x, s_y)
p
p
p
p

Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapters 8 & 10

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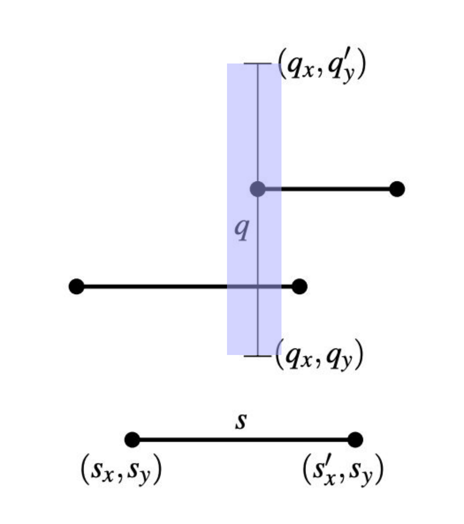
2D Interval Tree + Range Tree Analysis

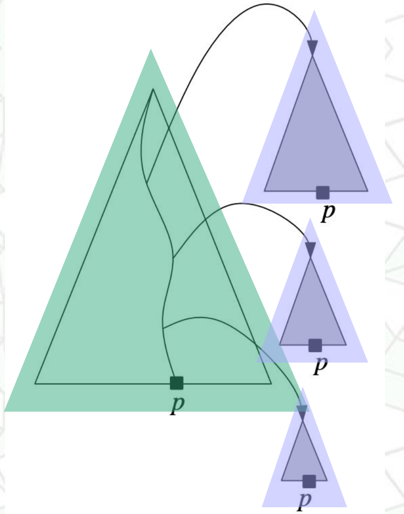
- For *n* horizontal input segments and a query segment that will return *k* items

- Memory Usage:

- Construction Time:

- Query Time:





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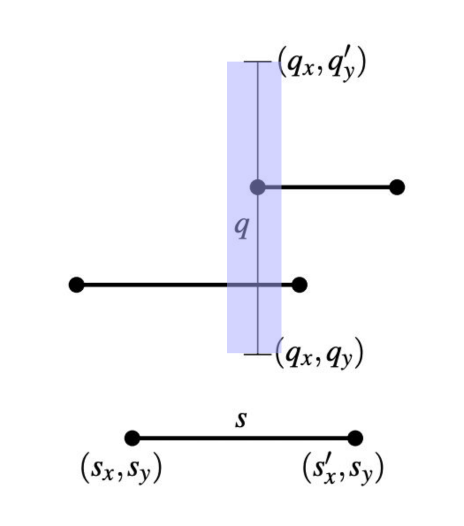
2D Interval Tree + Range Tree Analysis

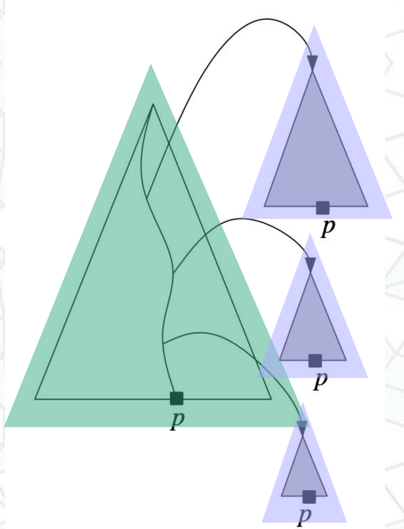
- For *n* horizontal input segments and a query segment that will return *k* items

- Memory Usage:
  → *O(n log n)*

- Construction Time:
  → *O(n log n)*

- Query Time:
  → *O(log n + k)*





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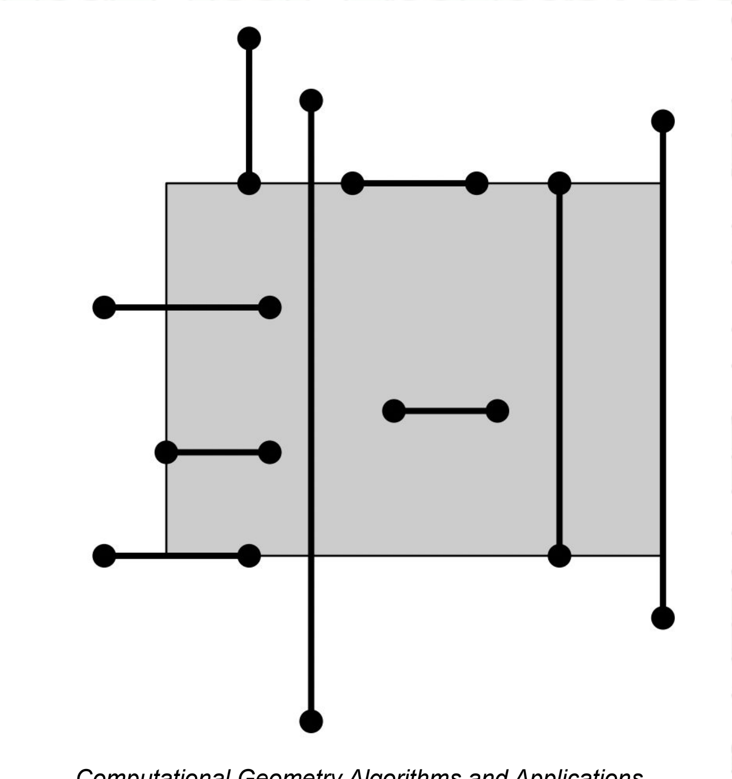
How to handle a 2D Axis-Aligned Query Box?

- Initially, let’s restrict to horizontal & vertical segments



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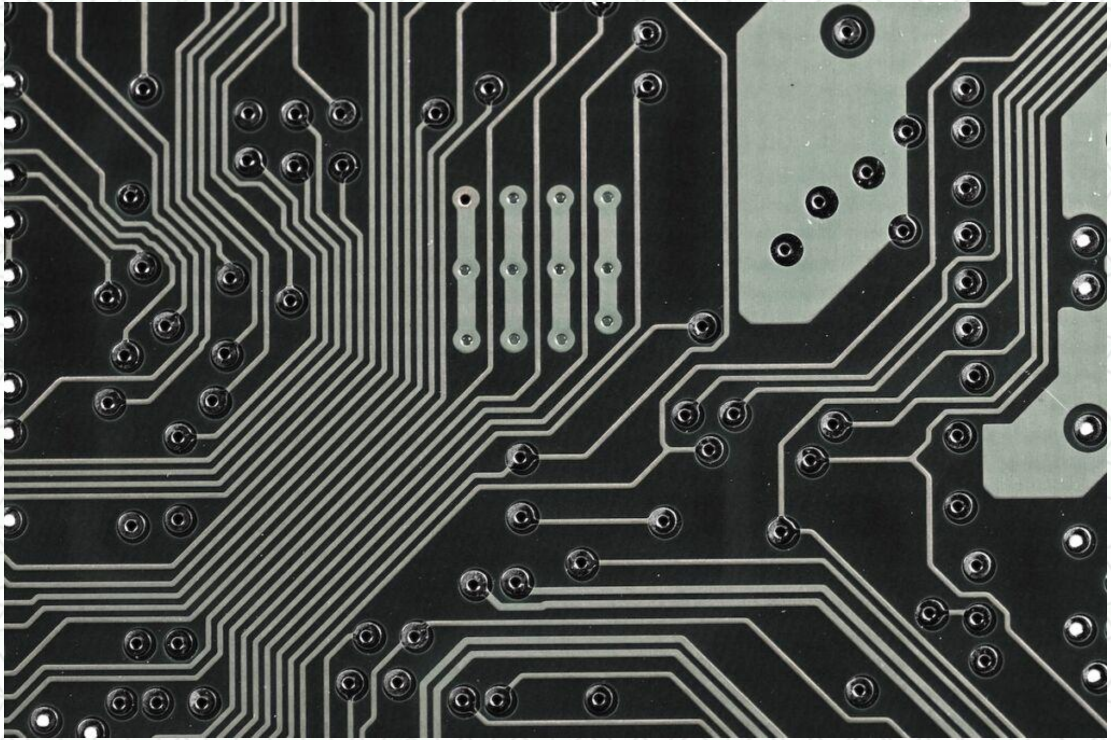
de Berg, Cheong, van Kreveld and Overmars, Chapter 10

How to handle a 2D Axis-Aligned Query Box?

- Why is Axis-Aligned Interesting?
- A bit later we will also restrict ourselves to non-overlapping segments…
  Why is that interesting?

Motivation:
Circuit board layouts →

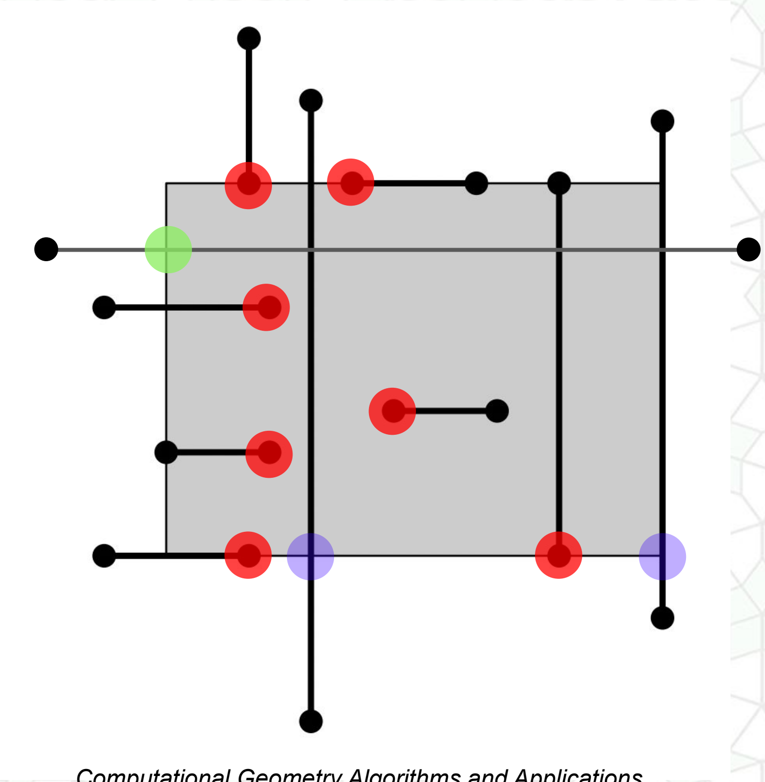


https://www.findlight.net/blog/evolution-of-laser-marking-and-engraving-technologies/

How to handle a 2D Axis-Aligned Query Box?

- Initially, let’s restrict to horizontal & vertical segments

- Case Analysis:
  Segments that touch the query box will:
  - Have one endpoint inside the box,
    OR
  - Will have both endpoints outside the box
    AND
    - Will be a horizontal segment that overlaps the left edge of the box
      OR
    - Will be a vertical segment that overlaps the bottom edge of the box



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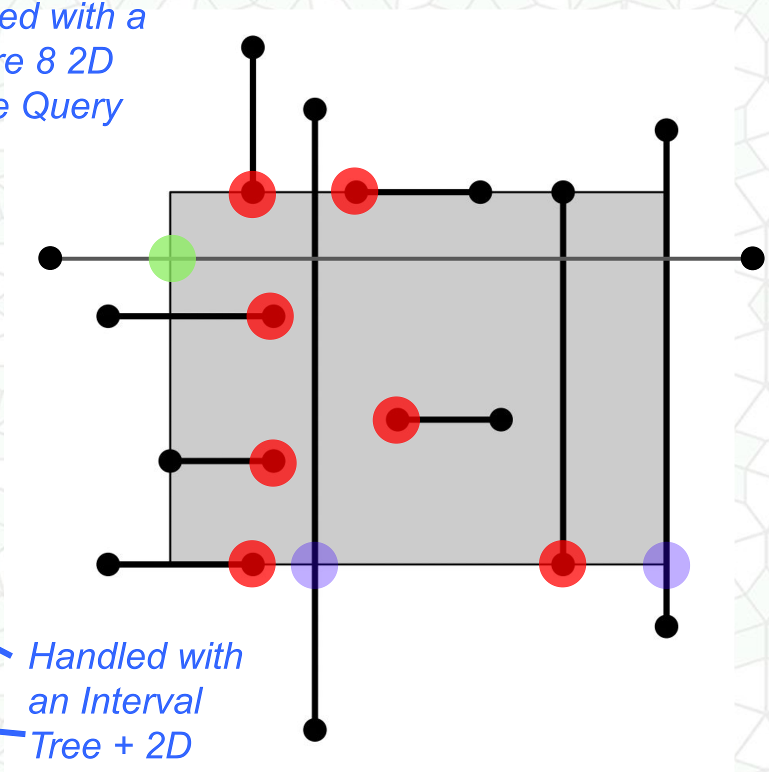
Handled with a Lecture 8 2D Range Query

Handled with an Interval Tree + 2D Range Query



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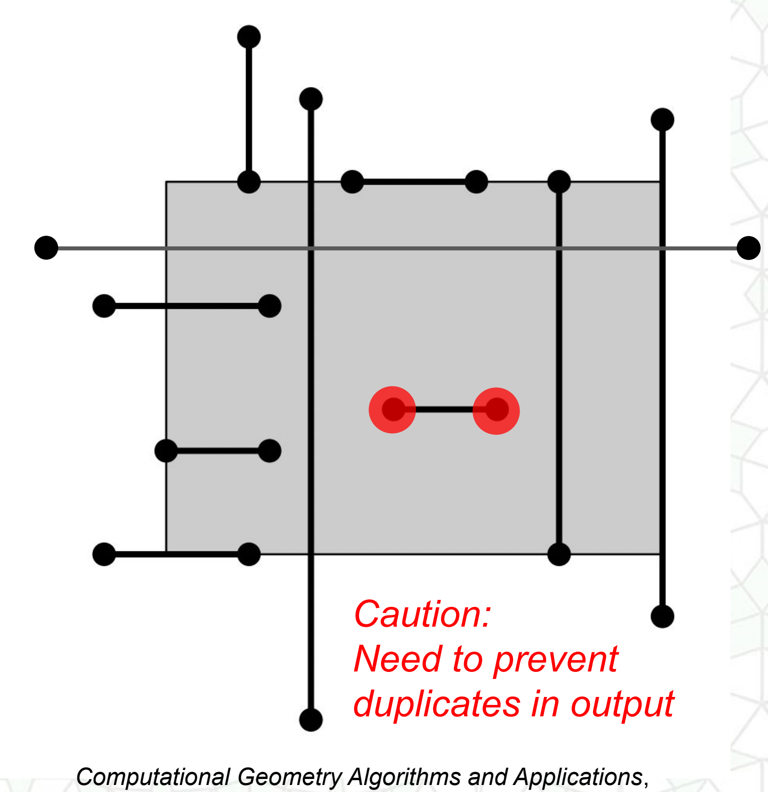

Caution: Need to prevent duplicates in output

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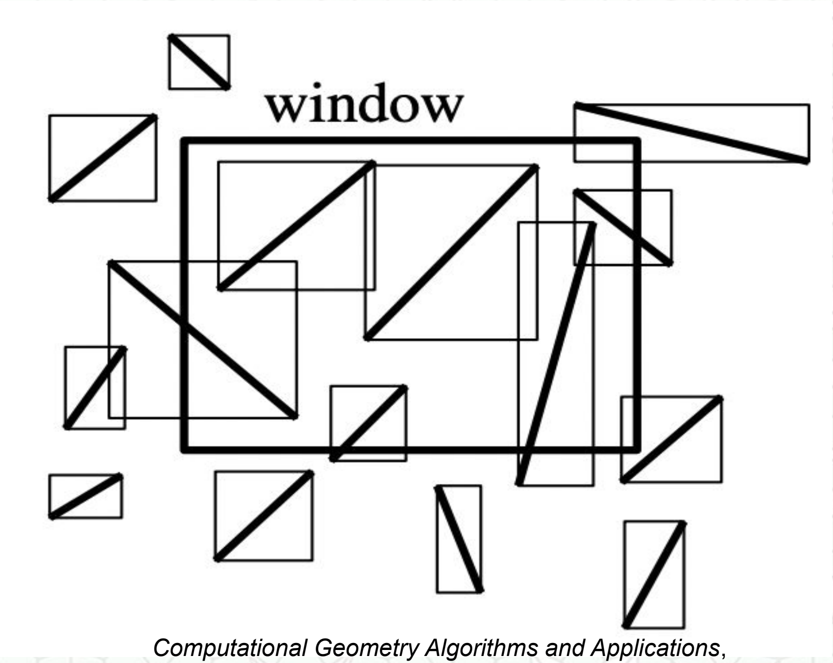
How do we handle General 2D Segments?

- Not restricted to horizontal & vertical segments!
- (Note: We will later insist that the segments do not cross…)

How do we handle General 2D Segments?

- *Do the (sloppy?) Computer Graphics thing…*

  Output the segment if its bounding overlaps the axis-aligned query box



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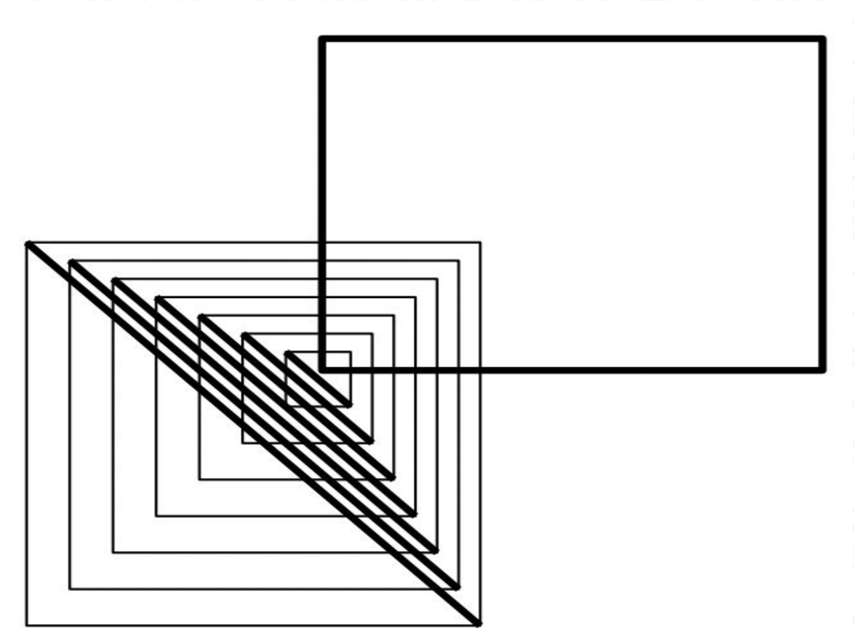
- ***We might have LOTS of false positives!***



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How do we handle General 2D Segments?

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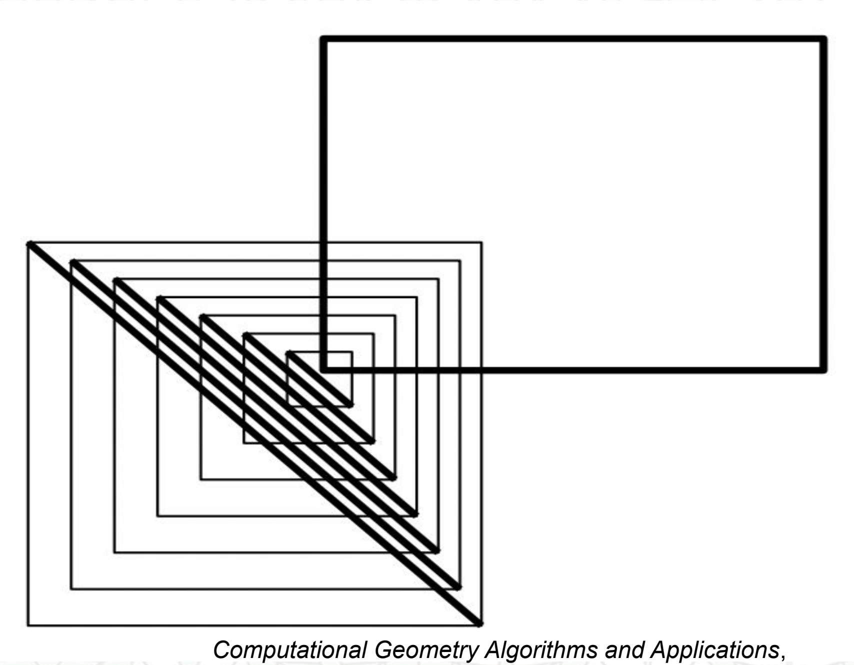
  Output the segment if its bounding overlaps the axis-aligned query box

- ***We might have LOTS of false positives!***

- Can we do better?
  - Ensure good (output sensitive) Performance
    AND
  - Avoid false positives?



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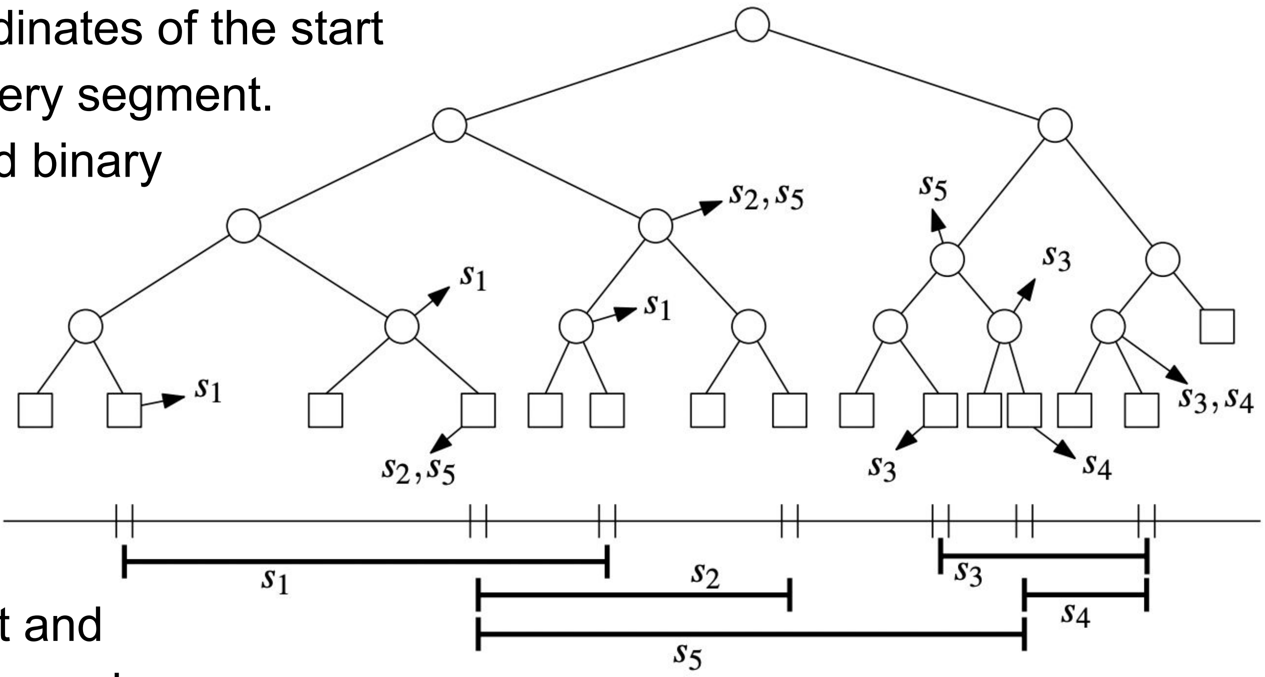
Segment Tree - First Dimension (*x*)

- First, sort the *x* coordinates of the start and end points of every segment.
- Construct a balanced binary search tree with these *x* values.
- Insert every segment into the structure
- If a segment overlaps both the left and right subranges of the node store it at the node (do not recurse)



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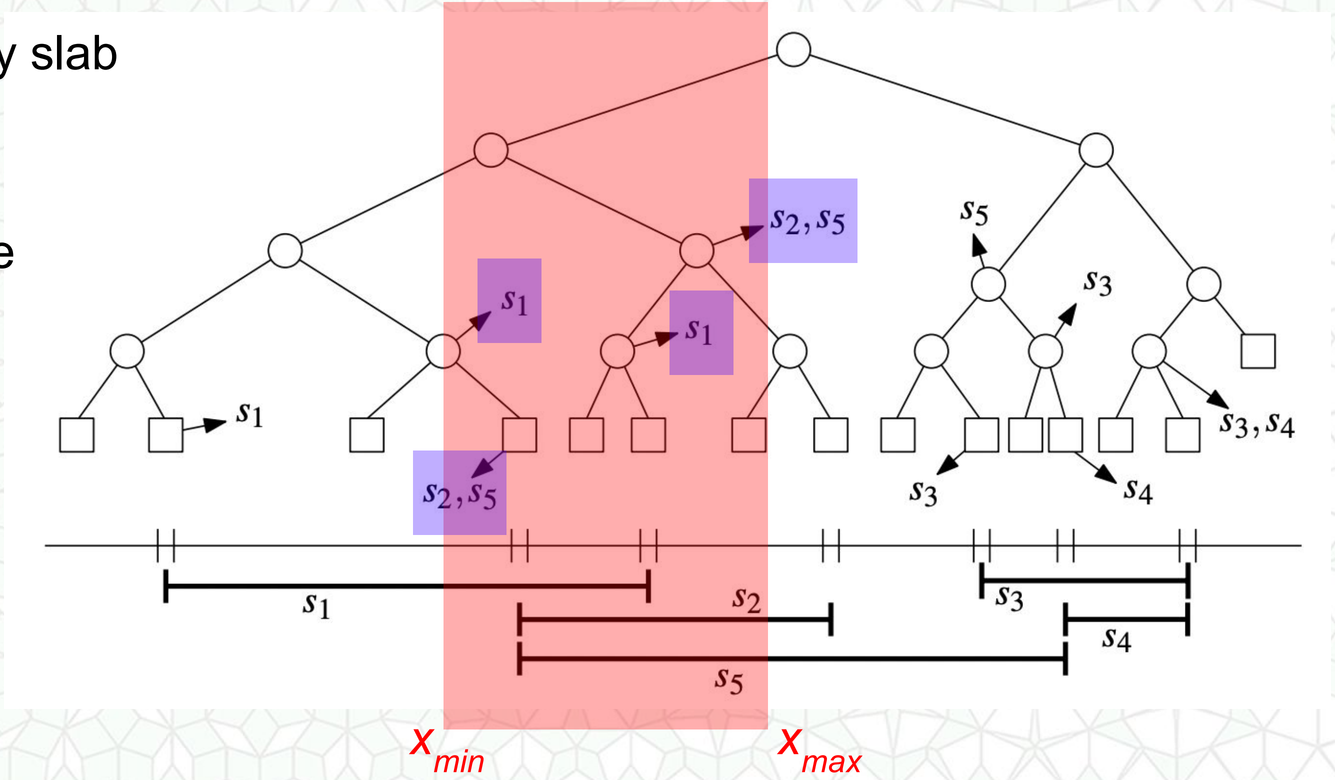
Segment Tree - First Dimension (*x*)

- For a vertical query slab (x_{min}, x_{max})

- Walk down the tree
- If the node is in range, return all items at that node
- Recurse left and/or right as appropriate

- & filter duplicates…



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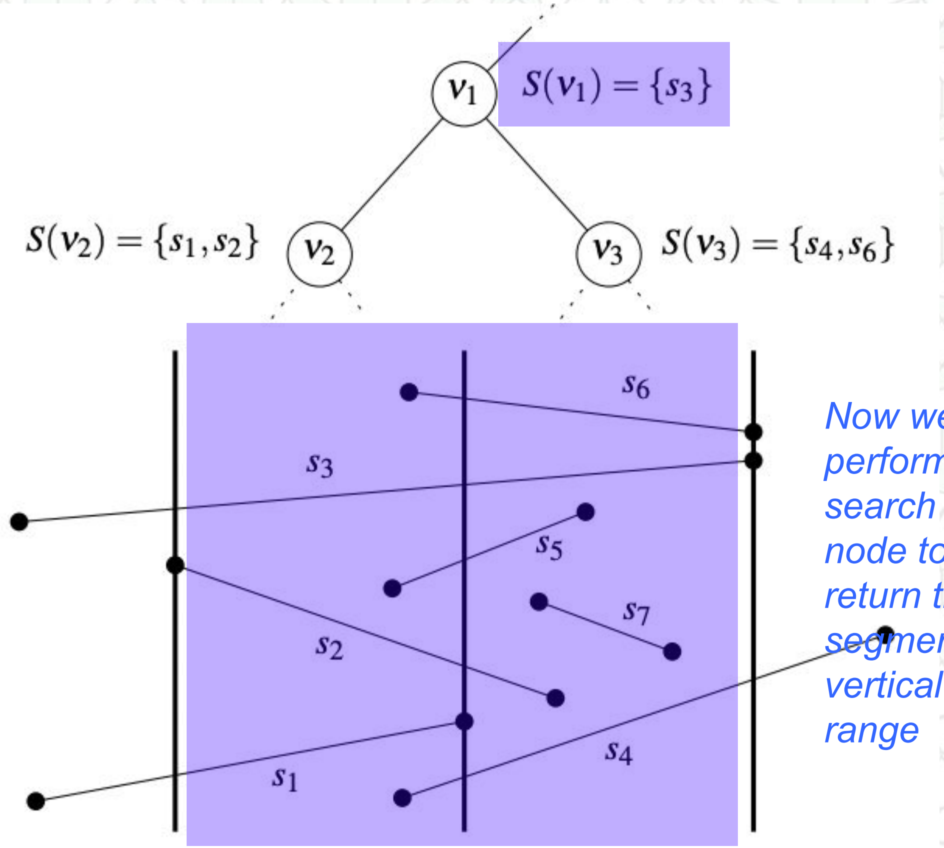
Segment Tree - Second Dimension (*y*)

- To efficiently query a vertical range in addition to the horizontal range:
- Sort the segments stored at each node by *y*
- *Remember: this is only the segments that completely overlaps the node's range*
- Note: this is why we require no crossings in the input segments



Now we can perform binary search at each node to only return the segments in the vertical query range

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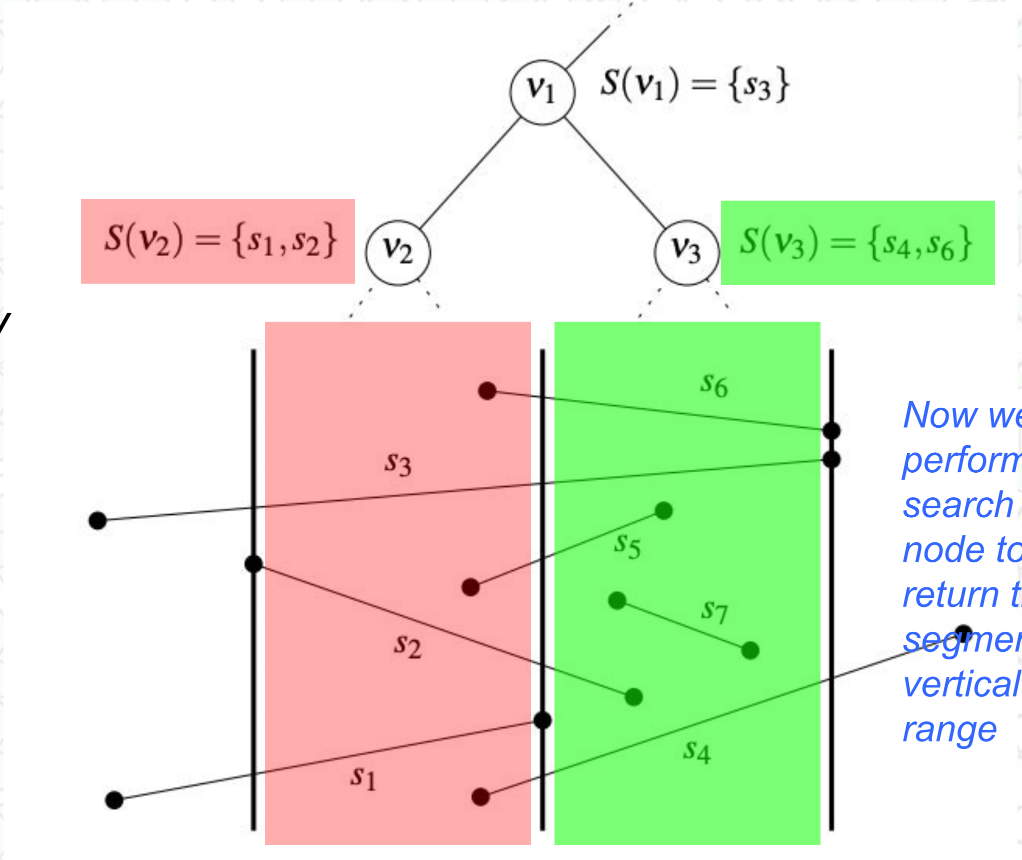

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Segment Tree - Analysis

- For *n* input segments, for a query that will return *k* segments

- Memory:
  Each segment is stored in
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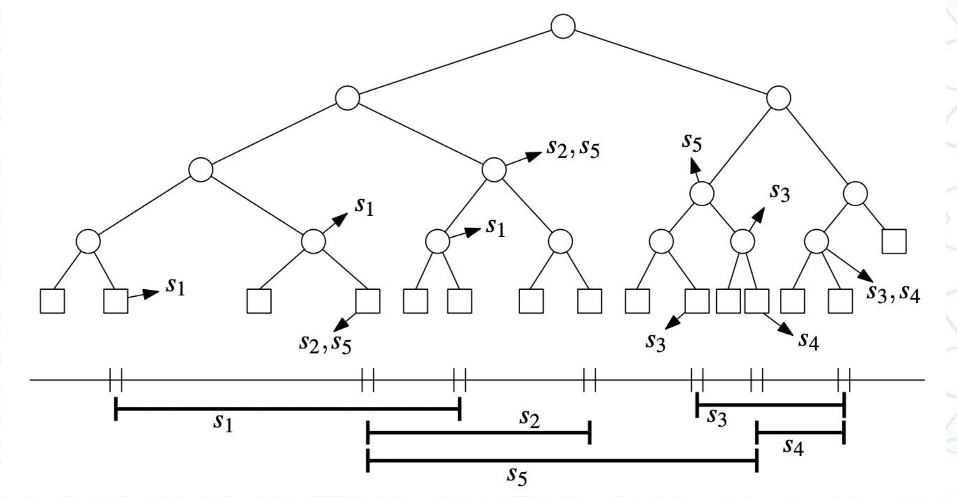
- Construction Time:
  Presort all endpoints
  by x & y *O(n log n)*

- Query Time:



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  → *O(n log n)*
- Construction Time:
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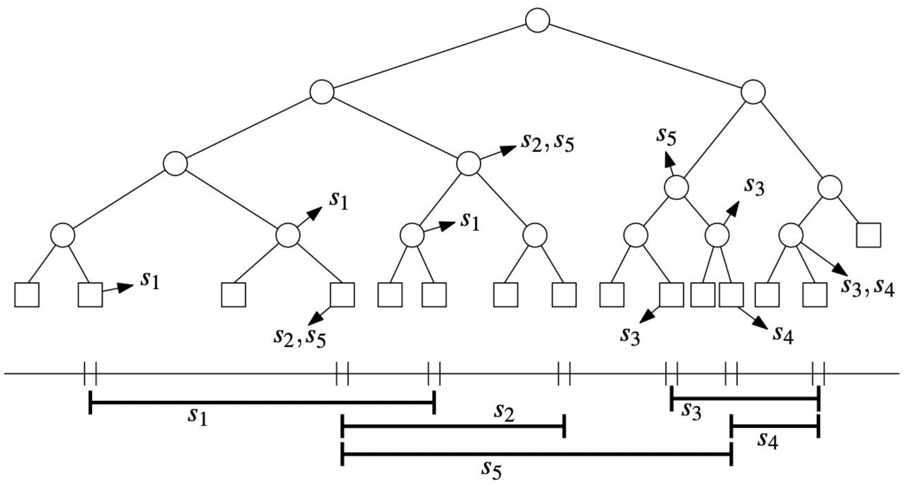
  → *O(n log n)*
- Query Time:
  → *O(log n * log n + k)*
  → *O(log ^{2} n + k)*



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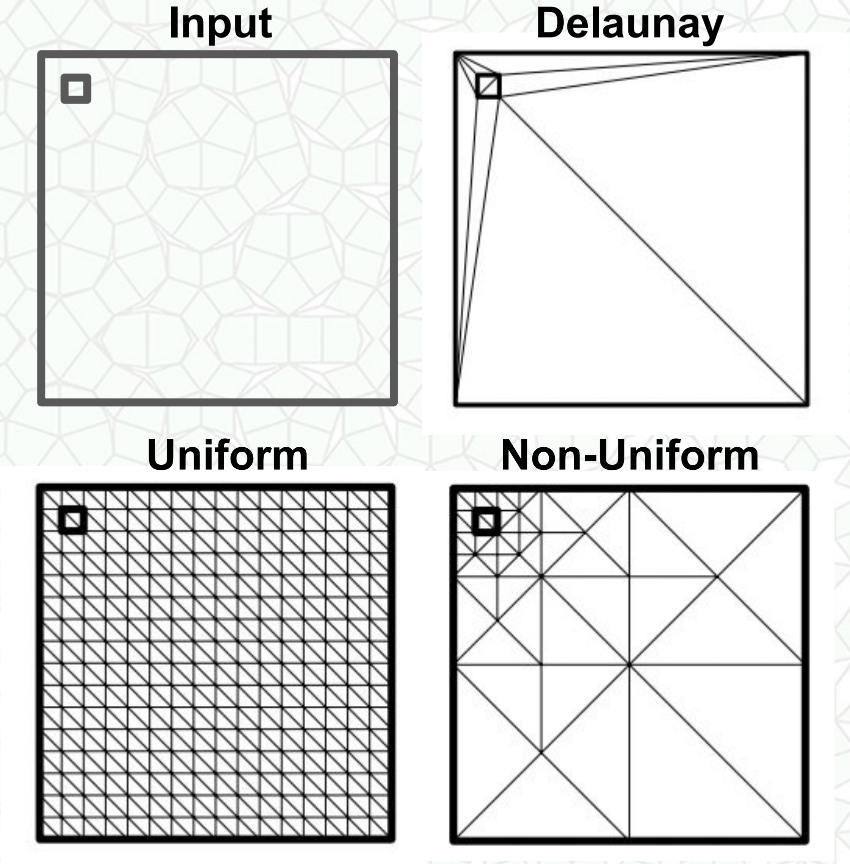
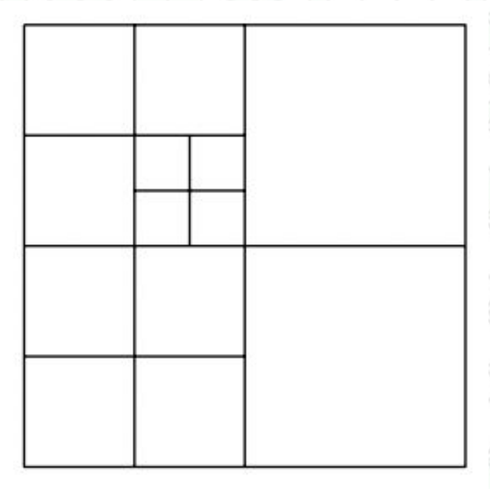


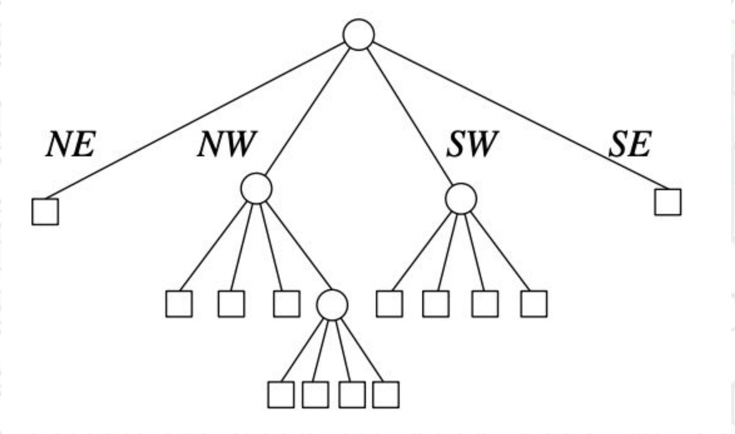
Input

Delaunay

Uniform

Non-Uniform





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