CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/F23/

Lecture 21: Polyominoes & Tiling

Outline for Today

- Final Project Questions?
- Last Time: Hidden Line Drawing, Painter's Algorithm, & BSP
- Polyominoes Terminology
- Counting Polyominoes
- Tiling / Packing Polyominoes
- Polyomino Themed Puzzles
- Next Time: More Tiling!

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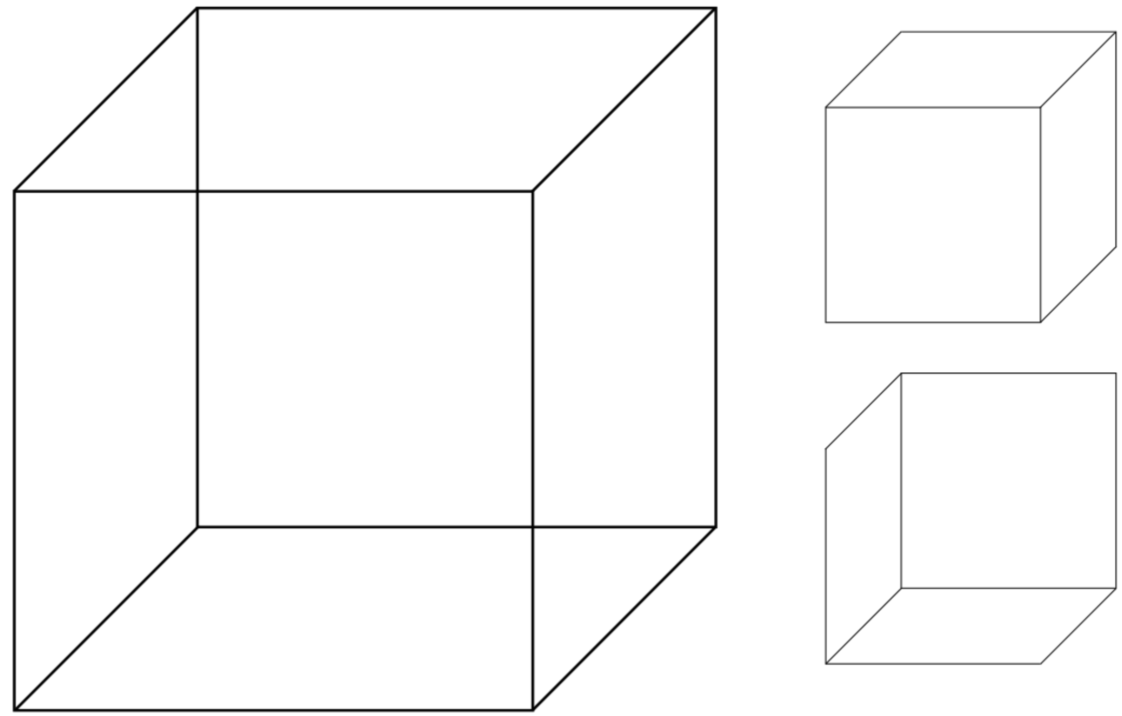
Necker Cube

- A two dimensional representation of a three dimensional wire frame cube

- Viewer's perception can flips back and forth between equally possible perspectives



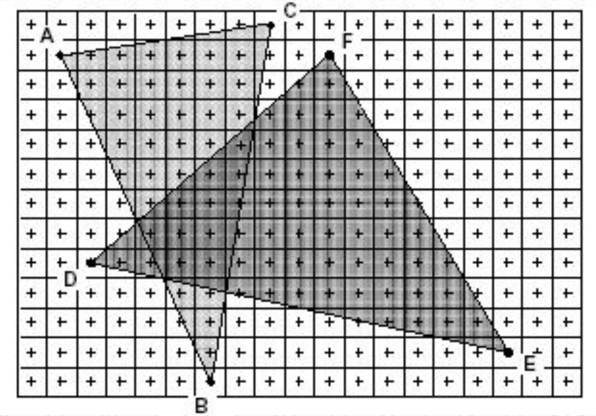
https://www.newworldencyclopedia.org/entry/necker_cube

https://commons.wikimedia.org/wiki/File:Necker%27s_cube.svg

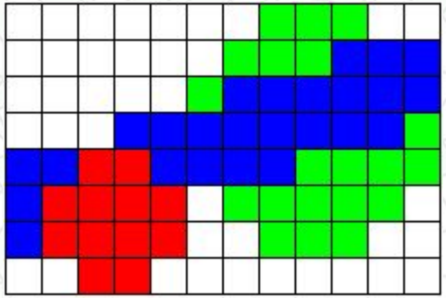
Hidden Line Drawing / Depth Buffer (z-Buffer)

- Given a primitive's vertices & the color / illumination at each vertex:
- Figure out which pixels to "turn on" to render the primitive
- Interpolate the color / illumination values to "fill in" the primitive
- At each pixel, keep track of the closest primitive (depth buffer / z-buffer)

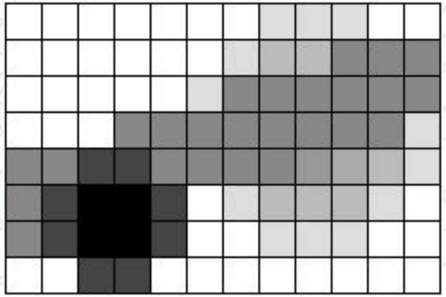
Triangles can be in any order!
A.k.a. "Polygon soup"




```
glBegin(GL_TRIANGLES)
glNormal3f(...)
glVertex3f(...)
glVertex3f(...)
glVertex3f(...)
glEnd();
```



frame buffer

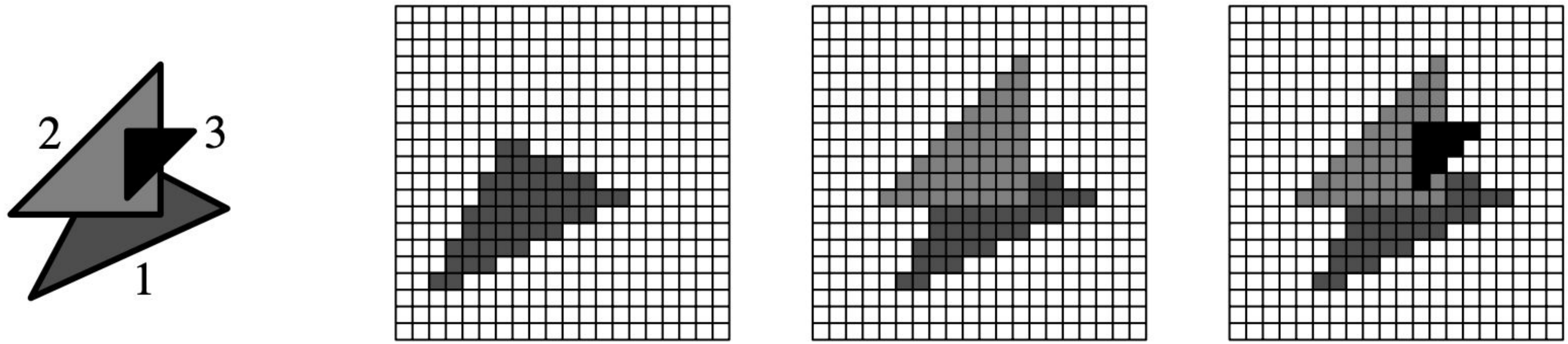


depth buffer

Hidden Line Drawing: Painter's Algorithm

- Let's order the primitives by how close they are to the camera
- Draw the primitives from back to front
- Then we don't need to keep track of the depth!

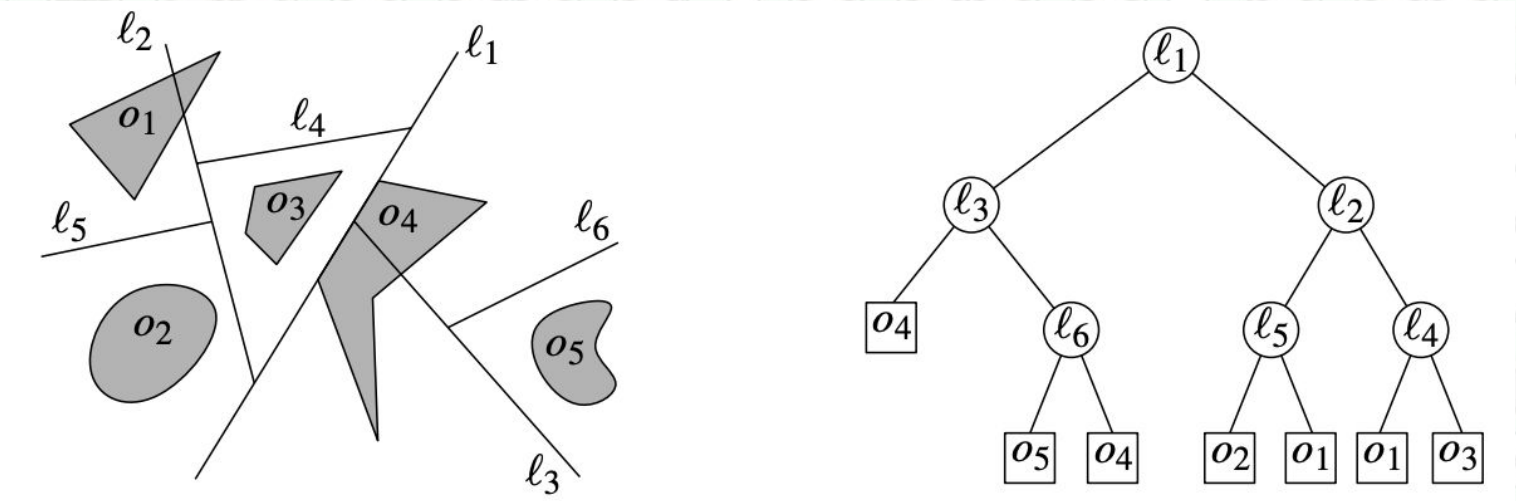
  Save memory!



Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 12

Definition: Binary Space Partition

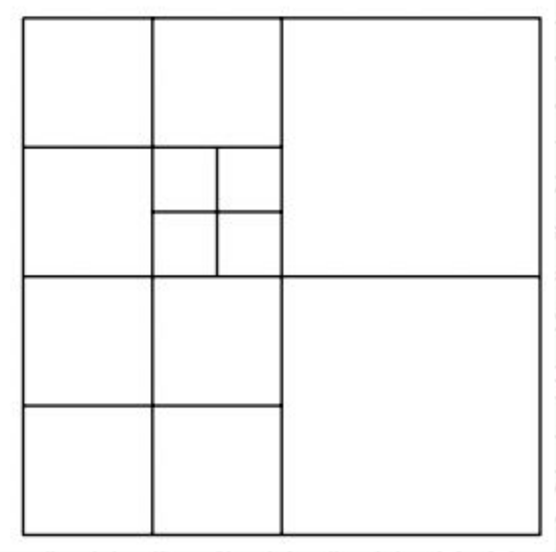
- Place items in a binary tree, each node stores a half plane
- Primitives that are collinear with the half plane are stored in the node
- Items overlapping a half plane are copied/split into two primitives
- We recurse until exactly one item is left, it is stored in the leaf



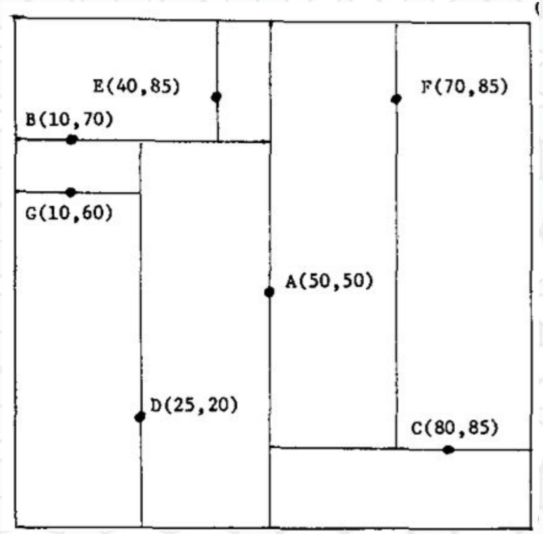
Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 12

Discussion - Quad Tree, kD Tree, BSP

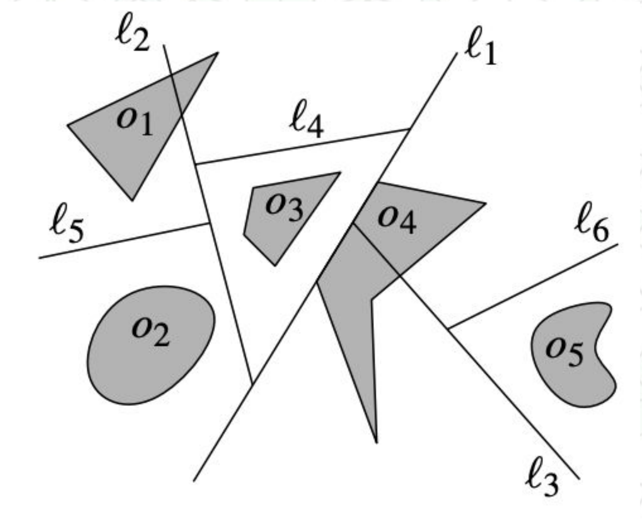
- k-D trees are a special case of BSP (where splits are always axis aligned)
- Quad trees are a special case of k-D trees
  (where splits are always at the midpoints)



Quad Tree



k-D Tree



BSP

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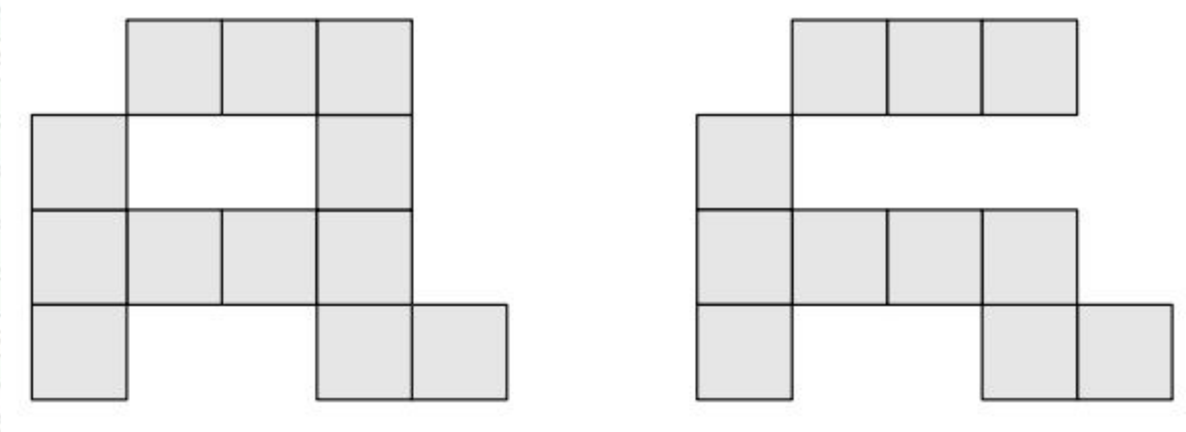
What is a Polyomino?

- An n-omino is a set of n cells on a square graph that is connected

is a polyomino **is NOT a polyomino**

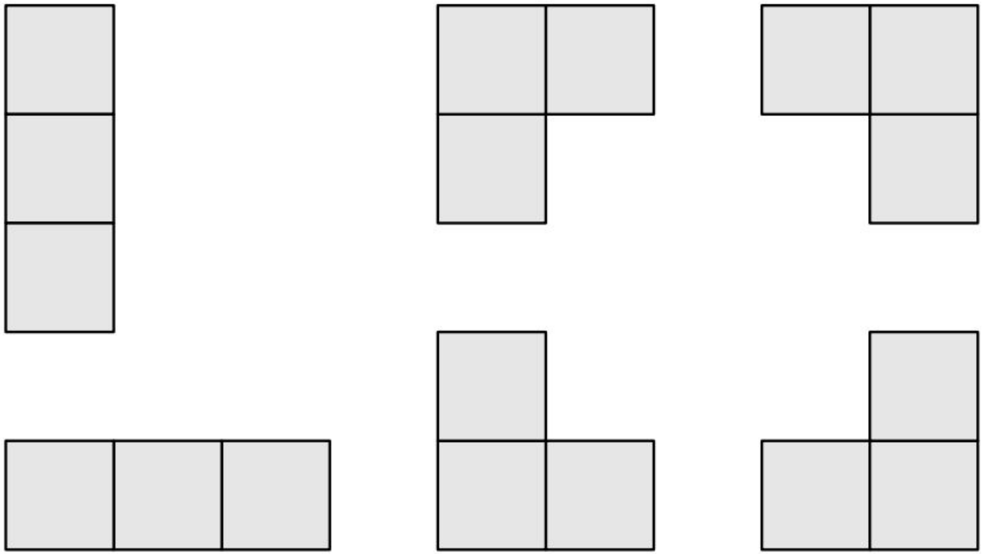


“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Translation-Equivalent / Fixed Polyomino

- Only left/right/up/down translation is allowed

- There are 6 unique Fixed 3-ominoes (a.k.a. trominoes):



“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Translation-Equivalent / Fixed Polyomino

- Only left/right/up/down translation is allowed

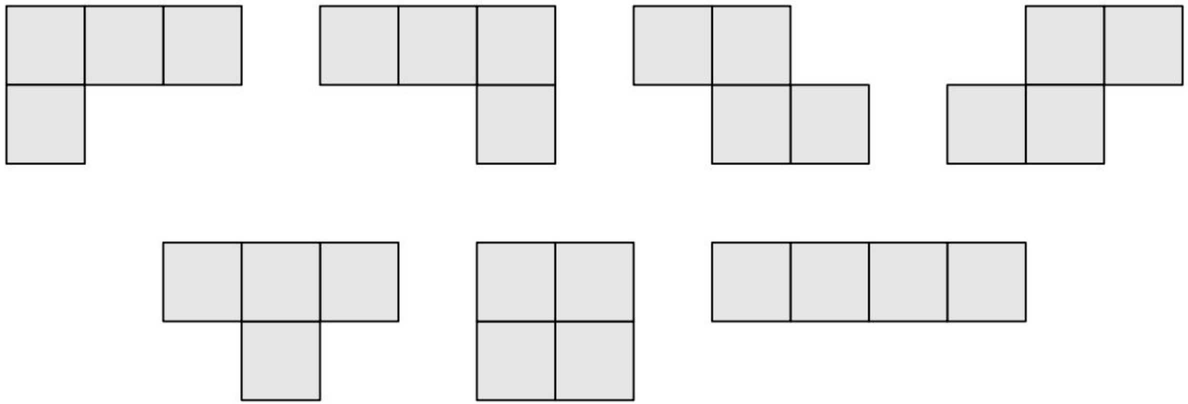
- *How many fixed 2-ominoes (a.k.a. dominoes) are there?*
- *Draw them!*

Rotation-Equivalent / Chiral Polyomino

Chiral: asymmetric in such a way that the structure and its mirror image are not superimposable

- left/right/up/down translation allowed
- 90°/180°/270° rotation allowed

- There are 7 unique chiral 4-ominoes (a.k.a. tetrominoes):

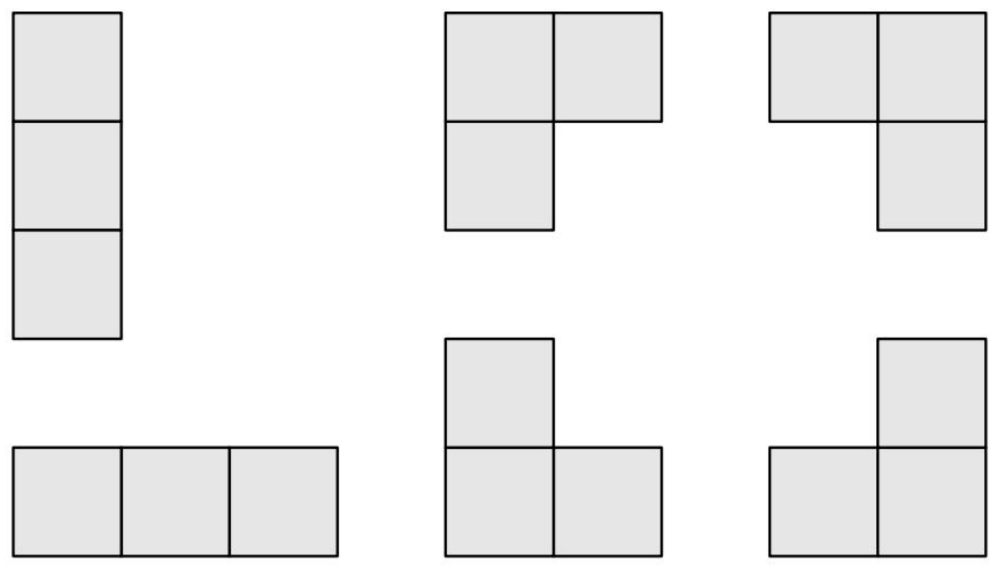


“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Rotation-Equivalent / Chiral Polyomino

- left/right/up/down translation allowed
- 90°/180°/270° rotation allowed

- *How many chiral 3-ominoes are there?*
- *Which of these shapes are rotationally-equivalent?*

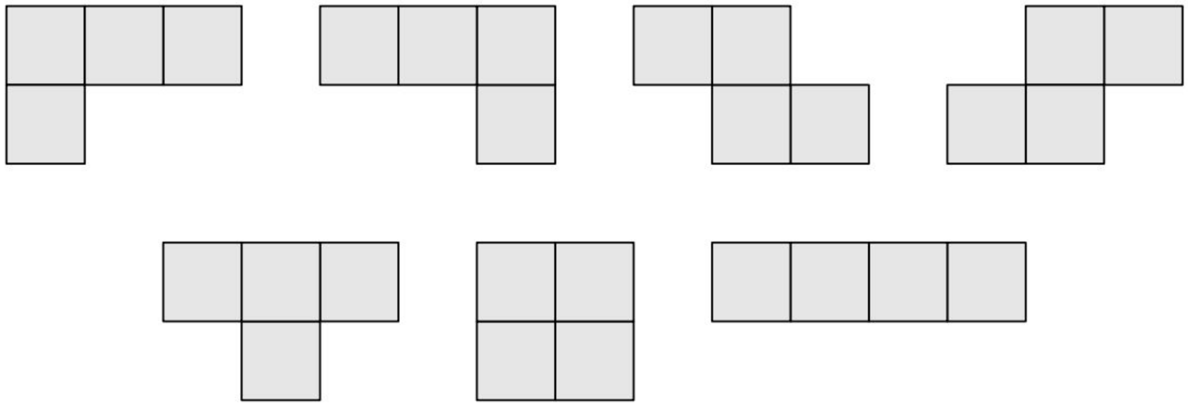


“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Translation-Equivalent / Fixed Polyomino

- Only left/right/up/down translation is allowed

- *How many fixed 4-ominoes are there?*
- *Which of these shapes are unique when rotated?*

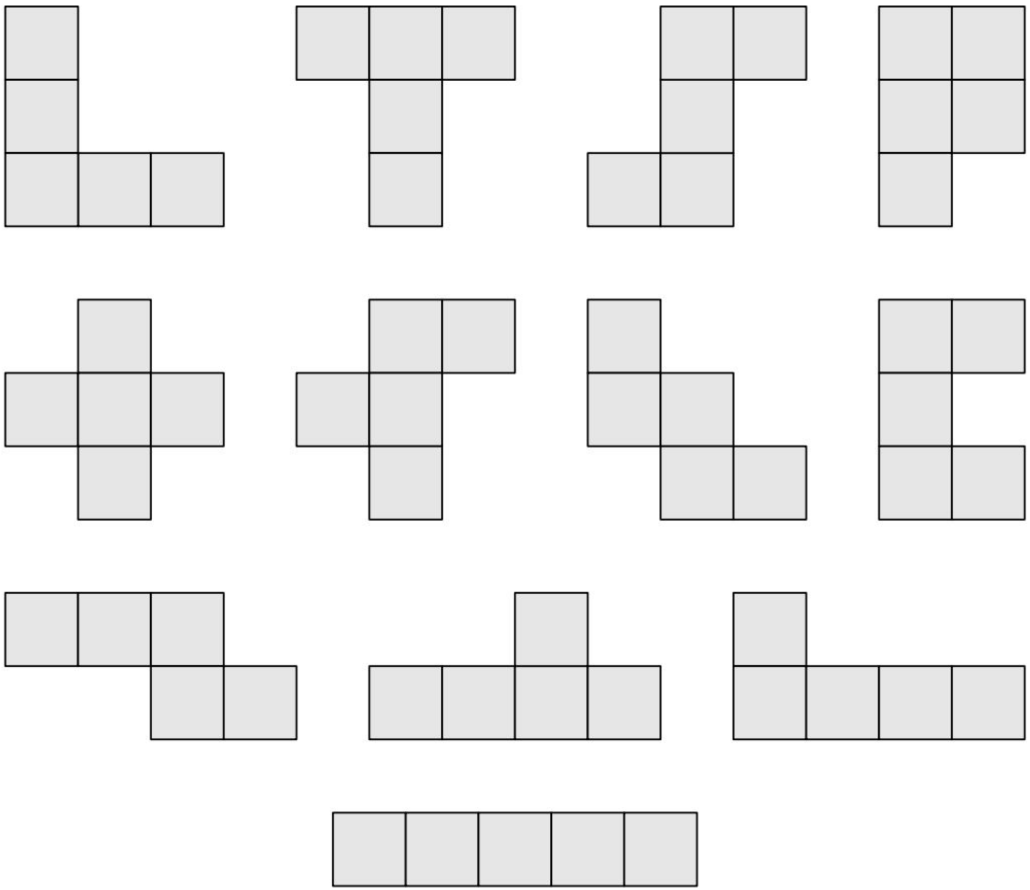


"Ch 14: Polyominoes", Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Free Polyomino

- Translation allowed
- Rotation allowed
- Reflection allowed

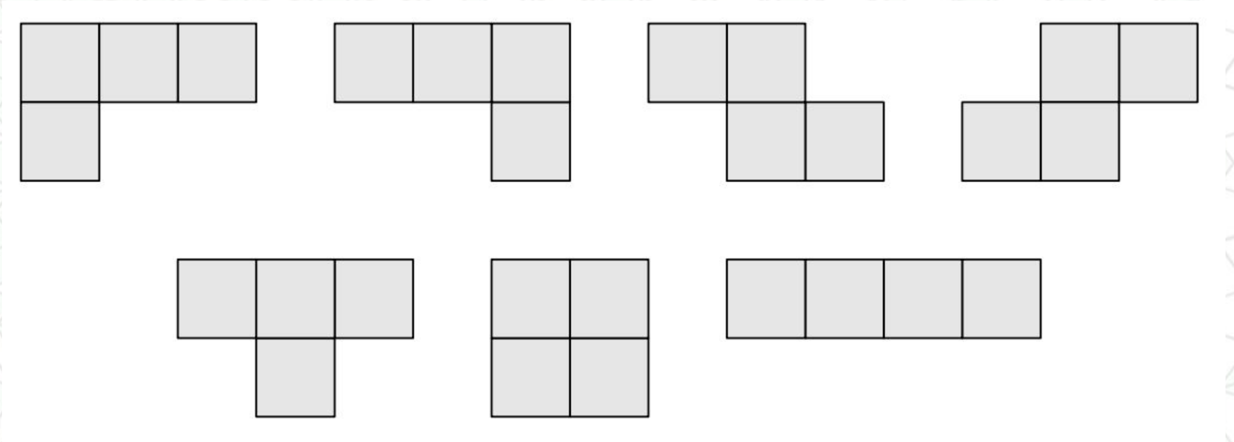
- There are 12 unique free 5-ominoes (a.k.a. pentominoes):



“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Congruent / Free Polyomino

- *How many free 4-ominoes are there?*
- *Which of these shapes are congruent? (duplicates when reflected)*

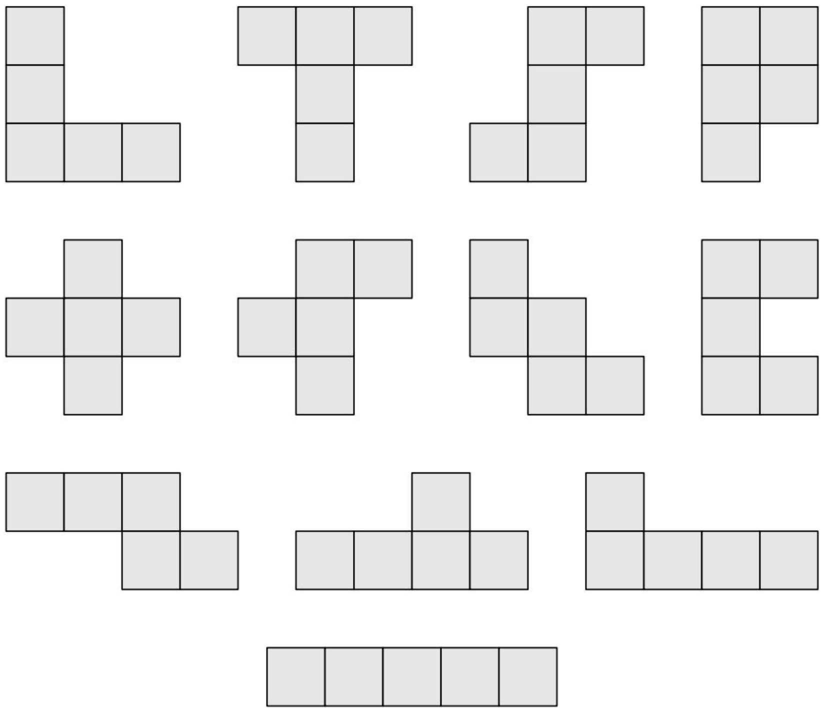


"Ch 14: Polyominoes", Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Rotation-Equivalent / Chiral Polyomino

- left/right/up/down translation allowed
- 90°/180°/270° rotation allowed

- *How many chiral 5-ominoes are there?*
- *Which of these shapes are unique when reflected?*

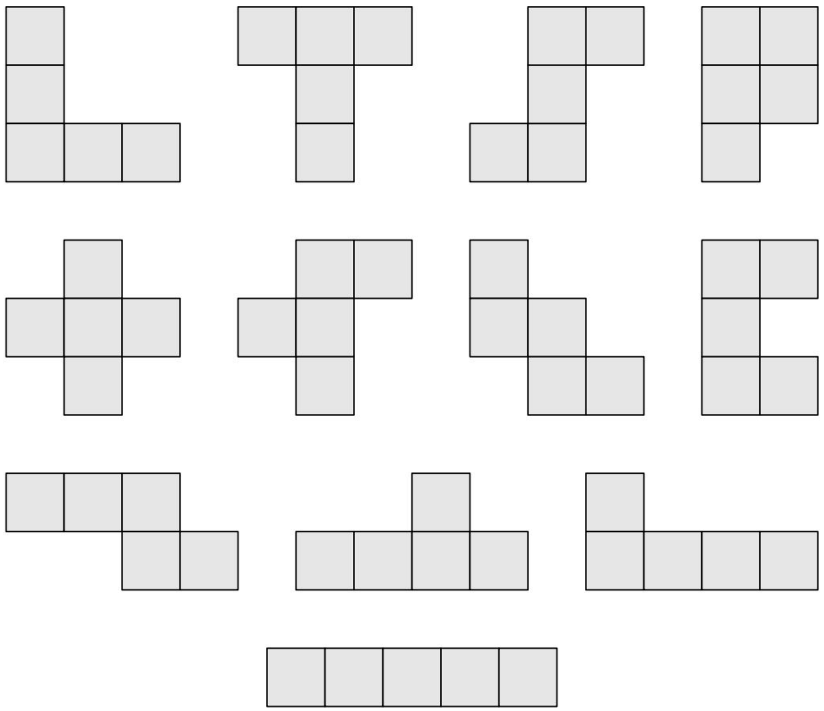


"Ch 14: Polyominoes", Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Translation-Equivalent / Fixed Polyomino

- Only left/right/up/down translation is allowed

- *How many fixed 5-ominoes are there?*
- *Which of these shapes are unique when rotated and/or reflected?*



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Counting Fixed, Chiral, and Free Polyominoes

	fixed	**chiral**	**free**
	translation-only	translation & rotation (no reflection)	translation, rotation, & reflection

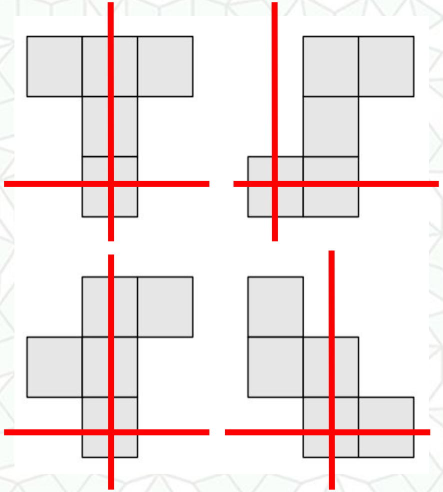
n	$t(n)$	$r(n)$	$s(n)$
1	1	1	1
2	2	1	1
3	6	2	2
4	19	7	5
5	63	18	12

"Ch 14: Polyominoes", Barequet, Golomb, & Klarner,
Handbook of Discrete and Computational Geometry, 2018

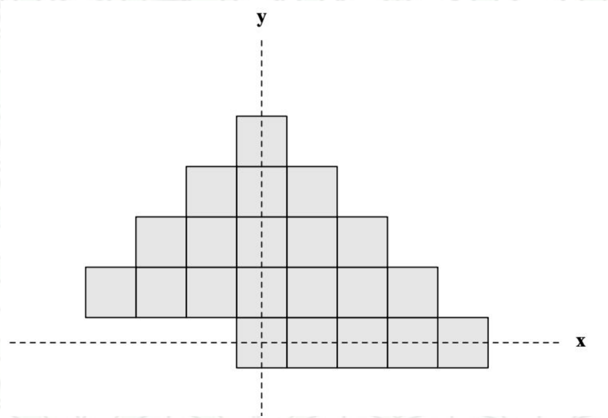
Counting Polyominoes

- *n*-omino Standard Position: Translate to place the leftmost cell in the bottom row at the origin.
- Enumerate all combinations of all possible cells
- Eliminate disconnected & duplicate ominoes
- # possible cells?

- Max # *n*-ominos?



all possible cells for 5-ominos



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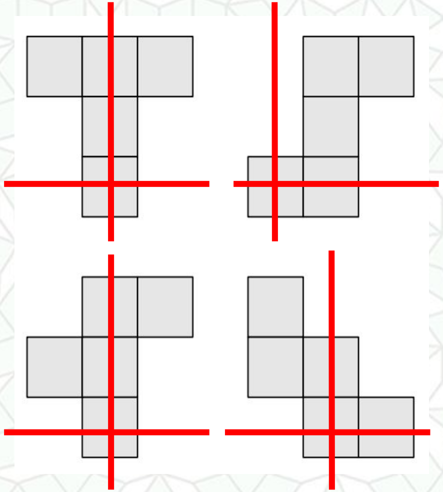
Counting Polyominoes

- *n*-omino Standard Position: Translate to place the leftmost cell in the bottom row at the origin.
- Enumerate all combinations of all possible cells
- Eliminate disconnected & duplicate ominoes
- # possible cells?

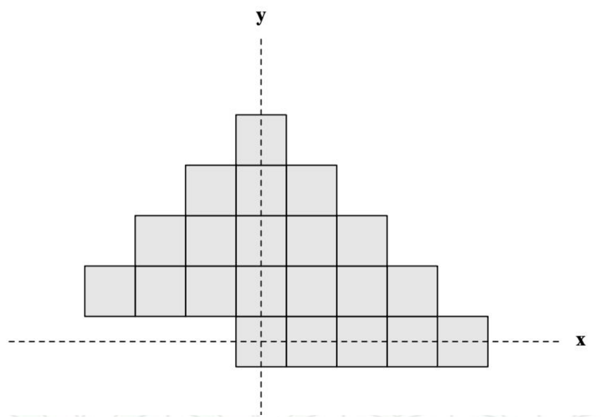
  $n(n-1) + 1$
- Max # *n*-ominos?

  $$\binom{n(n-1)}{n-1}$$
- Can show it's at most:

  $$\binom{3(n-1)}{n-1}$$



all possible cells for 5-ominos



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Counting Polyominoes

- What is the relationship *(e.g., inequalities < > = ≤ ≥)* between *t(n)*, *r(n)*, and *s(n)*?

	fixed	chiral	free
n	$t(n)$	$r(n)$	$s(n)$
1	1	1	1
2	2	1	1
3	6	2	2
4	19	7	5
5	63	18	12
6			
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22			
23			
24			

"Ch 14: Polyominoes", Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Counting Polyominoes

- What is the relationship *(e.g., inequalities < > = ≤ ≥)* between *t(n)*, *r(n)*, and *s(n)*?

$$\frac{t(n)}{8} \leq s(n) \leq r(n) \leq t(n)$$

“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

	fixed	chiral	free
n	$t(n)$	$r(n)$	$s(n)$
1	1	1	1
2	2	1	1
3	6	2	2
4	19	7	5
5	63	18	12
6	216	60	35
7	760	196	108
8	2725	704	369
9	9910	2500	1285
10	36446	9189	4655
11	135268	33896	17073
12	505861	126759	63600
13	1903890	476270	238591
14	7204874	1802312	901971
15	27394666	6849777	3426576
16	104592937	26152418	13079255
17	400795844	100203194	50107909
18	1540820542	385221143	192622052
19	5940738676	1485200848	742624232
20	22964779660	5741256764	2870671950
21	88983512783	22245940545	11123060678
22	345532572678	86383382827	43191857688
23	1344372335524	336093325058	168047007728
24	5239988770268	1309998125640	654999700403

Counting Polyominoes

- The number of polyominoes, *t(n)* is exponential in *n*.

  Current unproved estimate ≈ 4.06^{n}

- The running time of the current best algorithm to count *t(n)* is also exponential (but smaller)

  $O(3^{n/2}) \approx O(1.73^{n})$

- Can t(n) be computed in poly time?
  Open problem!!

	fixed	chiral	free
n	$t(n)$	$r(n)$	$s(n)$
1	1	1	1
2	2	1	1
3	6	2	2
4	19	7	5
5	63	18	12
6	216	60	35
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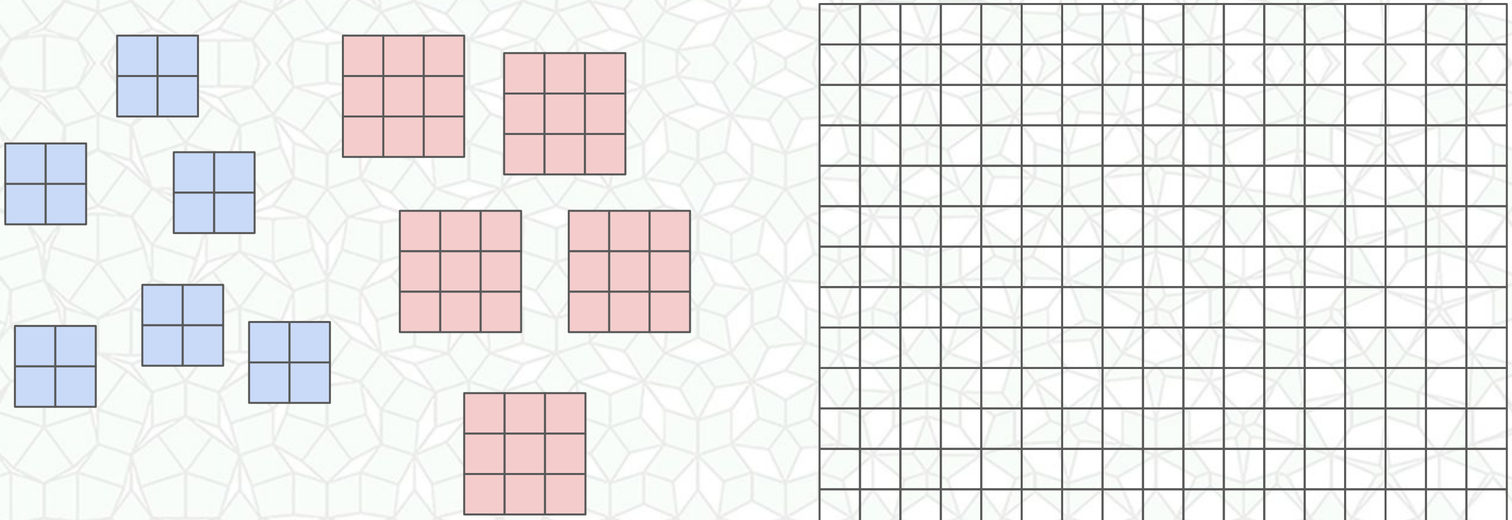
Packing Polyominoes

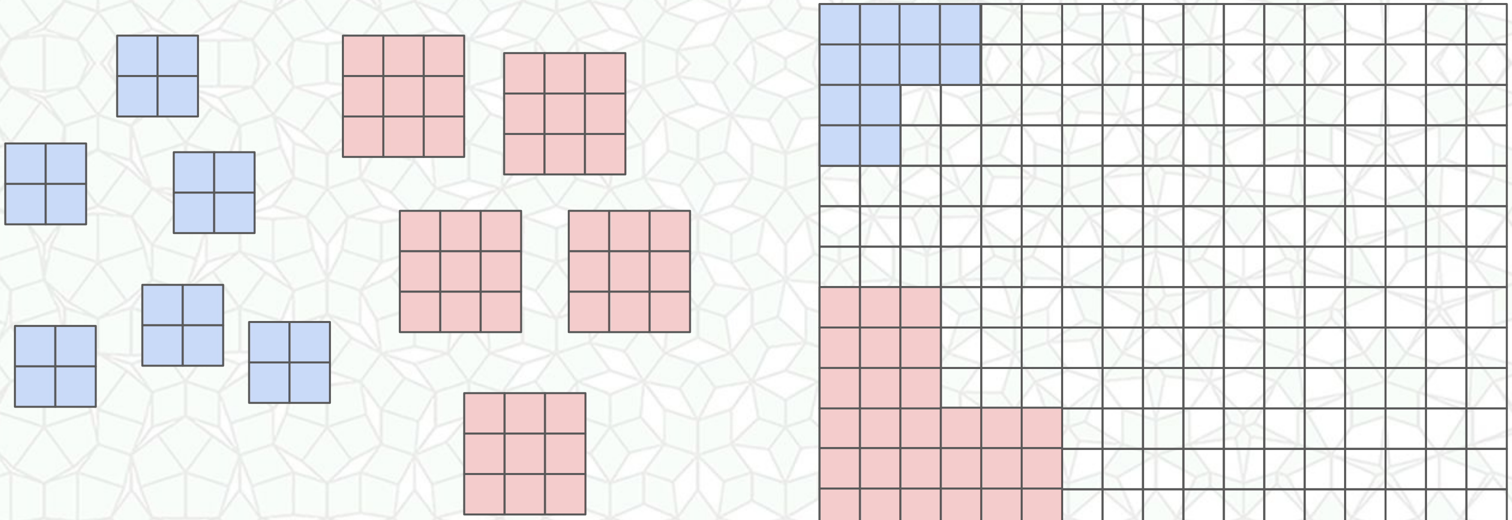
- Can we use 2x2 square 4-ominoes and 3x3 square 9-ominoes to cover (without overlaps) a 13x17 rectangle?



Packing Polyominoes

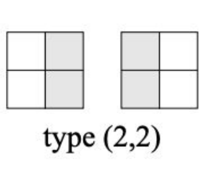
- Can we use 2x2 square 4-ominoes and 3x3 square 9-ominoes to cover (without overlaps) a 13x17 rectangle?

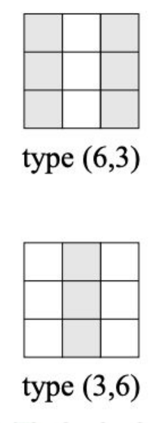
*Maybe…. counting cells: (17*4) + (17*9) = 17 * (9+4) = 17 * 13 = 221*

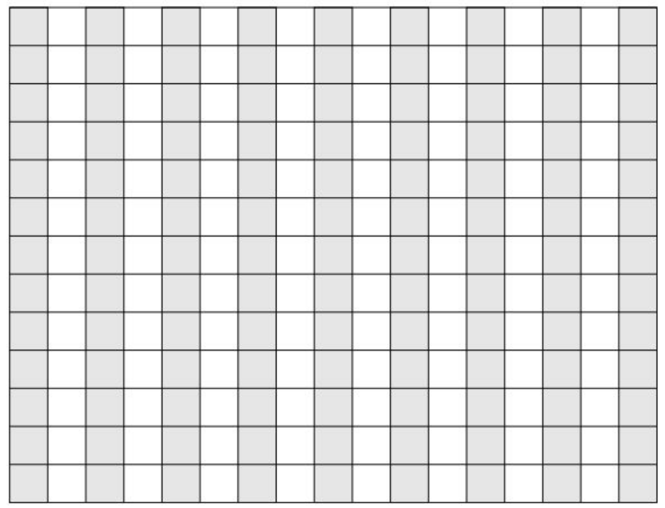


Packing Polyominoes

- Actually, this packing is not possible, and can be proven by contradiction using this coloring scheme







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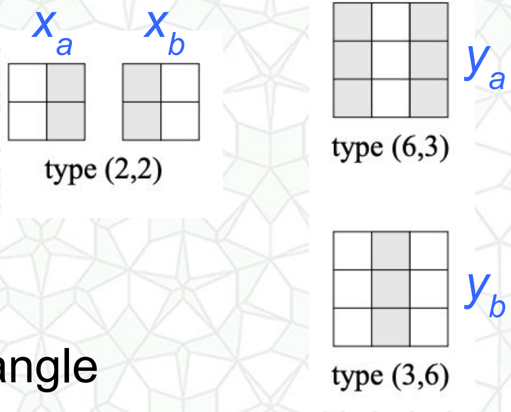
Packing Polyominoes



- Actually, this packing is not possible, and can be proven by contradiction using this coloring scheme

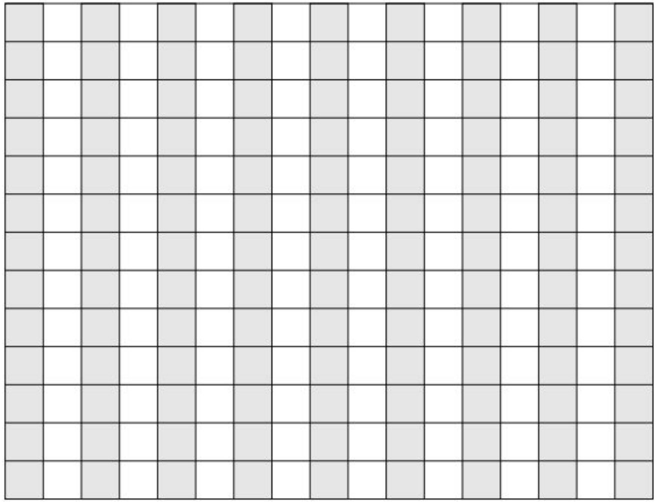
$13*9=117$ grey cells + $13*8=104$ white cells in the rectangle

$x_a*2 + x_b*2 + y_a*6 + y_b*3 = 117$ grey cells
$x_a*2 + x_b*2 + y_a*3 + y_b*6 = 104$ white cells
in the ominoes

$117 - y_a*6 - y_b*3 = 104 - y_a*3 - y_b*6$

$13 = y_a*3 - y_b*3$

$13 = 3 * (y_a - y_b)$ *no integer solutions!*



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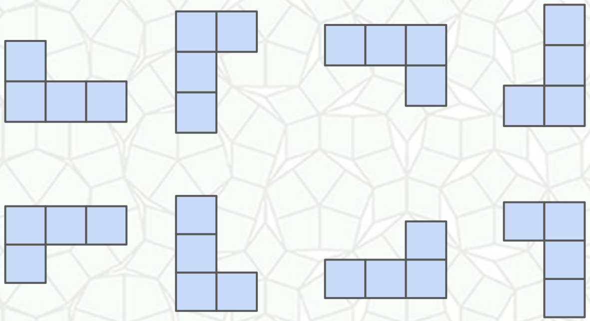
Packing Polyominoes

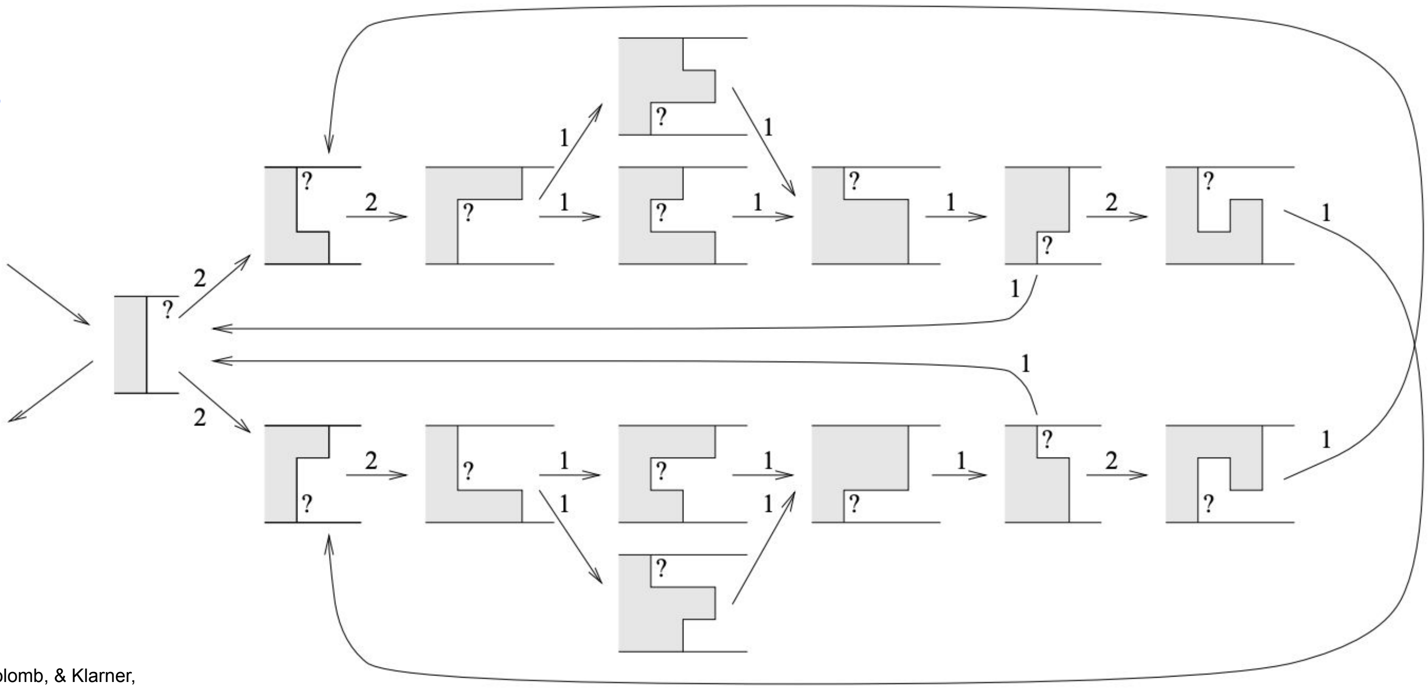
- Can we use the L-tetronimo, and all of its rotations and reflections to pack tile and infinite rectangle of height 3?



Packing Polyominoes

- Can we use the L-tetronimo, and all of its rotations and reflections to pack tile and infinite rectangle of height 3?

- *Yes, we can build the following automaton of all of states:*



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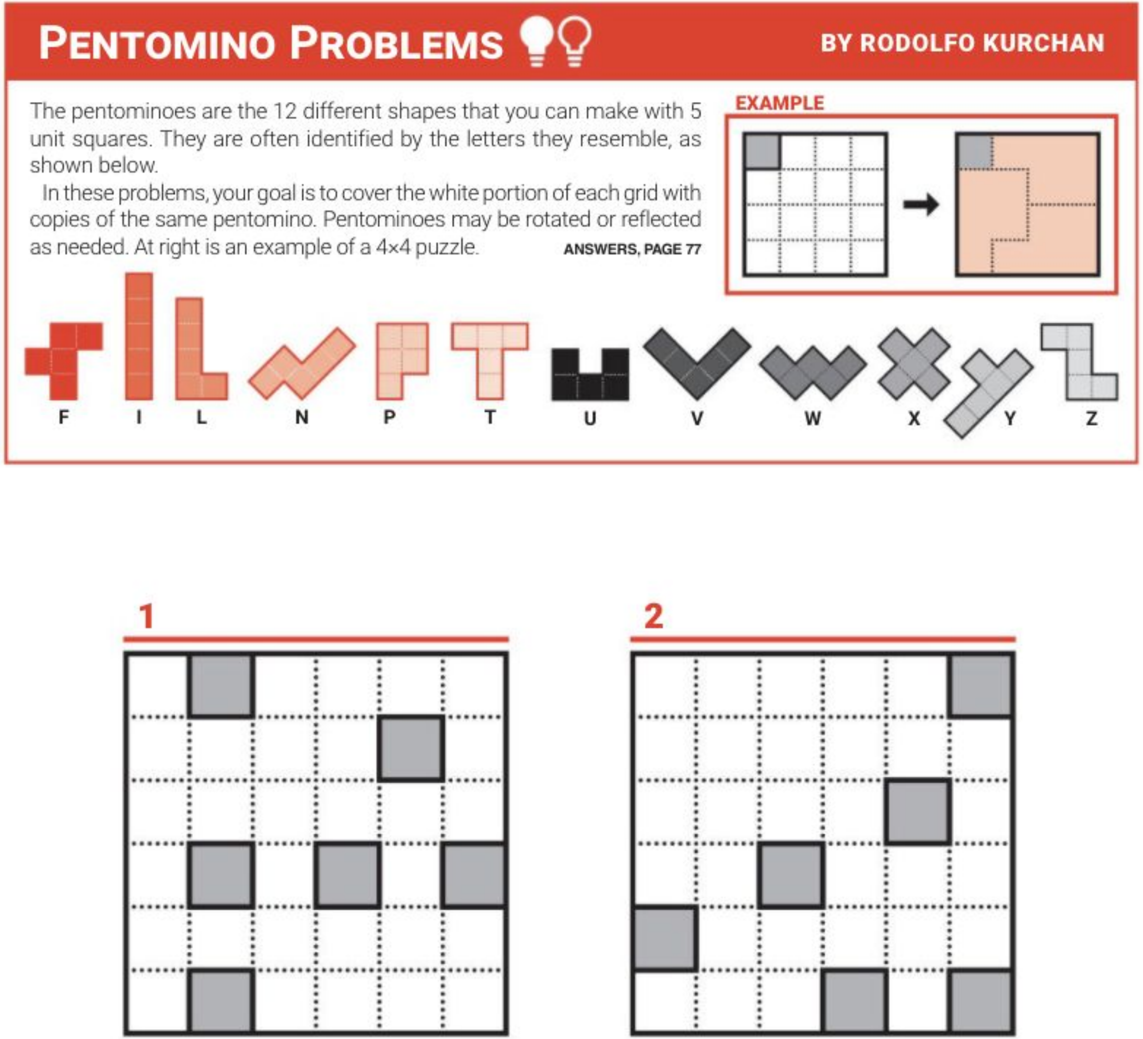
Puzzle from *Games Magazine* January 2022



PENTOMINO PROBLEMS

BY RODOLFO KURCHAN

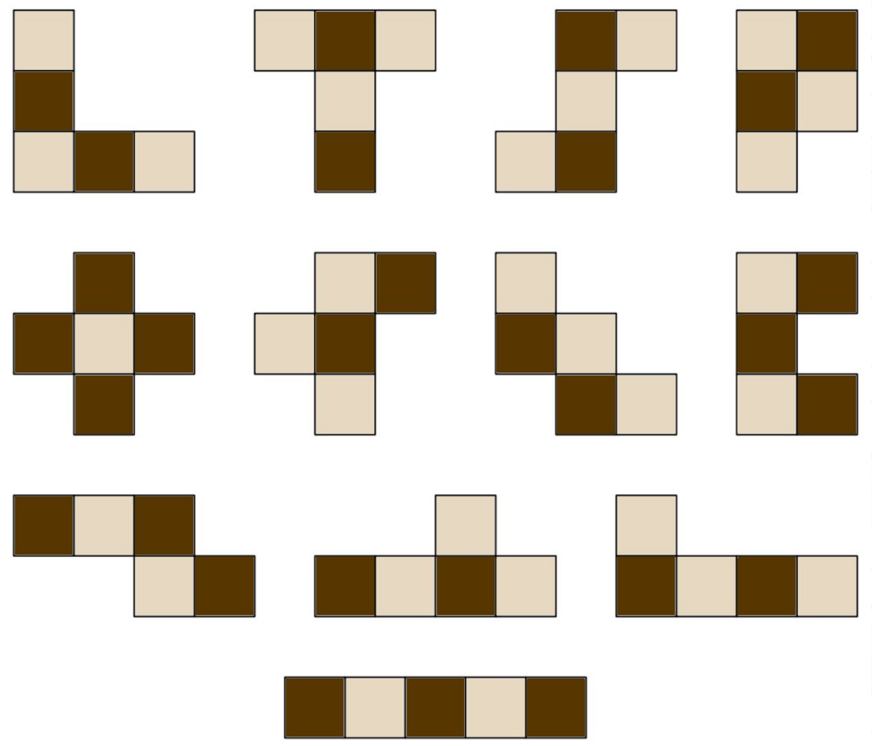
The pentominoes are the 12 different shapes that you can make with 5 unit squares. They are often identified by the letters they resemble, as shown below.

In these problems, your goal is to cover the white portion of each grid with copies of the same pentomino. Pentominoes may be rotated or reflected as needed. At right is an example of a 4×4 puzzle. ANSWERS, PAGE 77

3D Packing Puzzle: Bill's Checkerbox

https://billcutlerpuzzles.com/stock/checkerbox.html

3D Packing Puzzle: Bill’s Checkerbox

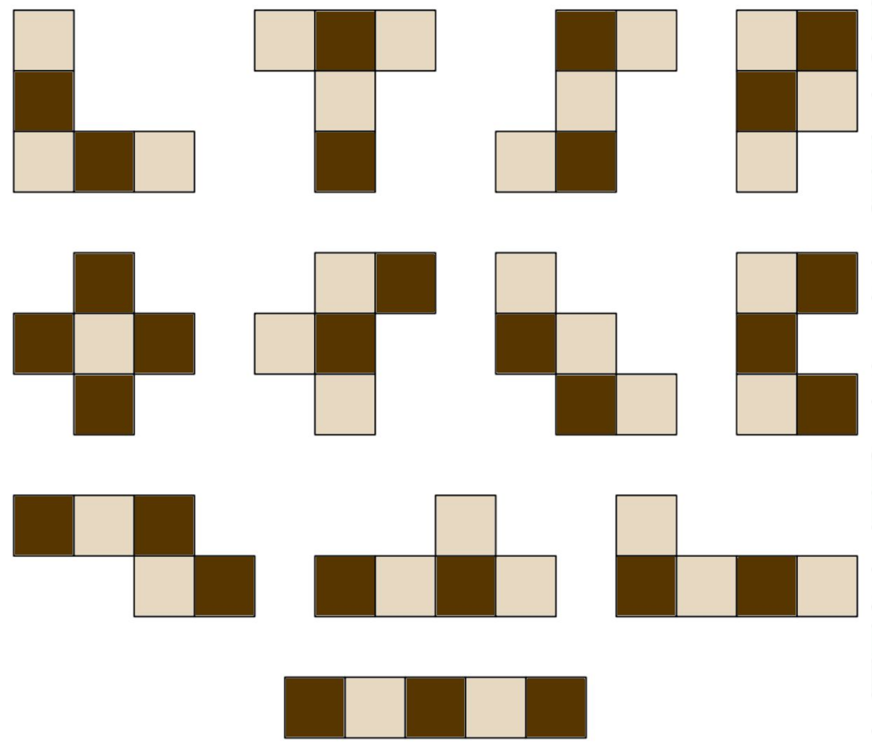


https://billcutlerpuzzles.com/stock/checkerbox.html

3D Packing Puzzle: Bill's Checkerbox

- How many cubes are needed to fill the box?
- How many dark cubes from the pieces?
- How many light cubes from the pieces?

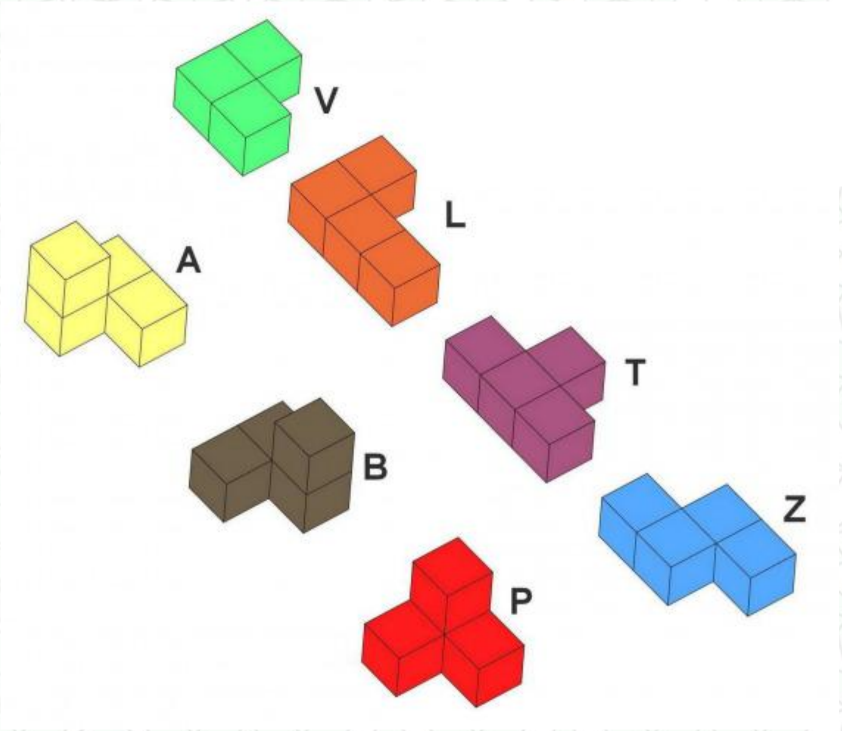
- *Many ways to pack the pieces into the box ignoring the checkerboard pattern.*

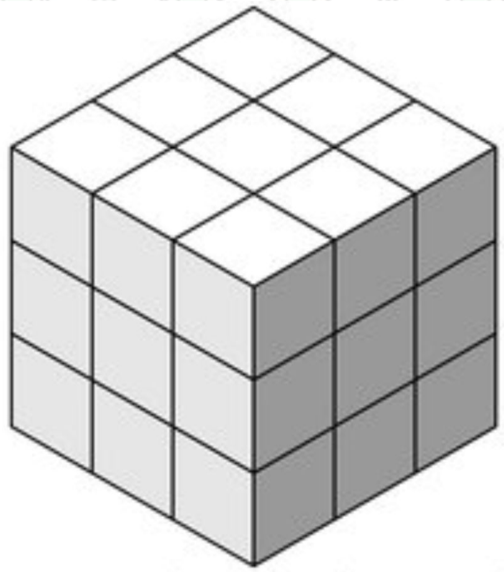


https://billcutlerpuzzles.com/stock/checkerbox.html

3D Packing Puzzle: Soma Cube

all possible combinations of three or four unit cubes, joined at their faces, such that at least one inside corner is formed.



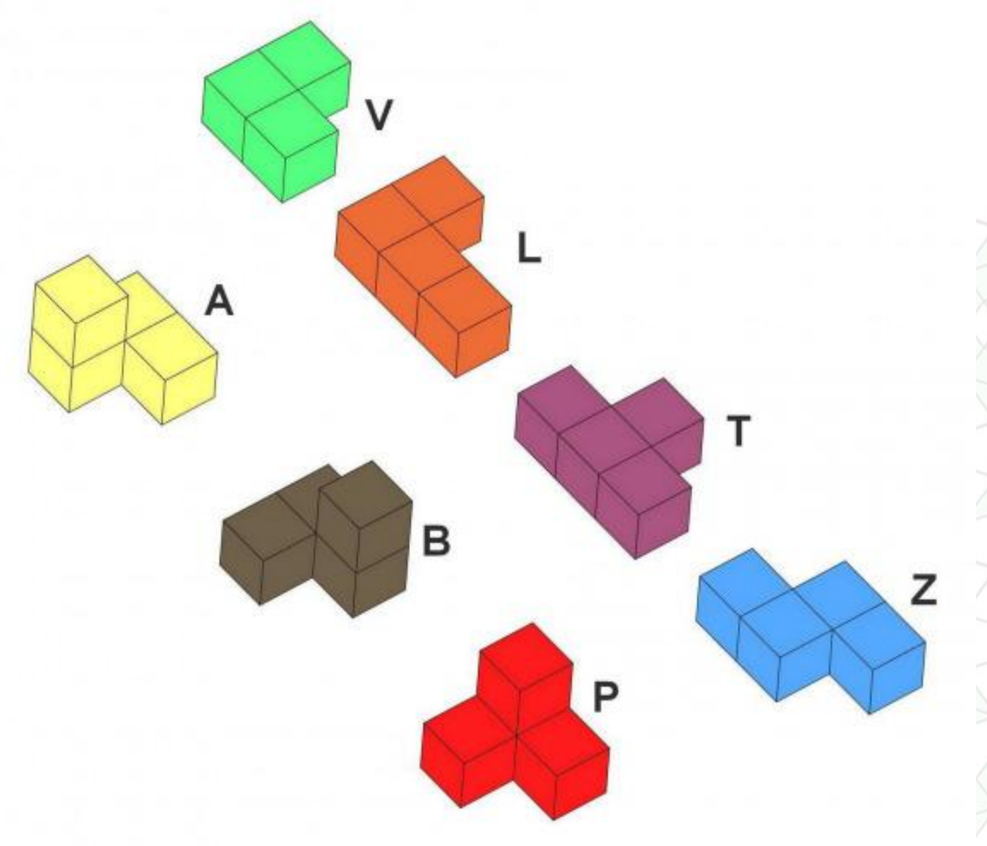


Pack into a 3x3x3 box

3D Packing Puzzle: Soma Cube

- Let’s count corners…
- For each piece,
  for each possible
  placement,

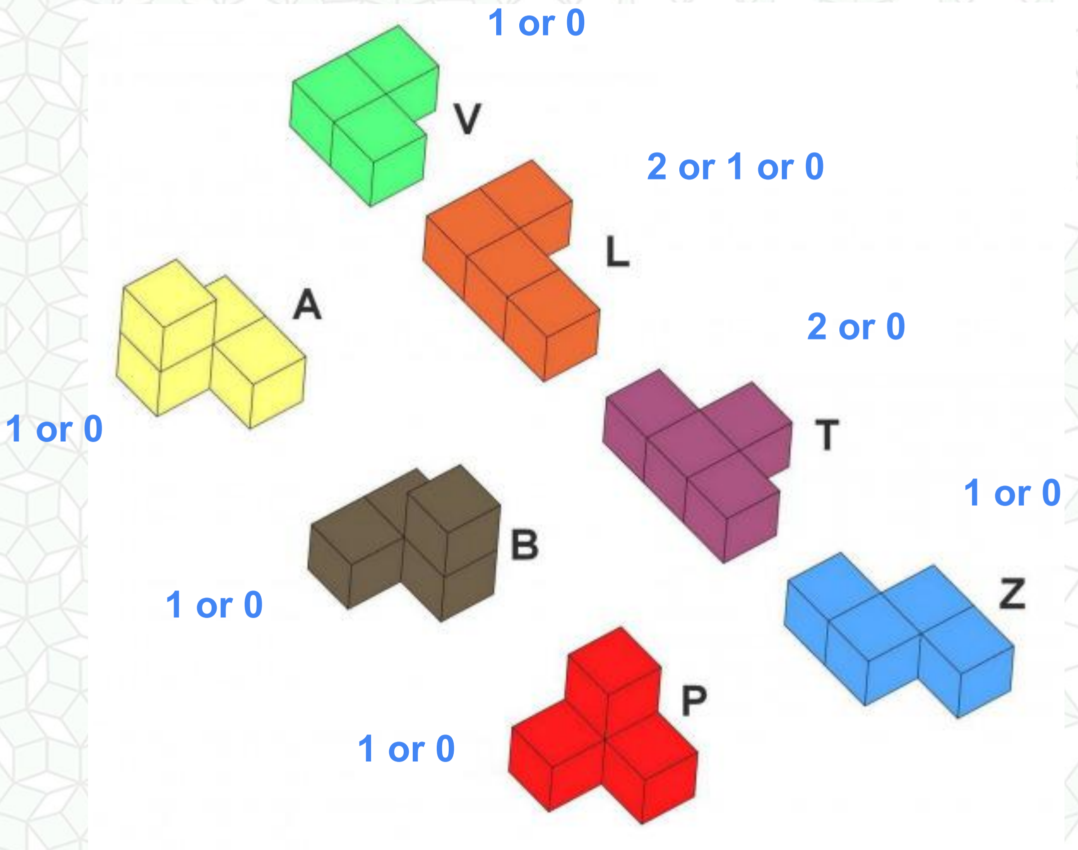
  How many of the
  8 box corners
  can it cover?



3D Packing Puzzle: Soma Cube

- Let's count corners…
- For each piece, for each possible placement,

  How many of the 8 box corners can it cover?

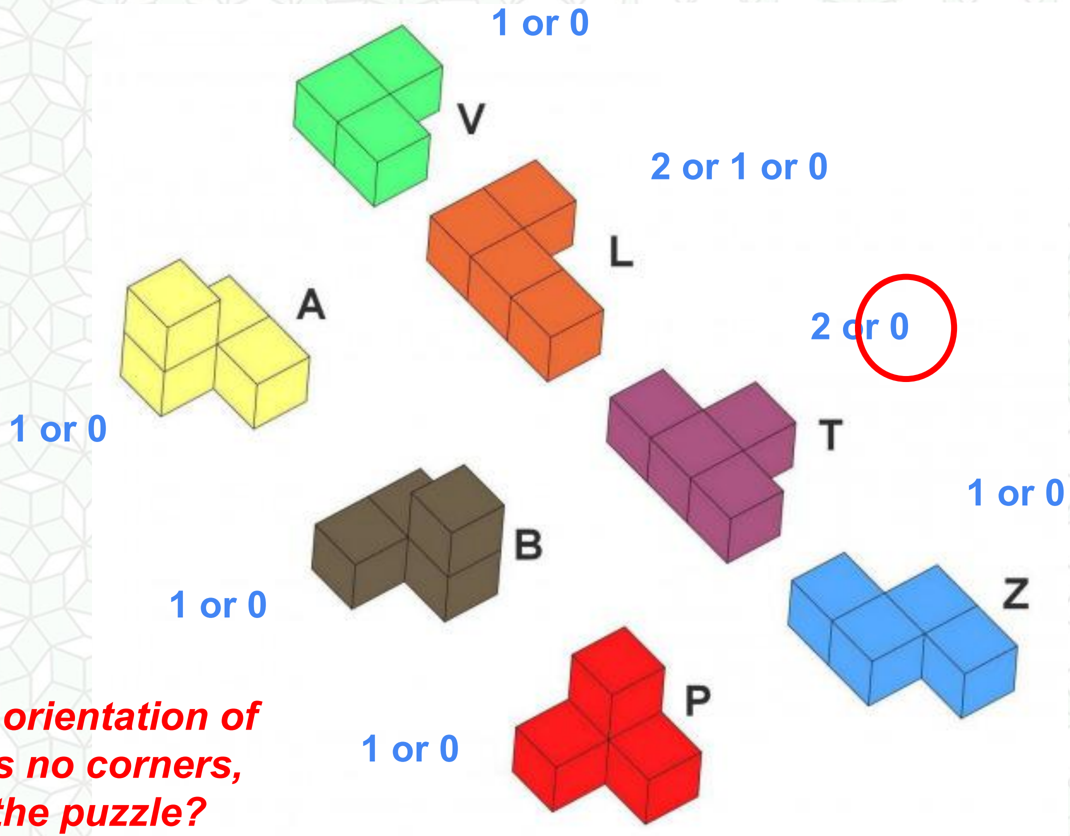


3D Packing Puzzle: Soma Cube

- Let's count corners…

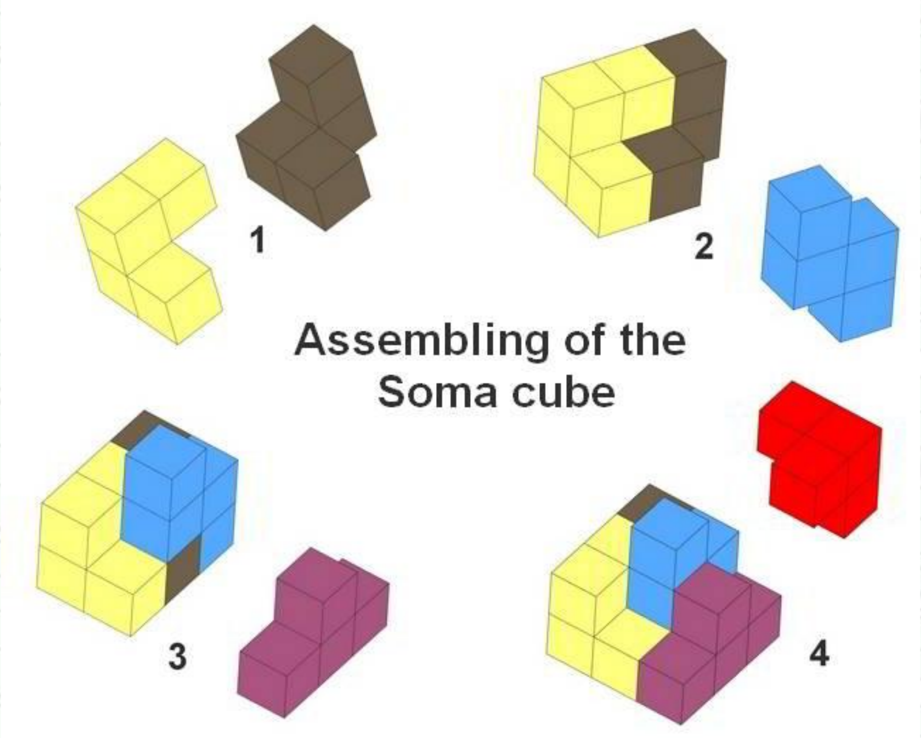
- For each piece, for each possible placement,

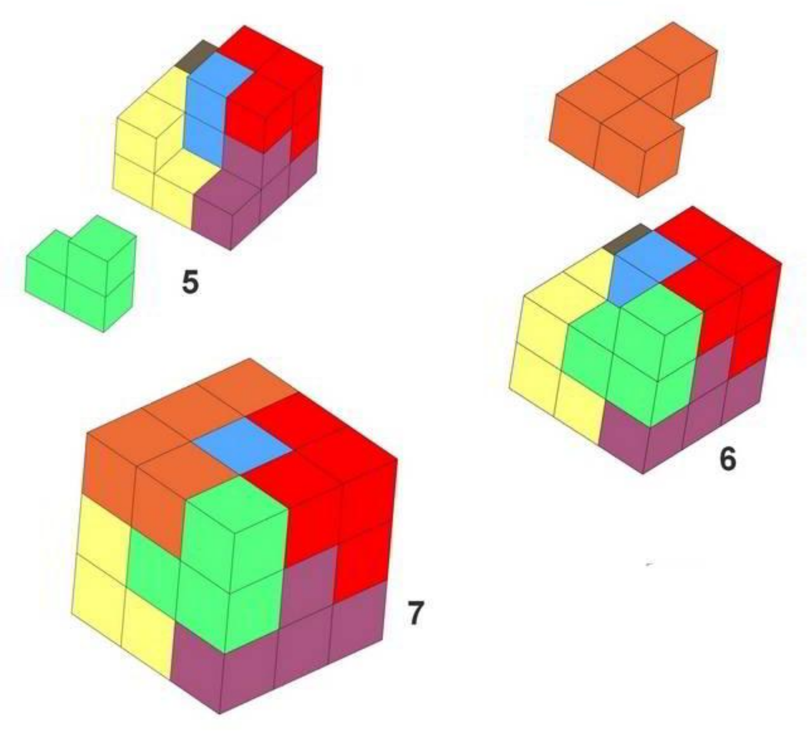
  How many of the 8 box corners can it cover?



If we choose the orientation of the T that covers no corners, can we solve the puzzle?

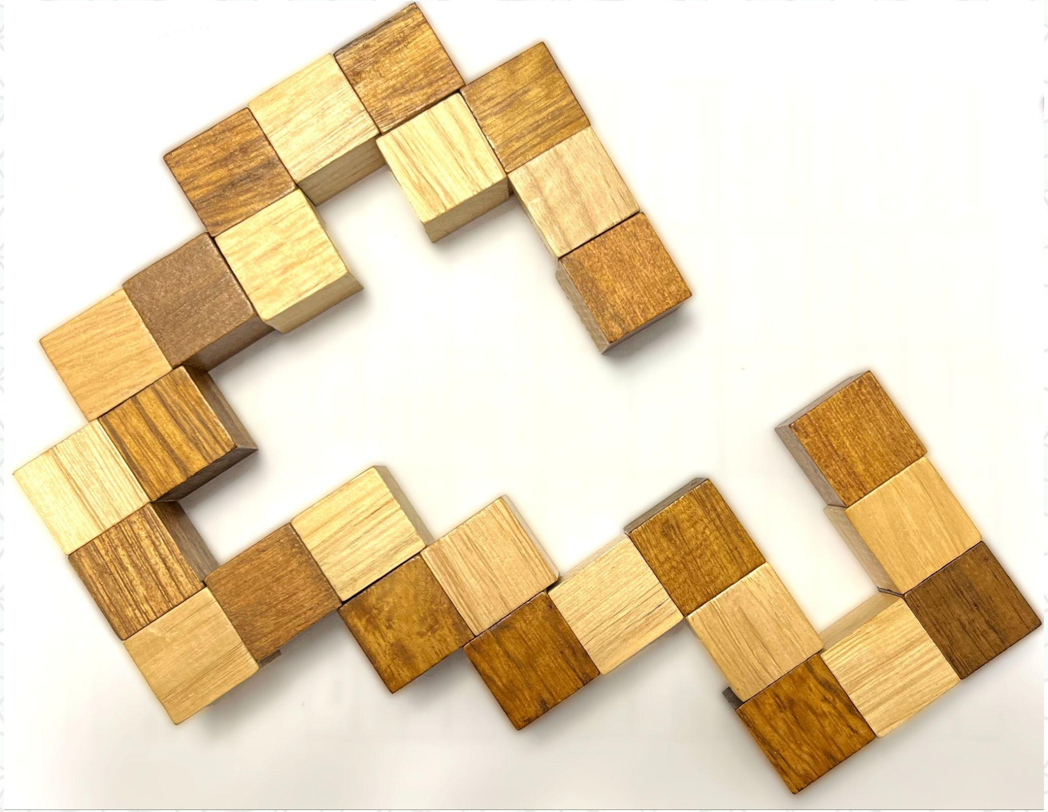
3D Packing Puzzle: Soma Cube



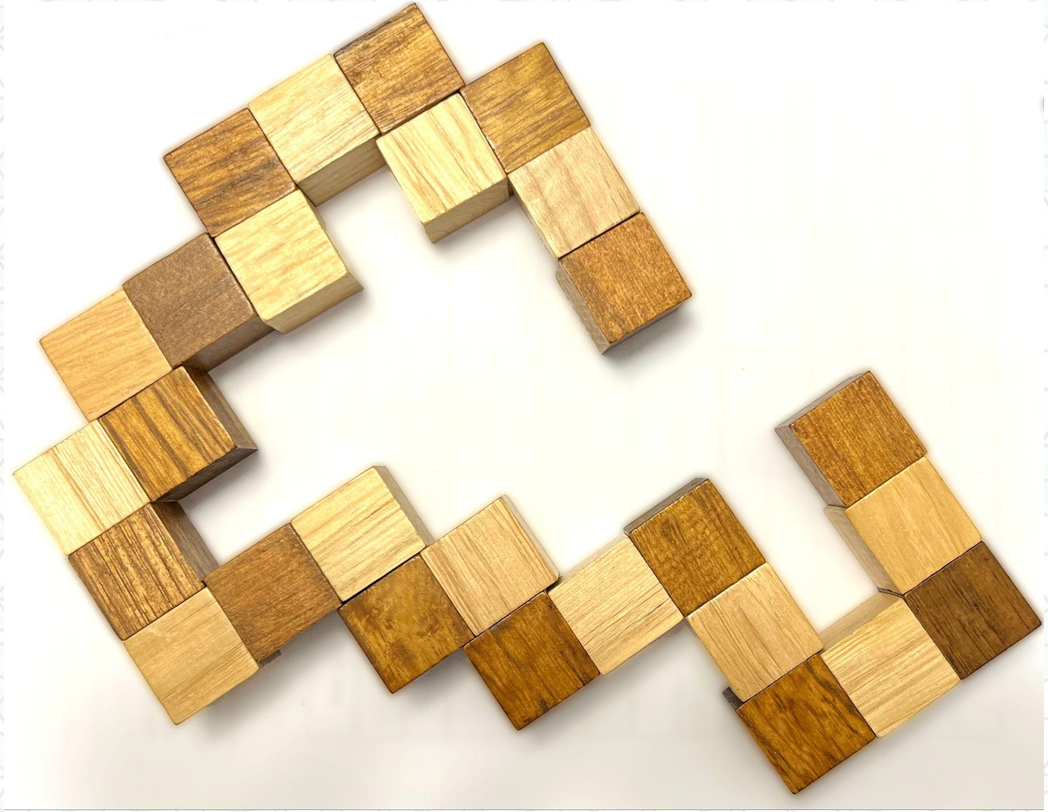


https://www.craftsmanspace.com/free-projects/make-a-soma-cube-puzzle.html

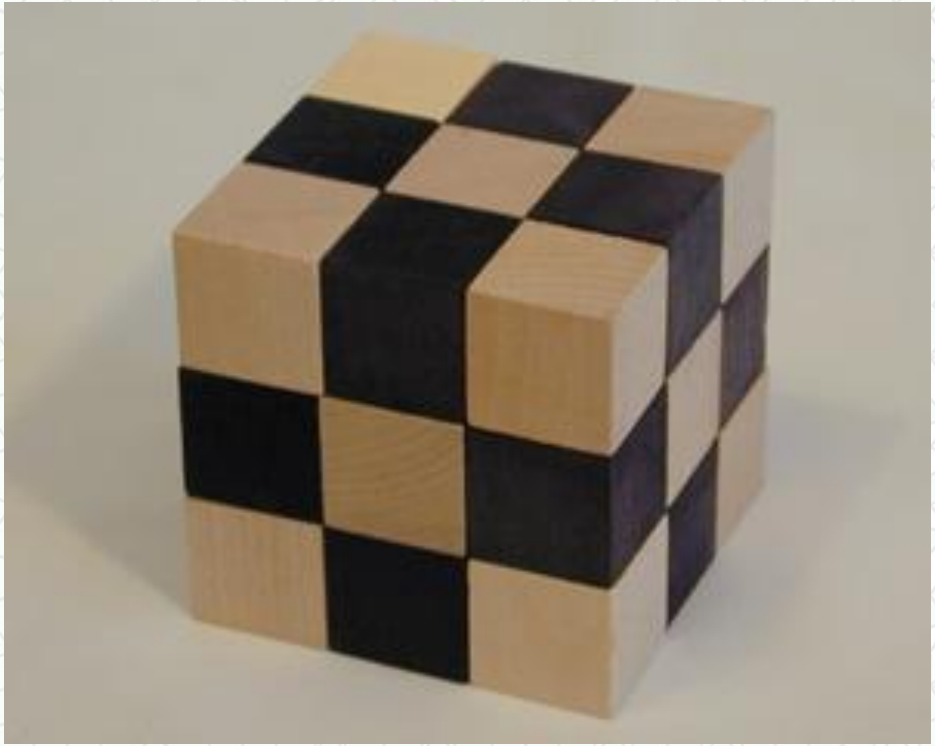
3D Packing Puzzle: Snake Cube



3D Packing Puzzle: Snake Cube



3D Packing Puzzle: Splitting Headache



http://billcutlerpuzzles.com/stock/splittingheadache.html

3D Packing Puzzle: Splitting Headache

http://billcutlerpuzzles.com/stock/splittingheadache.html

Outline for Today

- Final Project Questions?
- Last Time: Hidden Line Drawing, Painter’s Algorithm, & BSP
- Polyominoes Terminology
- Counting Polyominoes
- Tiling / Packing Polyominoes
- Polyomino Themed Puzzles
- Next Time: More Tiling!