CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/F23/

Lecture 23: Curves, Sketching, & Polyline Simplification

Outline for Today

- Last Time: Periodic & Non-Periodic Tiling
- Curve/Surface Continuity & Bezier Curves
- Polyline Simplification
- A Fun COVID Lockdown Project: Long Tiny Loops
- Clothoid or Cornu/Euler Spiral
- Hand-Drawn Sketch Smoothing
- Curve/Surface Reconstruction
- Next Time: Robot Motion Planning

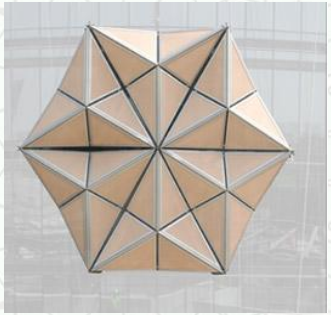
Zellij - Mosaic Tilework

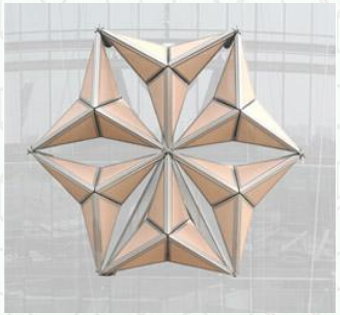
https://en.wikipedia.org/wiki/Zellij

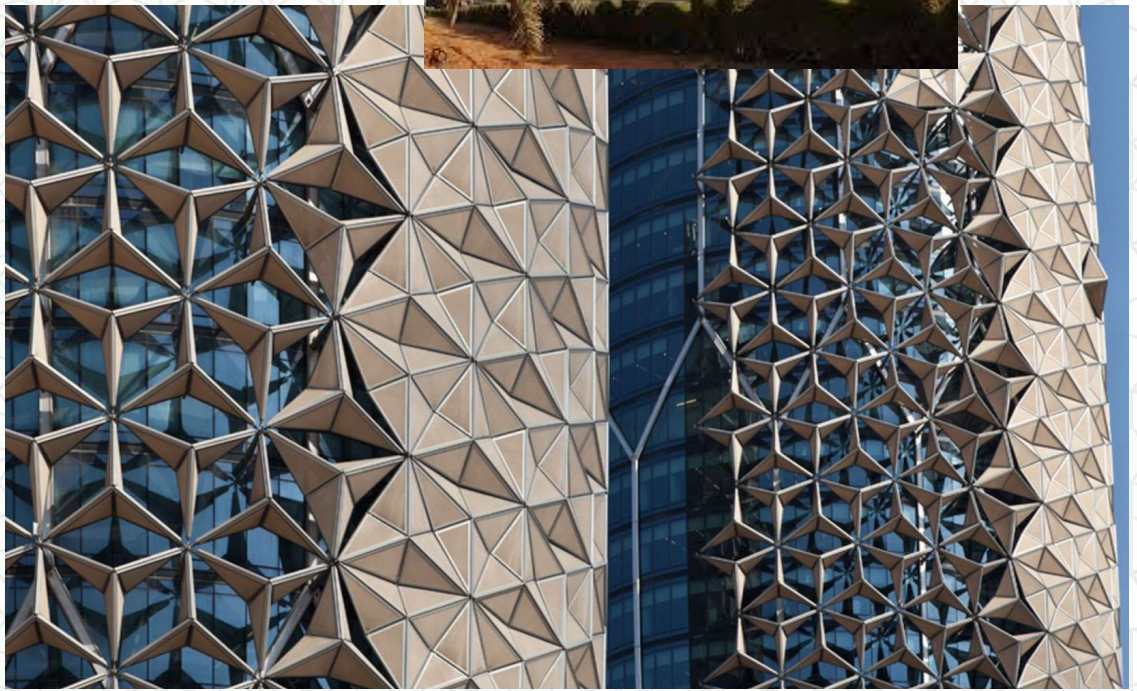
Kinetic Architecture

Al Bahar Towers, Abu Dhabi, UAE

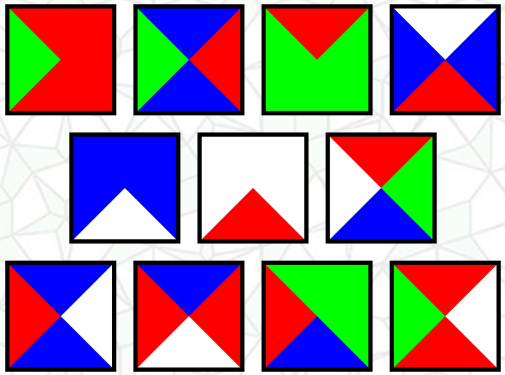
Aedas UK, Diar Consult, Arup, 2012



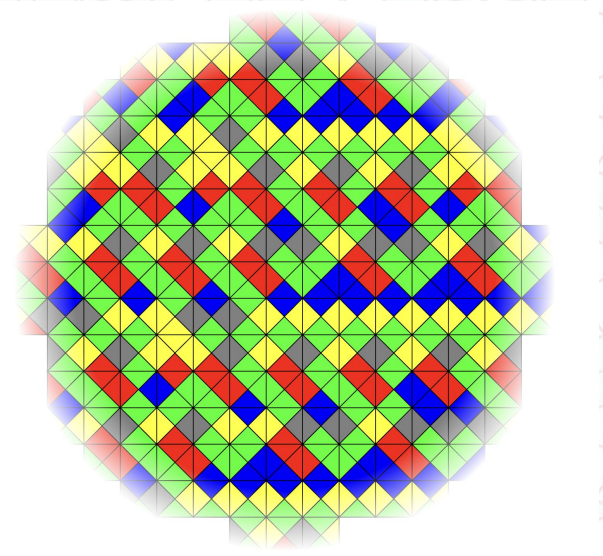




Wang Tiles / Wang Dominoes

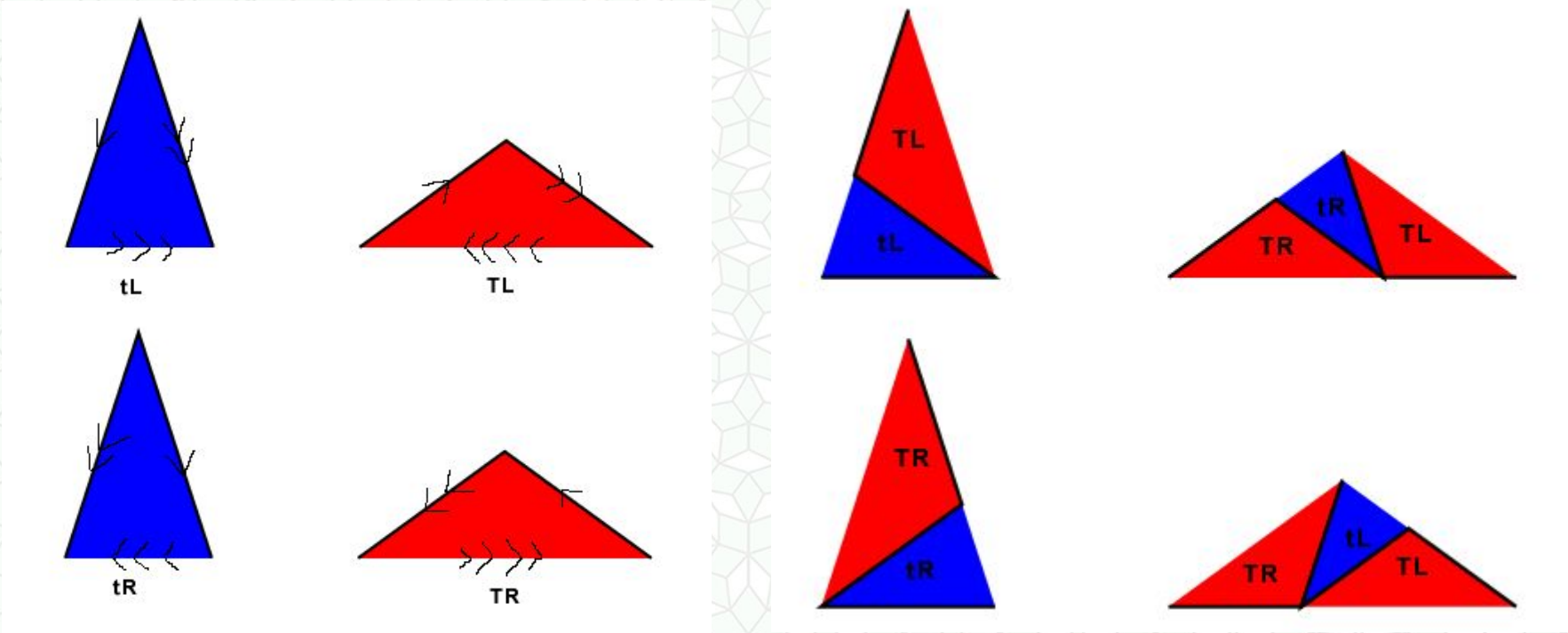


- Square tiles, edges labeled with colors, must be placed without rotation, with matching edges
- In 1961, Hao Wang conjectured that any finite set of tiles that could tile a plane infinitely, could be tiled periodically
- In 1966, Robert Berger proved that non-periodic Wang tile sets existed
- In 2015, Emmanuel Jeandel and Michael Rao proved that the smallest non-periodic Wang tile set was 11 tiles w/ 4 colors
- Applications: natural-looking, aperiodic synthesized texture, heightfields, & more



Penrose Tiling Subdivision

- *And conversely, this is how they are proved to be aperiodic!*



https://personal.math.ubc.ca/~cass/courses/m308-02b/projects/schweber/penrose.html

Penrose Tiling Subdivision

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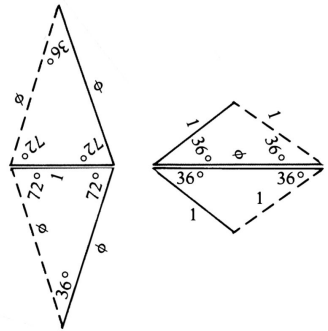

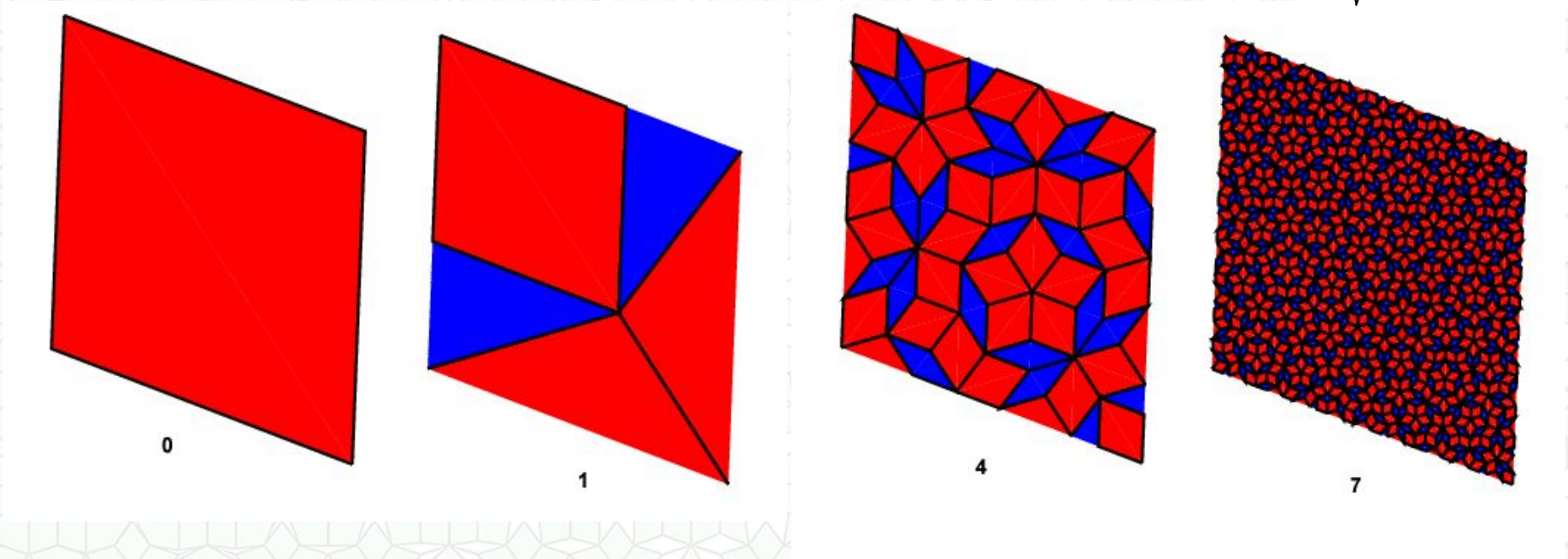

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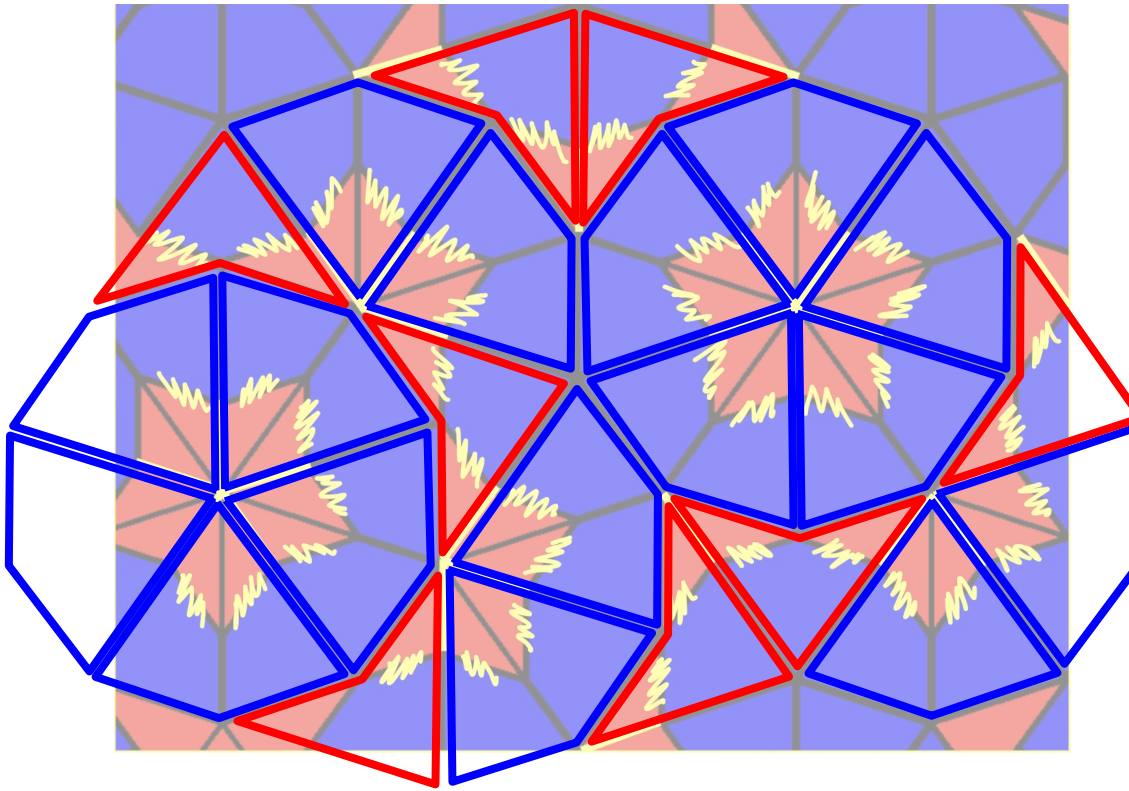
Penrose Tiling Inflation (Inverse of Subdivision)

- Cut each dart in half
- Merge each half dart with the kite along its short edge

- Now we have a new tiling with larger kites & darts
- & Repeat



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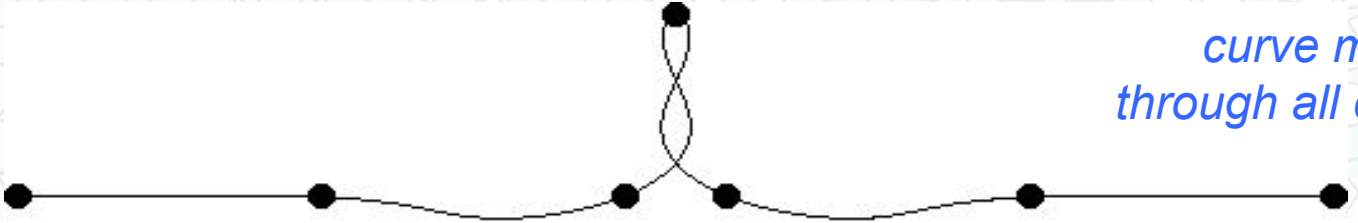
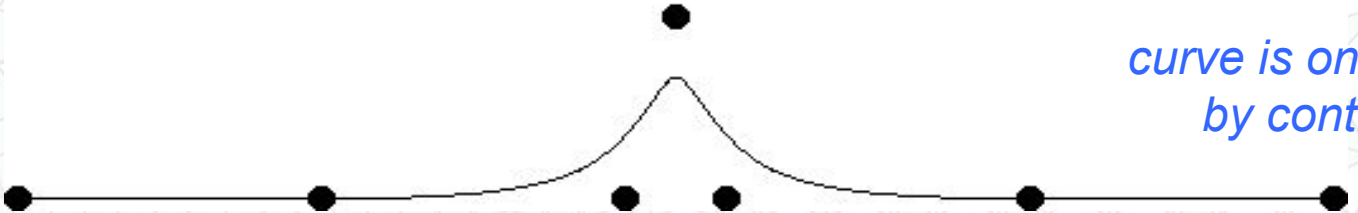
Interpolation vs. Approximation Curves

- Interpolation Curve – over constrained → lots of (undesirable?) oscillations



curve must pass through all control points

- Approximation Curve – more reasonable?



curve is only influenced by control points

Interpolation Curves / Splines

Used in boat/car/guitar drafting:
“spline” is a thin strip of wood
“ducks” are heavy lead weights

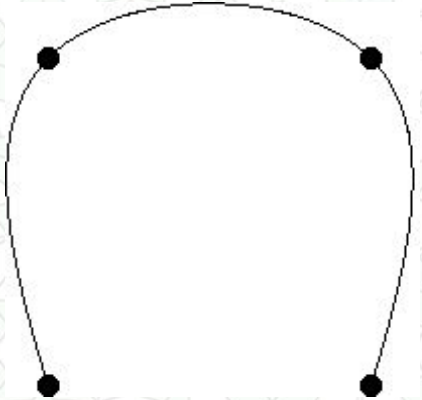
http://www.frets.com/FretsPages/Luthier/TipsTricks/DraftingWeights/draftweights.html

Interpolation Curves

- Curve is constrained to pass through all control points

- Given points P_0, P_1, ... P_n, find lowest degree polynomial which passes through the points

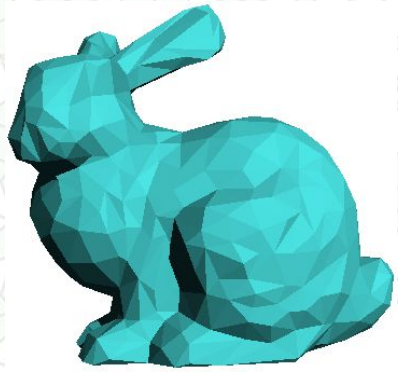
$$x(t) = a_{n-1}t^{n-1} + ... + a_2t^2 + a_1t + a_0$$
$$y(t) = b_{n-1}t^{n-1} + ... + b_2t^2 + b_1t + b_0$$

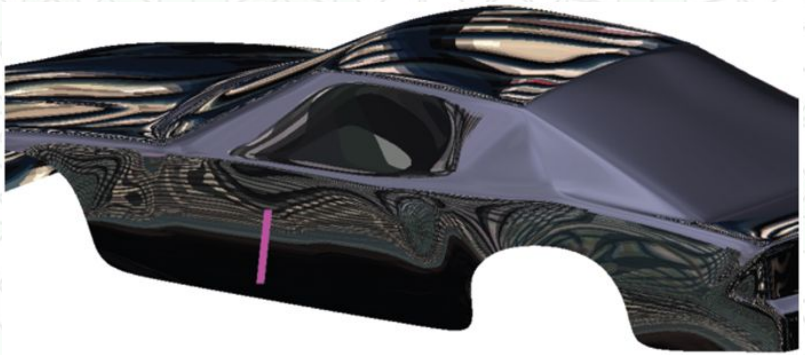


Impractical for large number of control points!

Continuity Definitions

- C^0 continuous:
  - curve/surface has no breaks/gaps/holes
- G^1 continuous:
  - tangent at joint has same direction
- C^1 continuous:
  - curve/surface derivative is continuous
  - tangent at joint has same direction *and* magnitude
- C^n continuous:
  - curve/surface through n^{th} derivative is continuous
  - important for shading



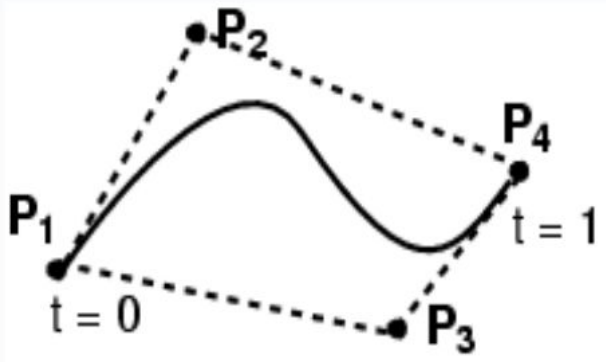


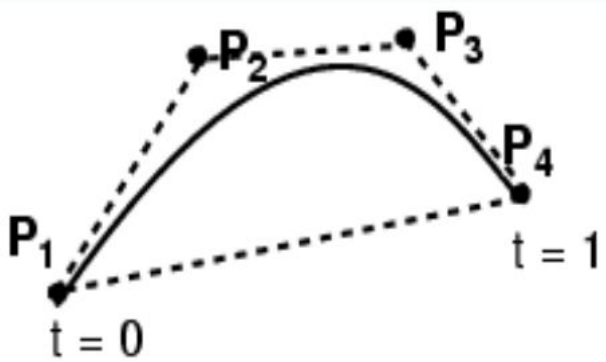
"Shape Optimization Using Reflection Lines", Tosun et al., 2007

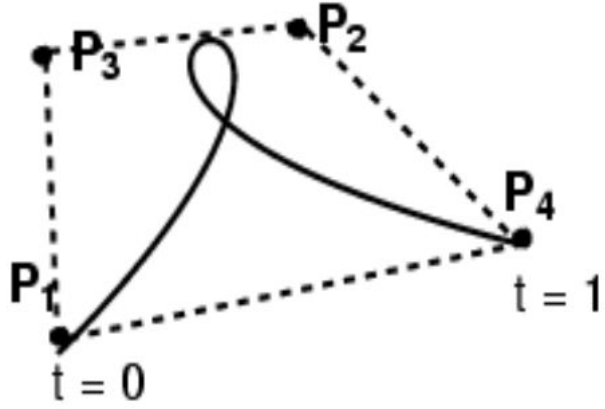
Cubic Bézier Curve

- 4 control points
- Curve passes through first & last control point
- Curve is tangent at $\mathbf{P}_1$ to $(\mathbf{P}_2\text{-}\mathbf{P}_1)$ and at $\mathbf{P}_4$ to $(\mathbf{P}_4\text{-}\mathbf{P}_3)$

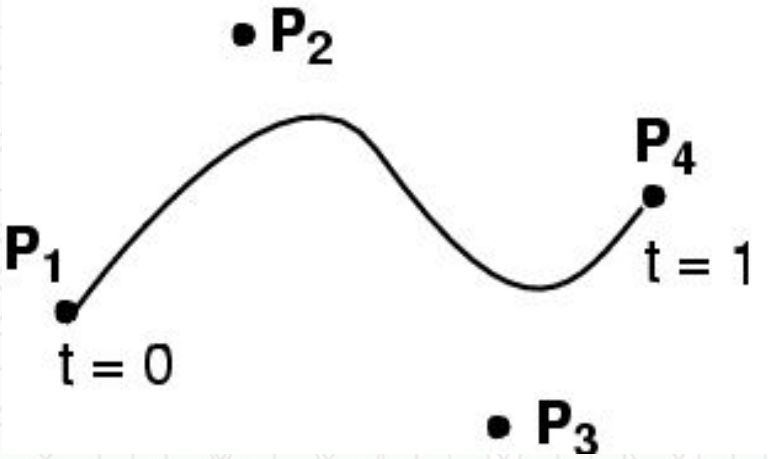
A Bézier curve is bounded by the convex hull of its control points.



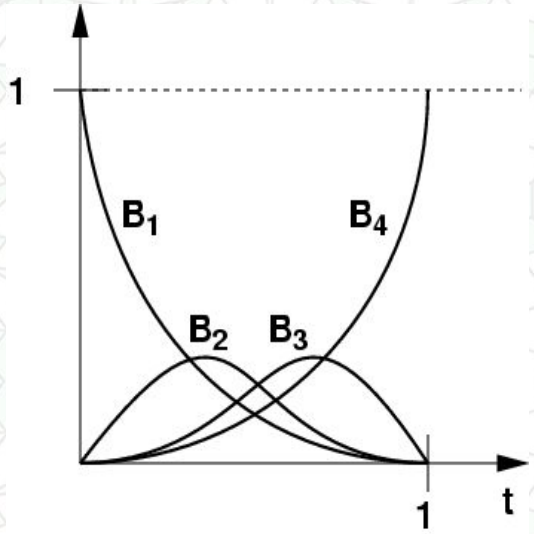




Cubic Bézier Curve



Asymmetric:
curve goes through
2 control points
and influenced by
2 control points



Parametric equation:
Function of *t*
t* varies *0* → *1

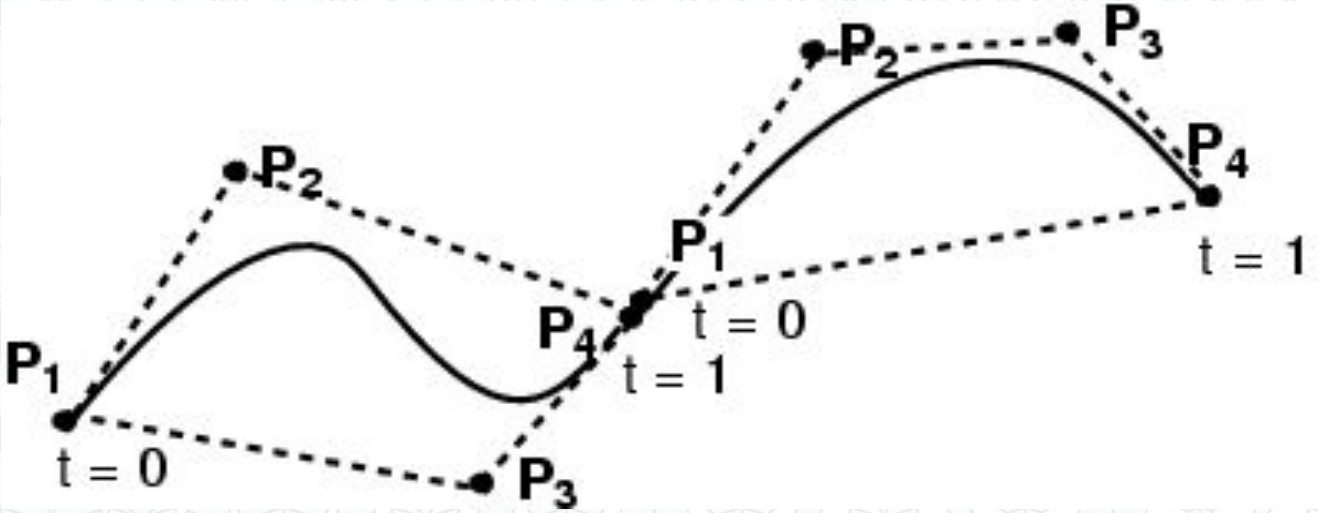
weights sum to 1

$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

control points

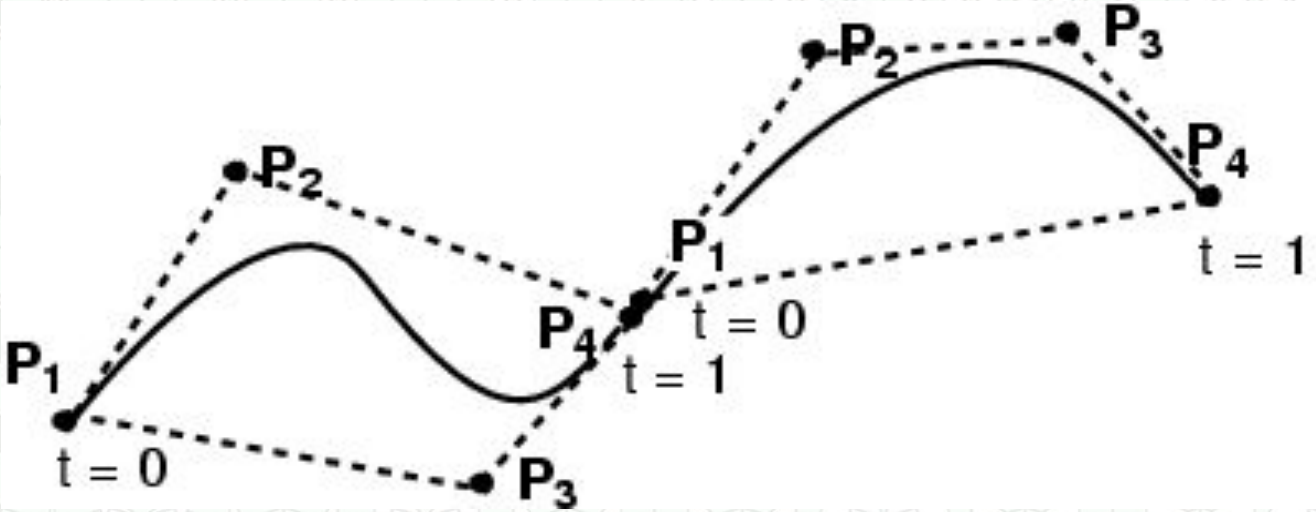
Connecting Cubic Bézier Curves

- How can we guarantee C^0 continuity?
- How can we guarantee G^1 continuity?
- How can we guarantee C^1 continuity?
- Can’t guarantee higher C^2 or higher continuity



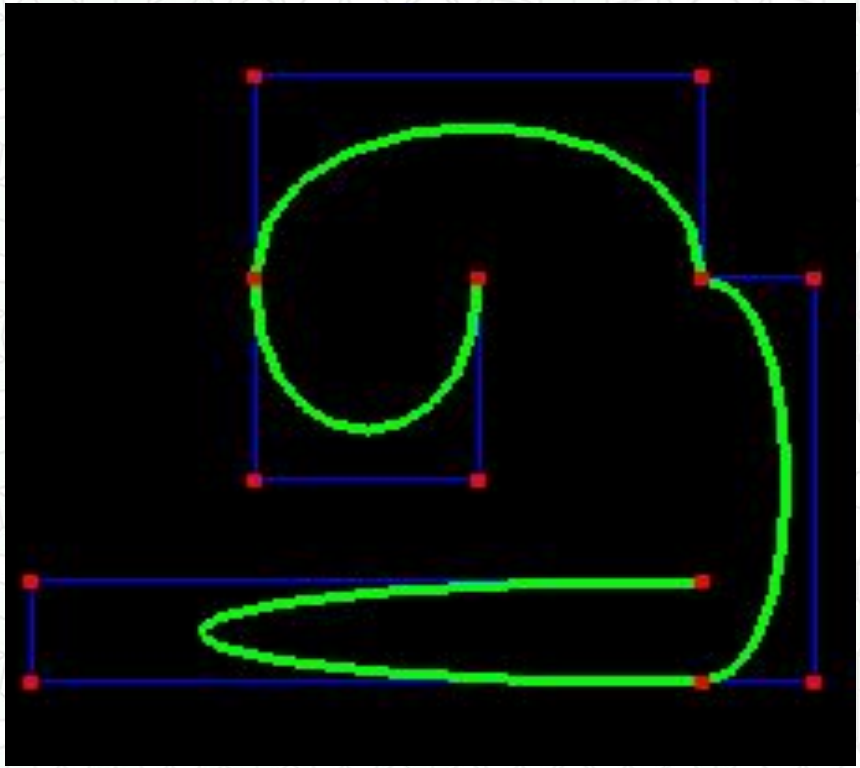
Connecting Cubic Bézier Curves

- How can we guarantee C^0 continuity? *P_4 of curve A = P_1 of curve B*
- How can we guarantee G^1 continuity? *P_3 & P4=P1 & P2 are collinear*
- How can we guarantee C^1 continuity? *$P_4 - P_3 = P_2 - P_1$*
- Can't guarantee higher C^2 or higher continuity



Connecting Cubic Bézier Curves

- Where is this curve:
  - C^0 continuous?
  - G^1 continuous?
  - C^1 continuous?
- What is the relationship between:
  - # of control points, &
  - # of Bezier subcurves?



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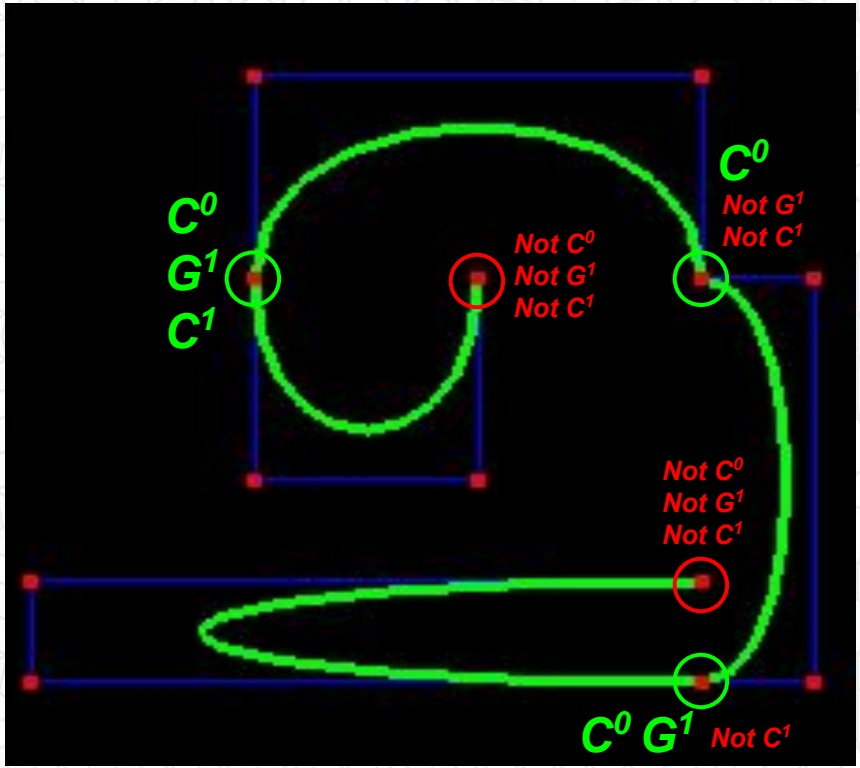
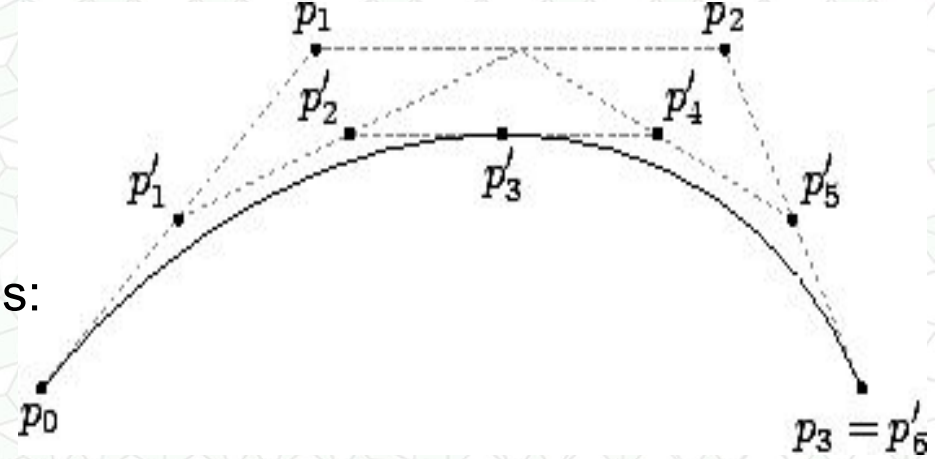
 # points = 3 # subcurves + 1*

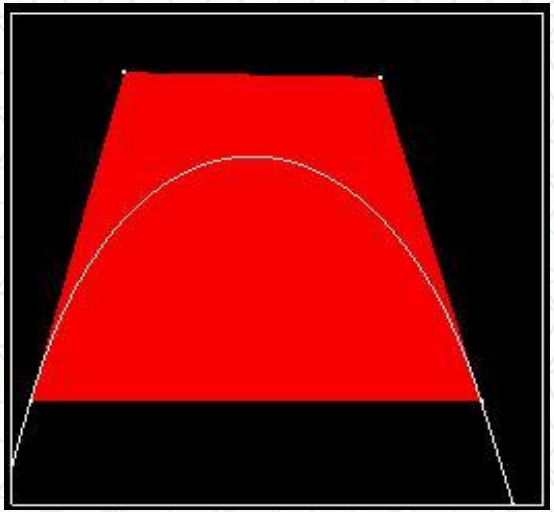
*13 = 3*4+1*

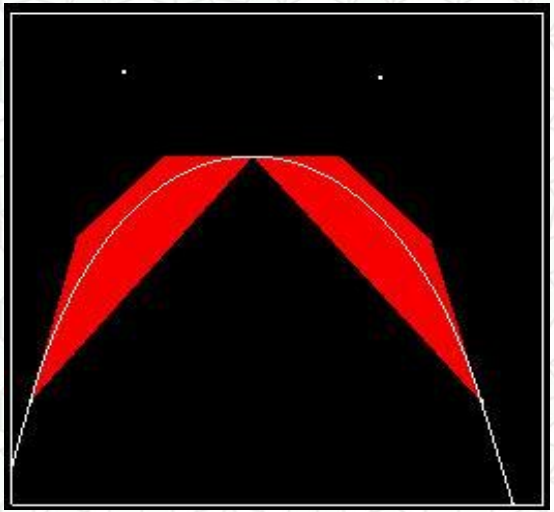


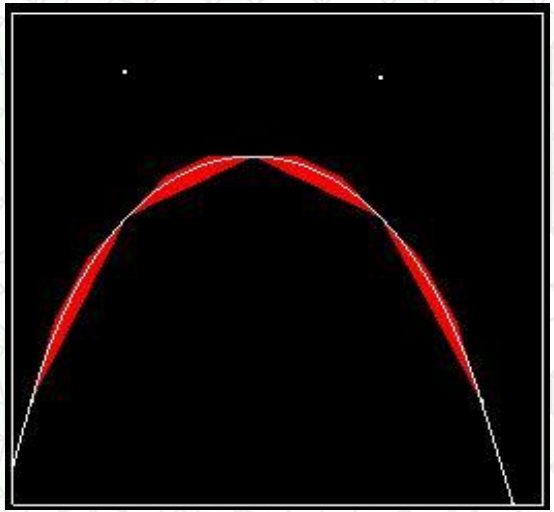
Neat Bezier Spline Trick

- A Bezier curve with 4 control points:
  - p_0 p_1 p_2 p_3
- Can be split into 2 new Bezier curves:
  - p_0 p'_1 p'_2 p'_3
  - p'_3 p'_4 p'_5 p_3









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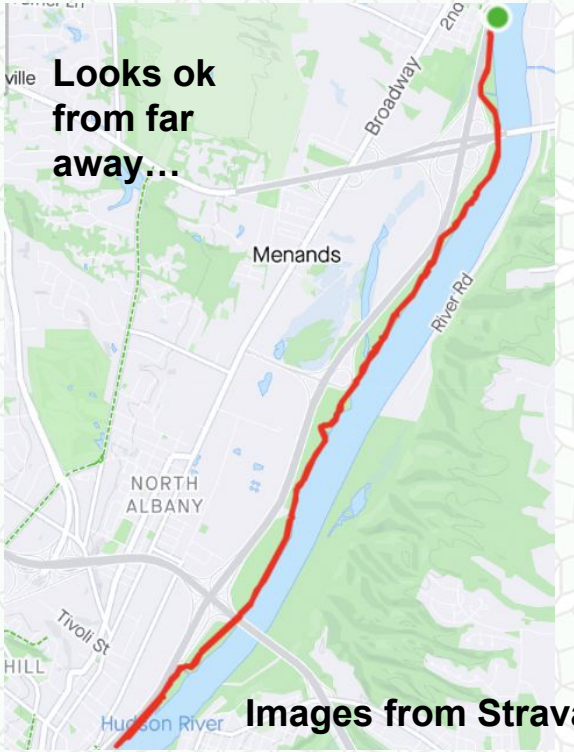
- Next Time: Robot Motion Planning

Noisy GPS Running Data

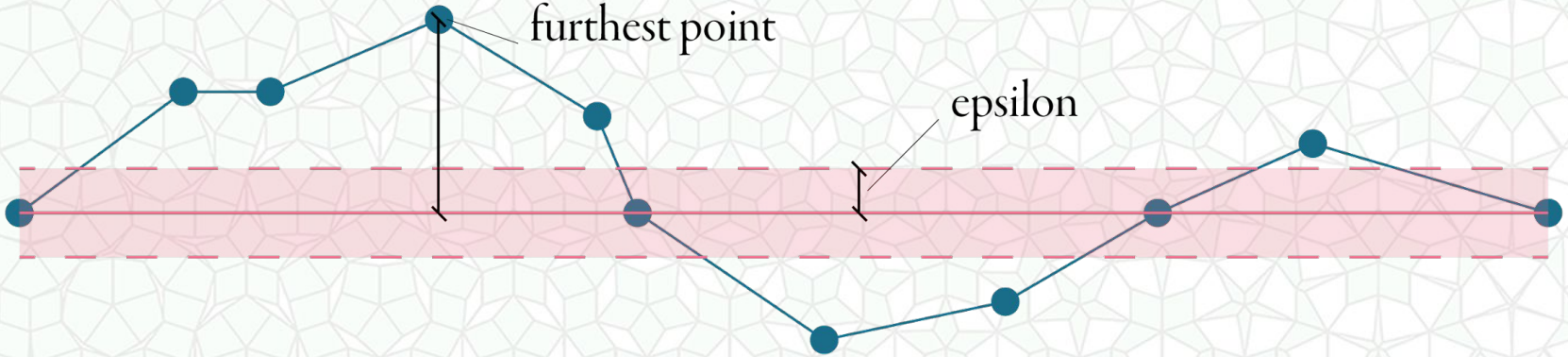
- Can overestimate distance by ~10% !!



Images from Strava

Polyline Simplification: Ramer–Douglas–Peucker

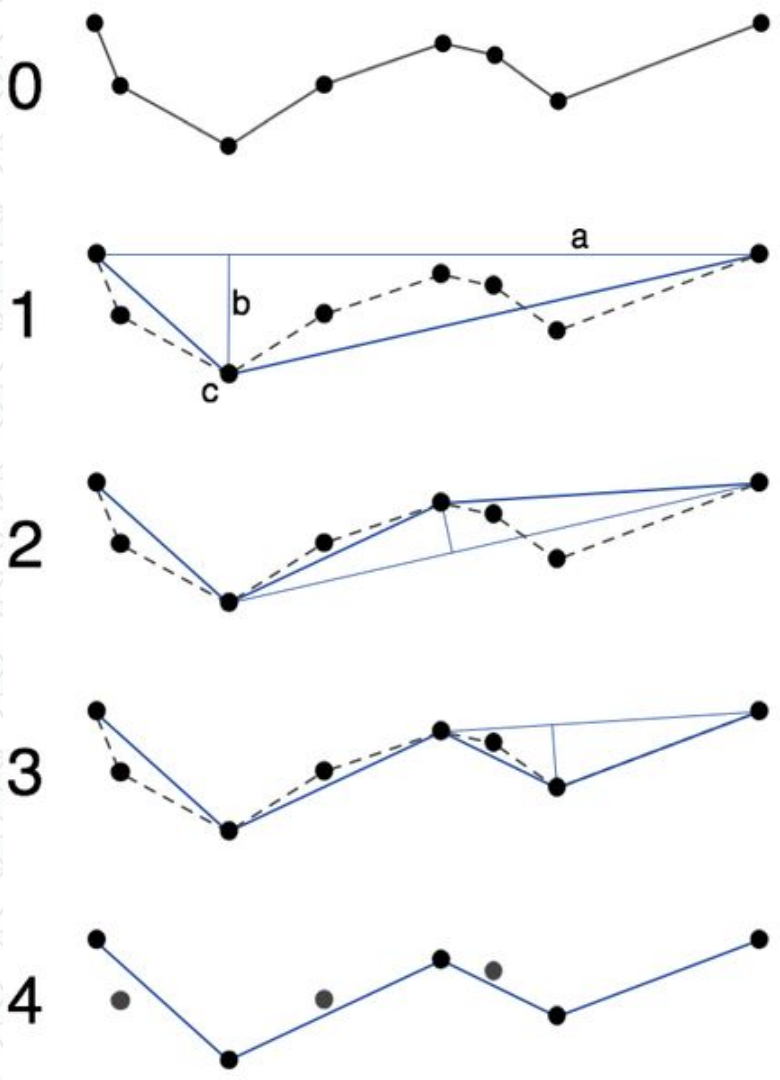
- Originally developed for cartography
- Identify most important points
- Discard points that are < ε from the simplified shape



https://martinfleischmann.net/line-simplification-algorithms/

Polyline Simplification: Ramer–Douglas–Peucker

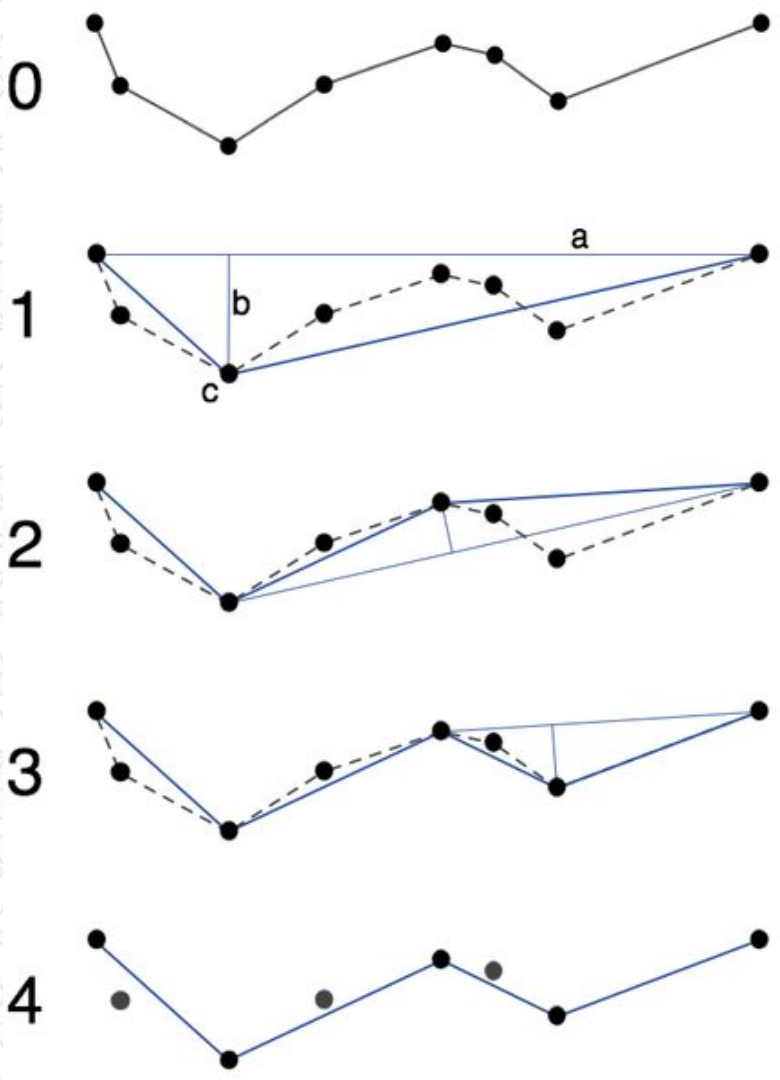
- Start with single line segment connecting original endpoints
- while (true)
  - Find original data point point that is furthest from the current polyline approximation
  - If it is < ε, break
  - Else add point & split segment into two subsegments and repeat



https://commons.wikimedia.org/wiki/File:Douglas_Peucker.png

Polyline Simplification: Ramer–Douglas–Peucker

Starts with two points, adds points incrementally



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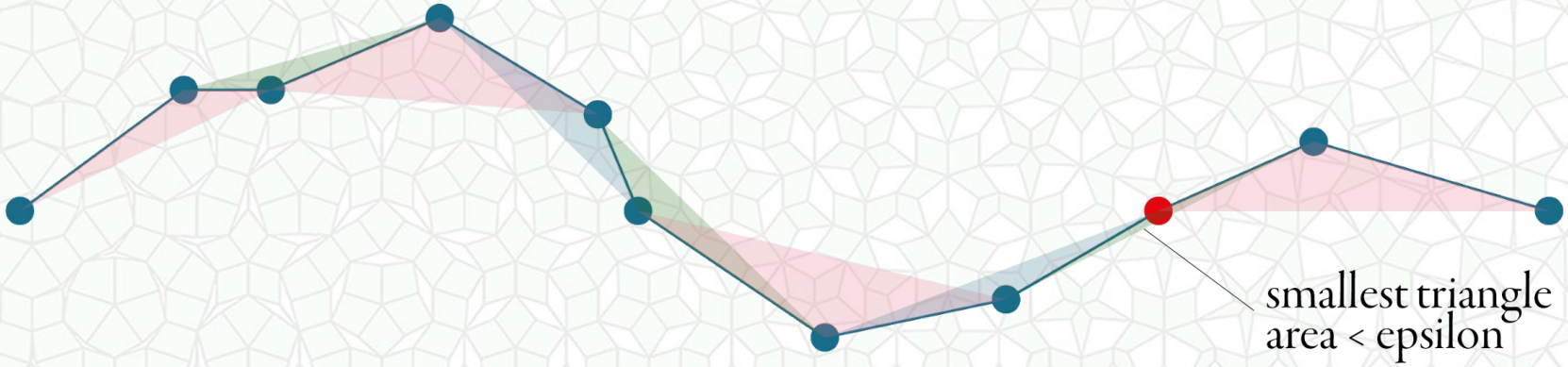
Polyline Simplification: Visvalingam-Whyatt

- Similar algorithm to Ramer-Douglas-Peucker (sometimes but not always the same result)
- Remove a point if the triangle formed by that point & two immediate neighbors has area < ε



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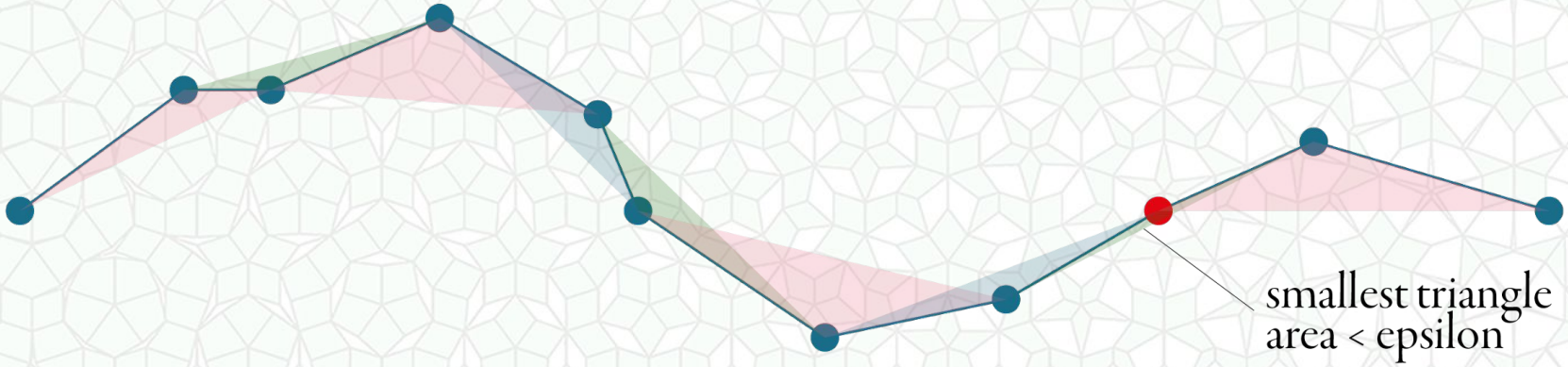
Polyline Simplification: Visvalingam-Whyatt

- Similar algorithm to Ramer-Douglas-Peucker (sometimes but not always the same result)
- Remove a point if the triangle formed by that point & two immediate neighbors has area < ε

Starts with all points, removes them one at at time



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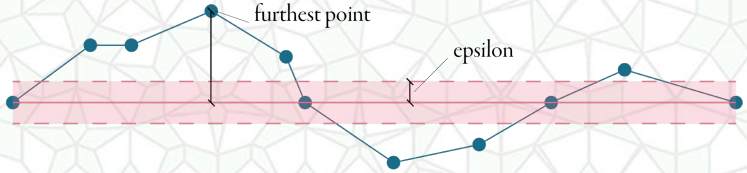
Polyline Simplification Analysis

- Ramer–Douglas–Peucker

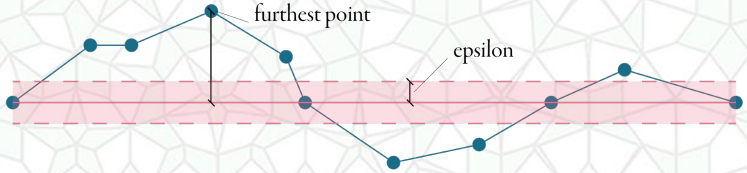


- Visvalingam-Whyatt

https://martinfleischmann.net/line-simplification-algorithms/

Polyline Simplification Analysis

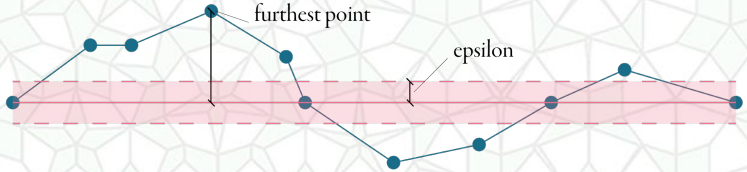
- Ramer–Douglas–Peucker
  - Connect endpoints, find split point furthest from current approximation:
  - Recurse on each side of split
  - Average case (even split):
  - Worst case (uneven split):
- Visvalingam-Whyatt
  - Compute all high resolution triangles:
    - Store in priority queue
  - Remove a point requires 2 new triangle computes
    - Priority queue update:
  - Overall:



https://martinfleischmann.net/line-simplification-algorithms/

Polyline Simplification Analysis

- Ramer–Douglas–Peucker
  - Connect endpoints, find split point furthest from current approximation: → *O(n)*
  - Recurse on each side of split
  - Average case (even split): → *O(n log n)*
  - Worst case (uneven split): → *O(n^2)*
- Visvalingam-Whyatt
  - Compute all high resolution triangles: → *O(n)*
    - Store in priority queue
  - Remove a point requires 2 new triangle computes
    - Priority queue update: → *O(log n)*
  - Overall: → *O(n log n)*



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Long Tiny Loops by Dan Aminzade

- Inspired by 2020 COVID lockdown
- How far can you run without repeating roads or intersections while staying close to home?

- GPS tracked run or bike
- Closed loop
- Streets or bike paths only
- Non-intersecting
- No repeated streets (even in opposite direction)
- Score = distance / max diameter

https://longtinyloop.com/faq

Long Tiny Loops by Dan Aminzade

- Extract GPS data from Strava API
- Ramer-Douglas-Peucker:
  Simplify input (remove false
  positive intersections due to noise)
- Verify closed loop
- Check for segment intersections
- Compute convex hull
- Rotating calipers maximum diameter

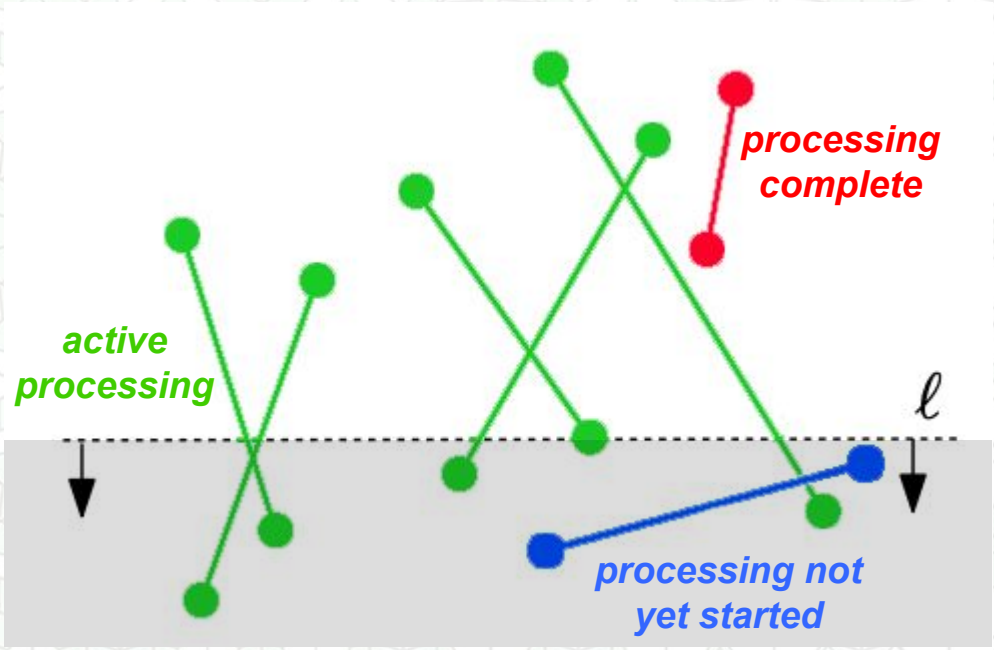
→ Compute final score
= distance / max diameter

https://longtinyloop.com/faq

Intersection Detection: Line-Sweep Algorithm

- (Review from Lecture 2)
- Sort all endpoints vertically
- Maintain horizontally sorted list of segments intersecting with current sweep line
- Check for intersections with adjacent segments only

- Overall: → *O((k+n) log n)*
  - *n* segments,
  - *k* intersections

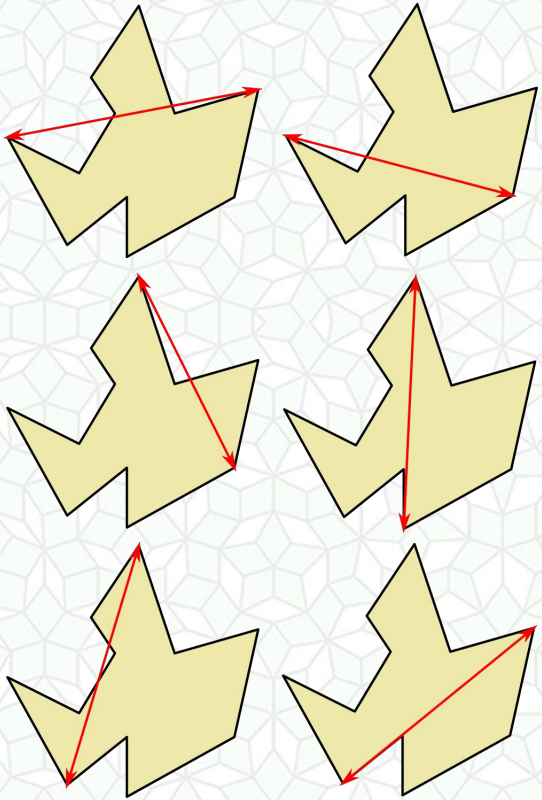


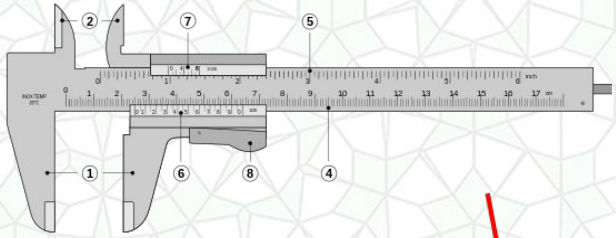
Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 2

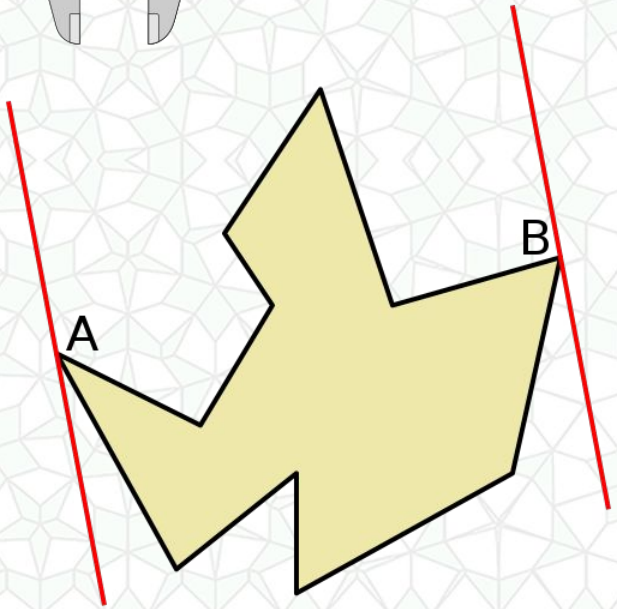
Maximum Diameter: Rotating Calipers

- Efficient algorithm to consider all pairs of *antipodal points*
  - also useful for other computations
- Return the maximum distance

- Analysis:





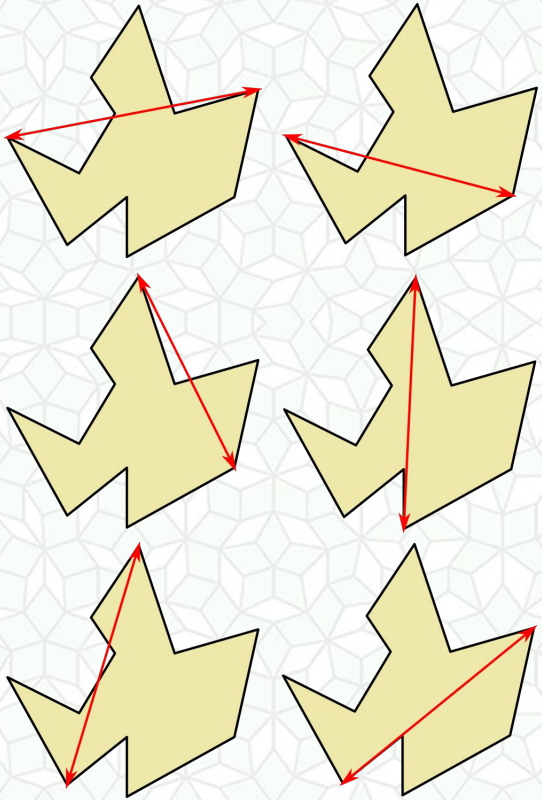


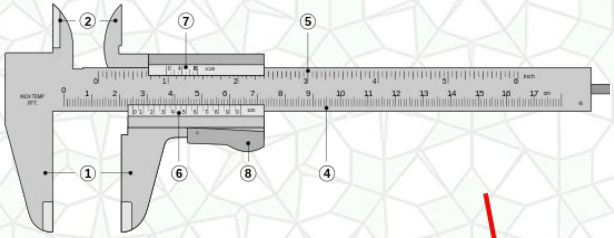
https://en.wikipedia.org/wiki/Rotating_calipers

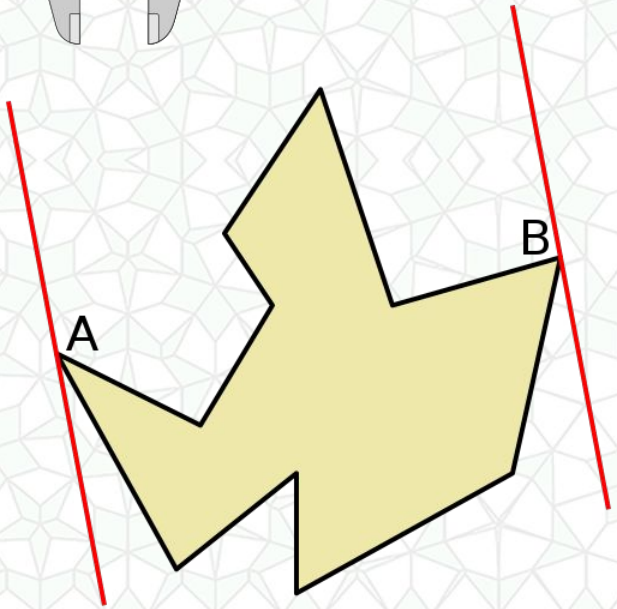
Maximum Diameter: Rotating Calipers

- Efficient algorithm to consider all pairs of *antipodal points*
  - also useful for other computations
- Return the maximum distance

- Analysis:
  → O(n)







https://en.wikipedia.org/wiki/Rotating_calipers

Long Tiny Loops

Current high score:
Nathan Rooy

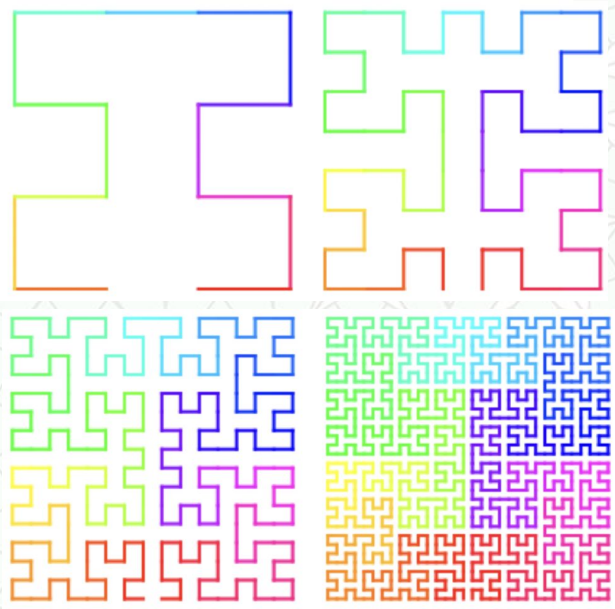
Distance:
190 km (118 miles)
Diameter:
4.7km (2.92 miles)

Score: 40.51

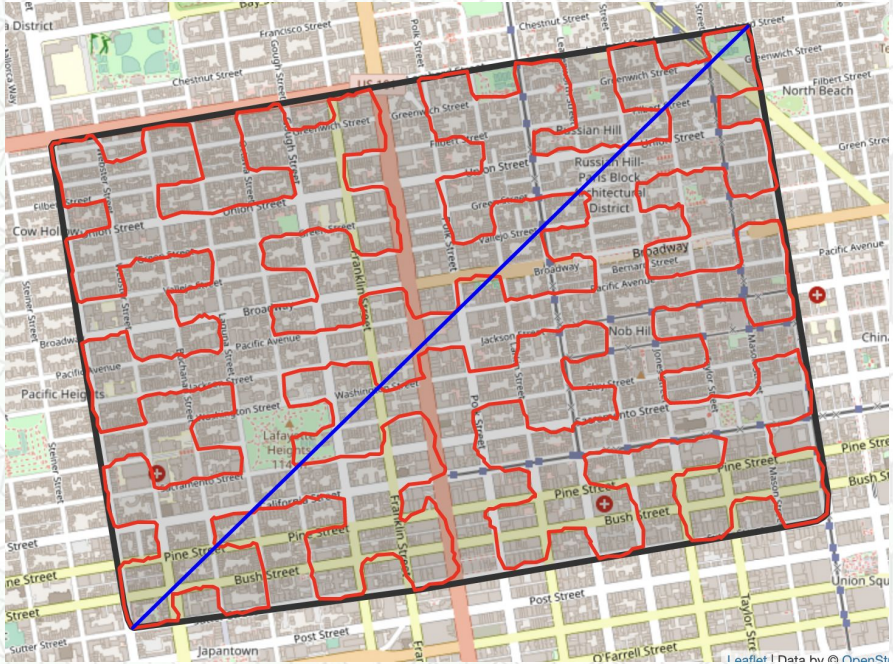
by Dan Aminzade
https://longtinyloop.com/

Space Filling Curve - A Fractal

- Peano curve (Guiseppe Peano, 1890)
- Hilbert Curve (David Hilbert, 1891)
- Moore Curve (E.H. Moore 1900)



path by Octavian Voicu
https://longtinyloop.com/



https://en.wikipedia.org/wiki/Moore_curve

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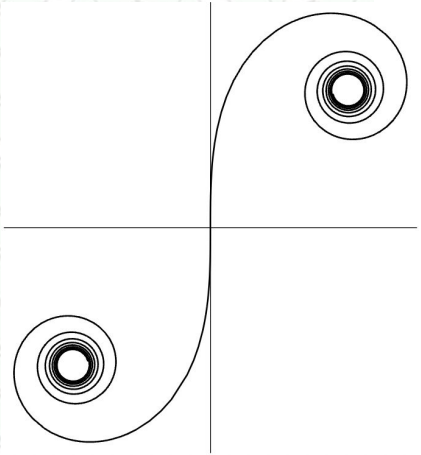
- Next Time: Robot Motion Planning

Clothoid or Cornu/Euler Spiral

"A new, simple and accurate transition curve type, for use in road and railway alignment design" European Transport Research Review, Eliou & Kaliabetsos, 2014

- For railroads, roads, rollercoasters, etc.
  - Avoid instantaneous curvature changes at high speed
- Linear correlation between curvature/radius & length



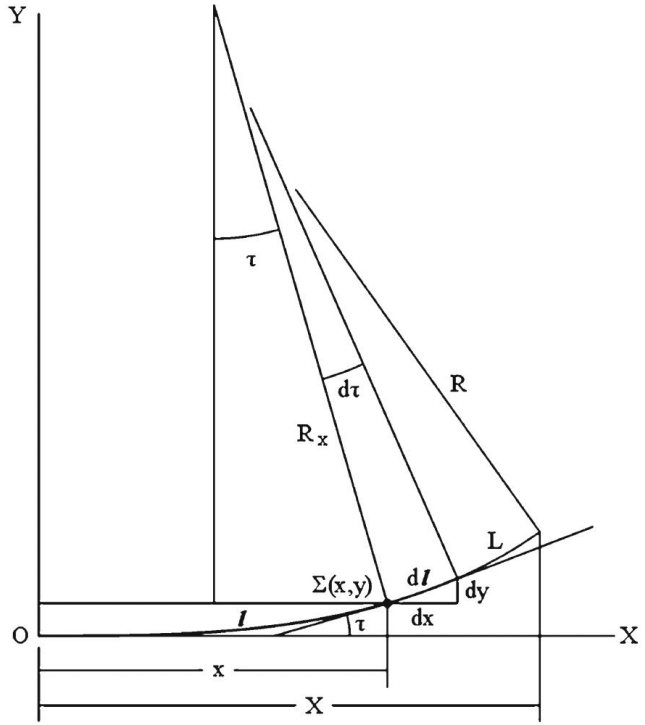
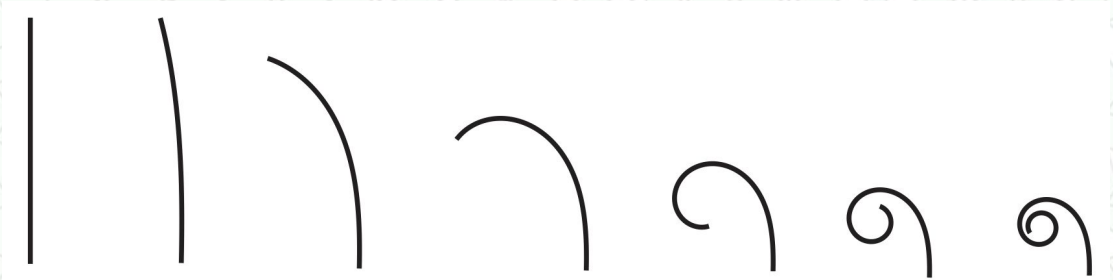


Fig. 2 Transition curve graph in detail



French Curve / Burmester Set

- Metal, wood, or plastic template
- For manual drafting/design
- Created from different segments of Clothoid or Cornu/Euler Spiral
- Invented by Ludwig Burmester

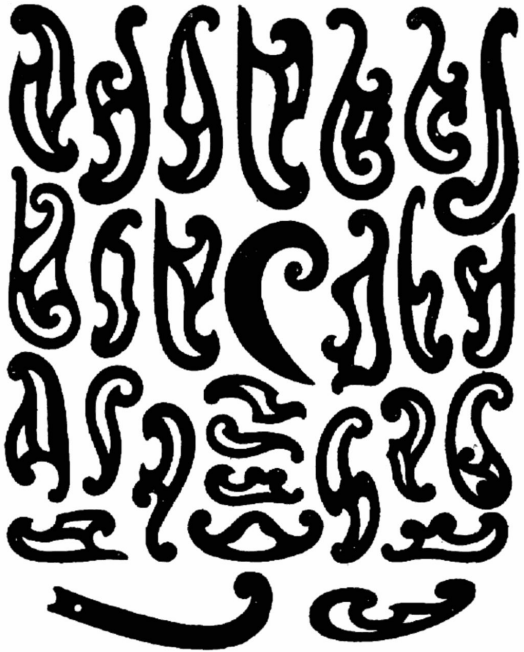


Fig. 6. Kurvenlineale von Gebrüder Wichmann, Berlin.

https://en.wikipedia.org/wiki/French_curve

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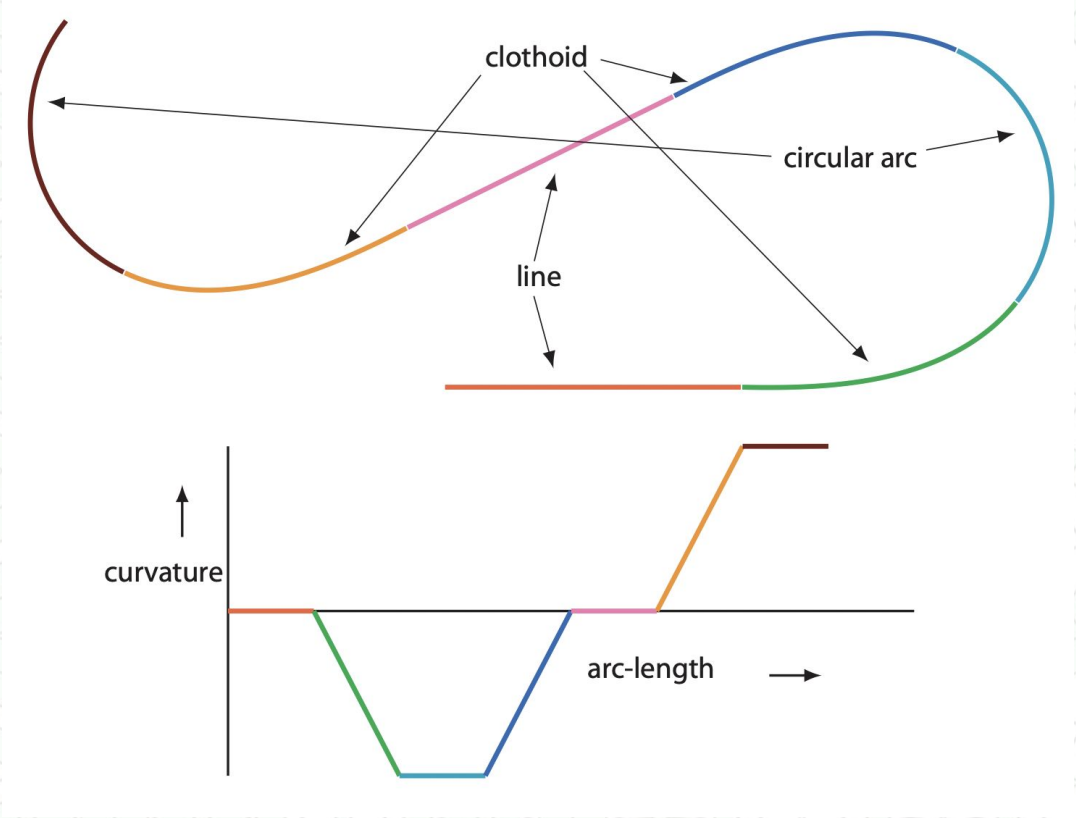
Piecewise Clothoid + Circular Arc + Line

- Aesthetically pleasing
- Fairness
- Can ensure G^2 or G^3 continuity
- Also model sharp discontinuities as appropriate



"Sketching Piecewise Clothoid Curves"
McCrae & Singh, 2008

Fairing (definition)

- Reduce undesirable, unaesthetic, unnecessary bumps and wiggles in a curve/surface
- Also: An additional part or structure added to an aircraft, tractor-trailer, etc. to smooth the outline and thus reduce drag

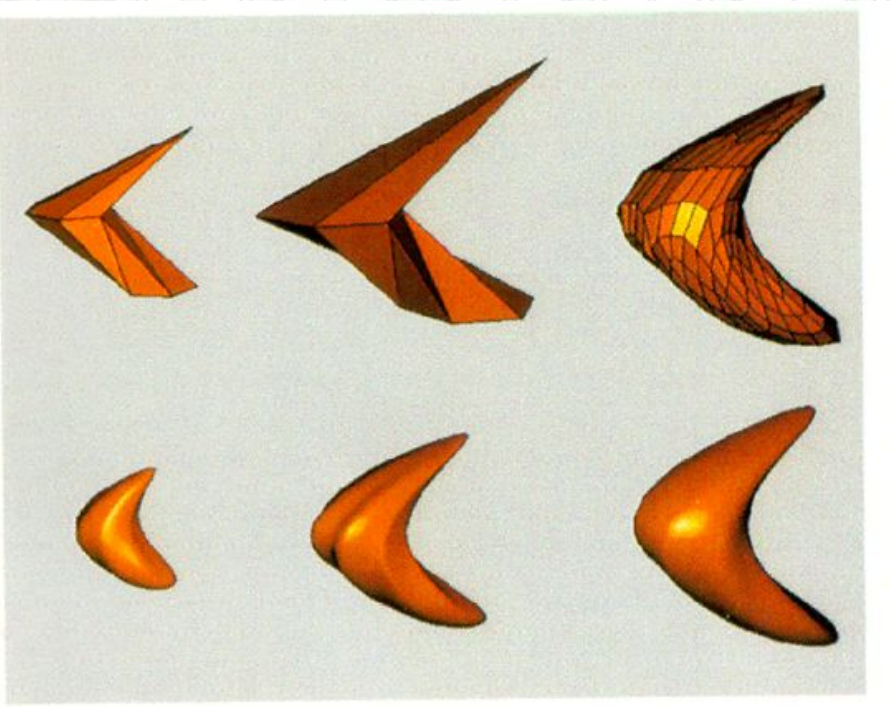


Figure 5: Top row: Original mesh, Interpolating mesh, Faired interpolating mesh. Bottom row: Corresponding Catmull-Clark surfaces. Interpolation introduces wiggles which are removed by fairing.

"Efficient, fair interpolation using Catmull-Clark surfaces", Halstead, Kass & DeRose, SIGGRAPH 1993

Advantages of Clothoids

- Interactive digital sketching data has noise and high frequency wiggles
- Clothoid fitting tends be smoothest and to minimize the *variation of curvature*

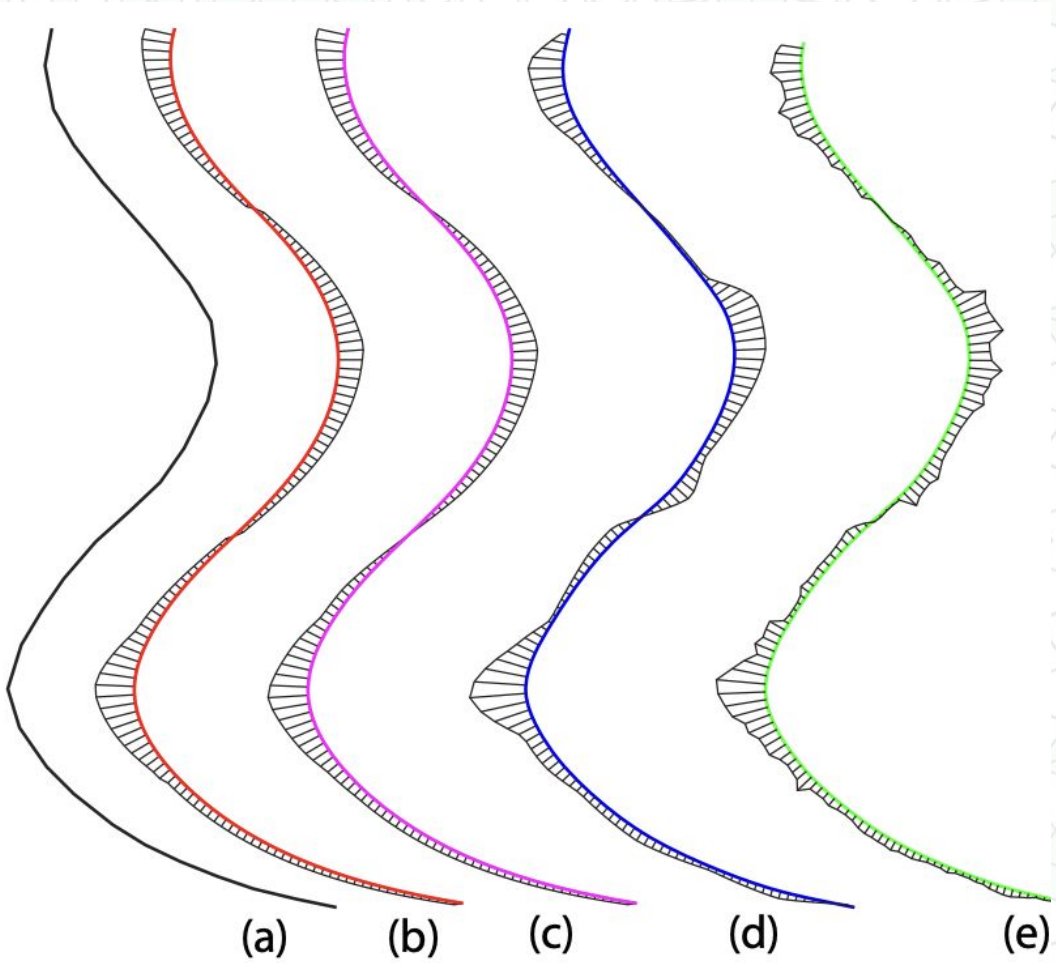
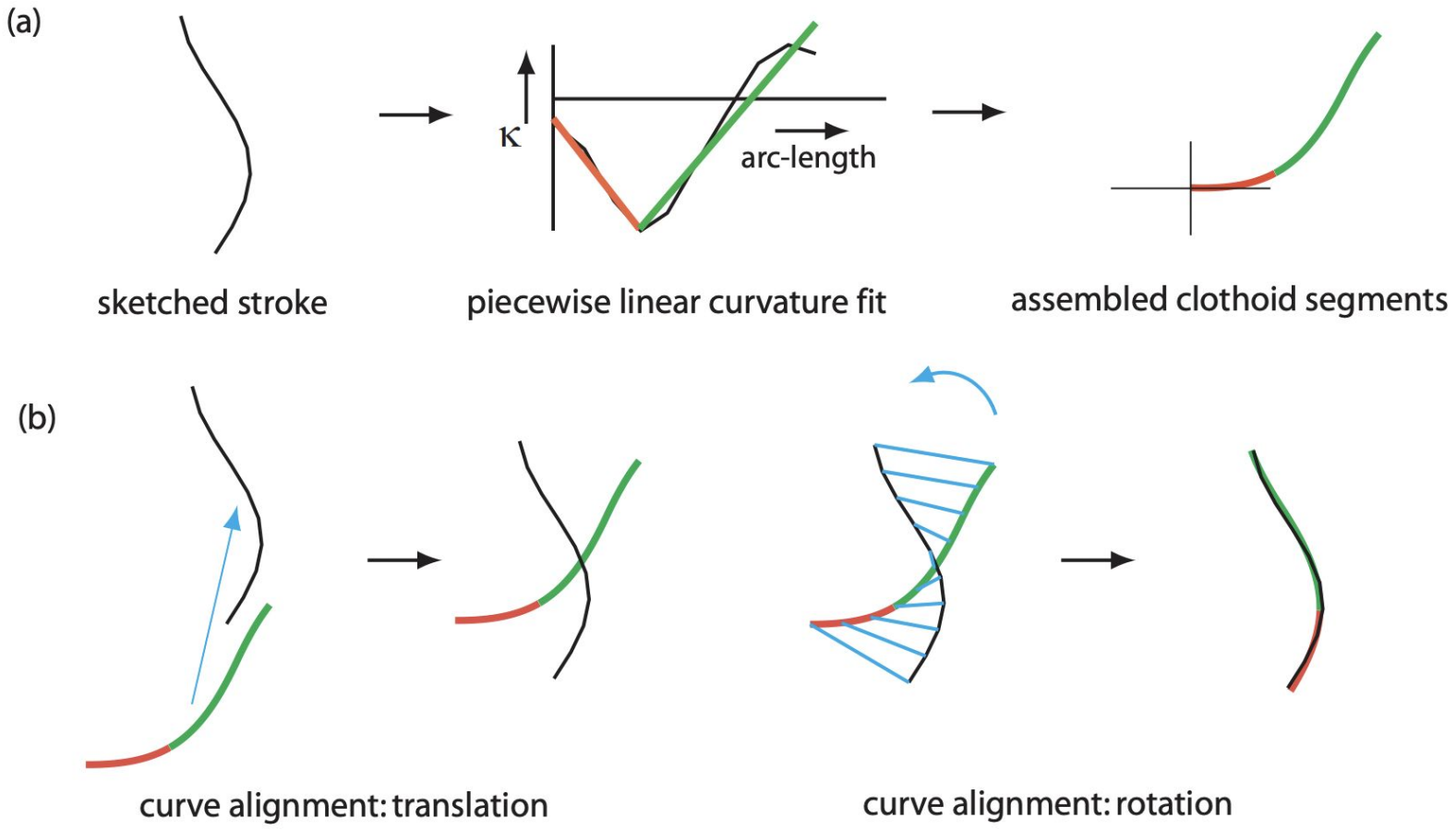
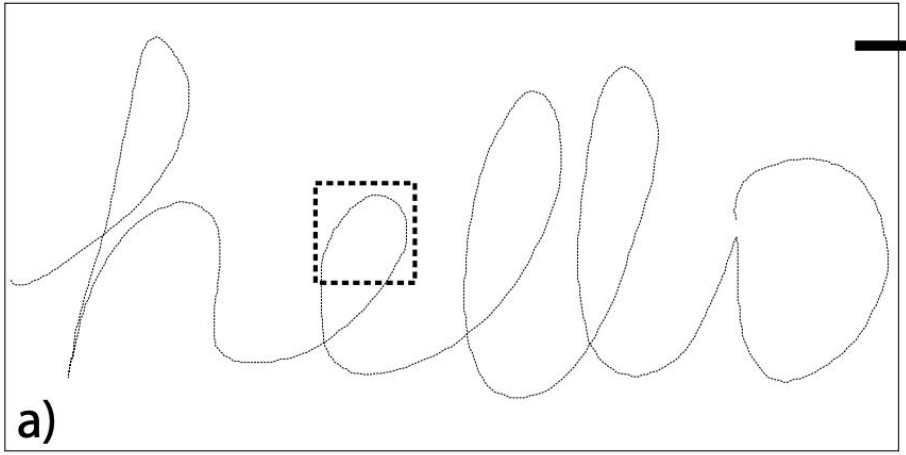


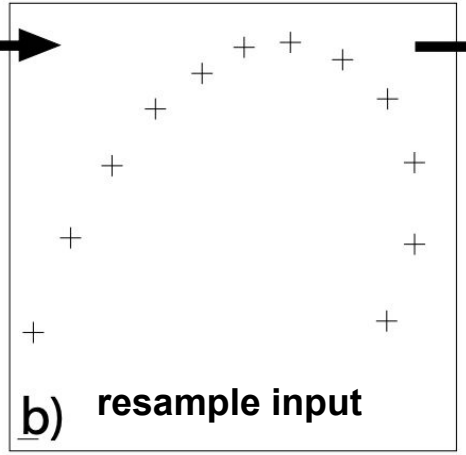
Figure 3: *Stroke fairing: (a) A sketched stroke. (b) Clothoid fitting the stroke (a). (c) Cubic spline fitting the clothoid curves in (b). (d) Cubic spline fitting the stroke (a). (e) Laplacian smoothing (4 iterations at 10%) the stroke (a). Curvatures are plotted uncolored along the length of processed strokes (b-d) to evaluate smoothness.*

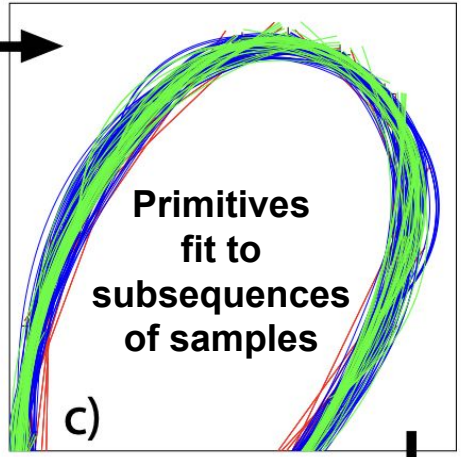
"Sketching Piecewise Clothoid Curves", McCrae & Singh, 2008

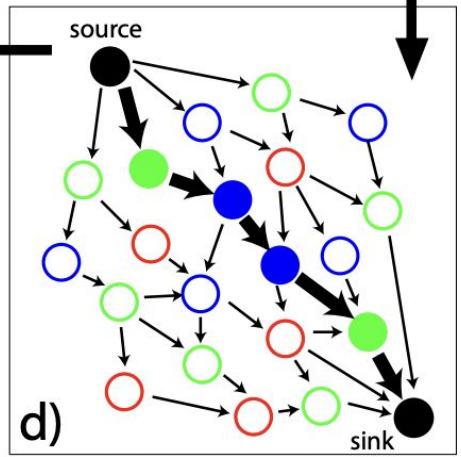


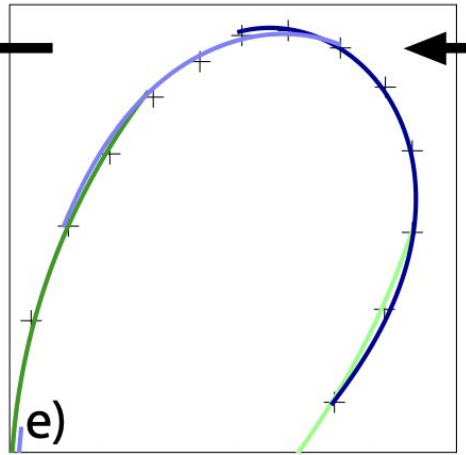
"Sketching Piecewise Clothoid Curves", McCrae & Singh, 2008

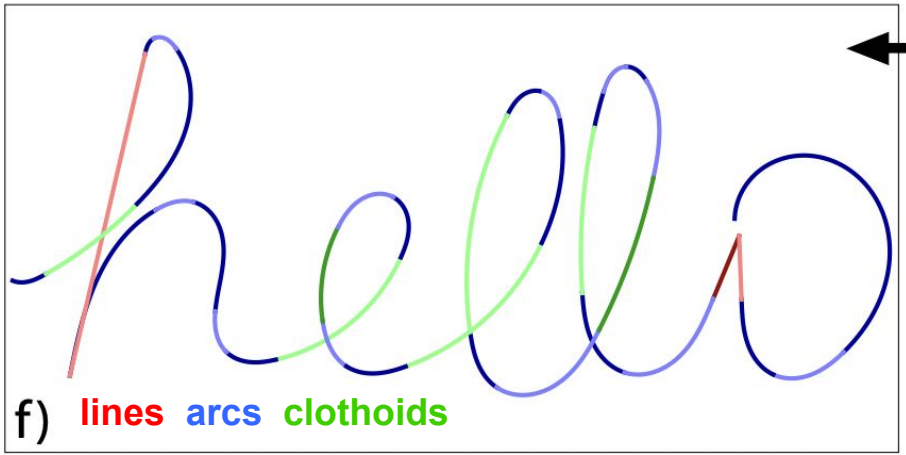












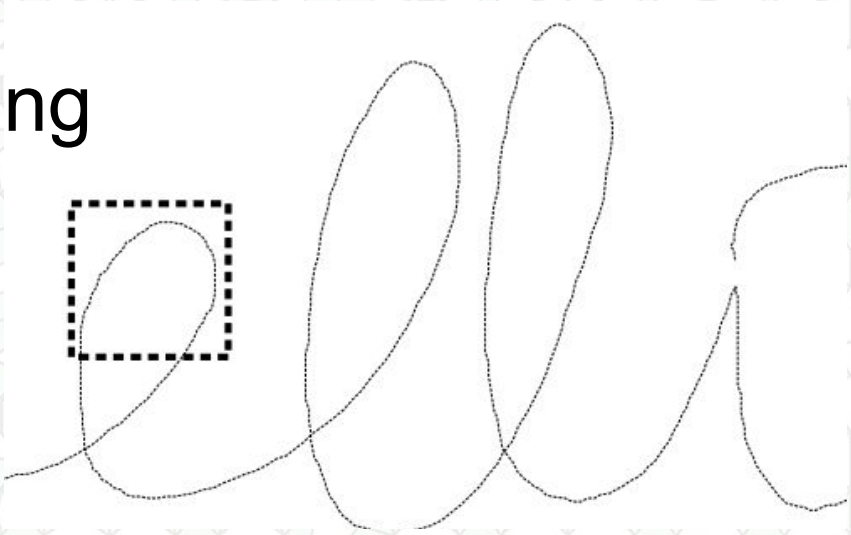
"Sketching Clothoid Splines Using Shortest Paths",
Baran, Lehtinen, & Popović, 2010

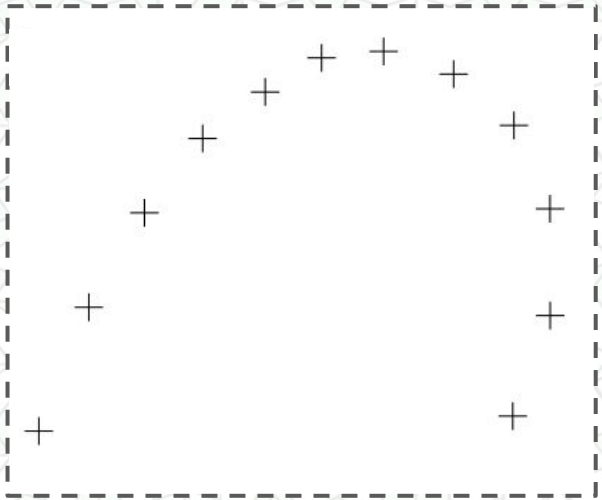
non-linear program to enforce continuity

find shortest path through all subsequences

Curvature-Based Resampling

- Raw sketch input usually has noise and overall too many samples
- Reduce total number
- Regularize the spacing of samples
- Have more samples where the curvature is higher





“Sketching Clothoid Splines Using Shortest Paths”,
Baran, Lehtinen, & Popović, 2010

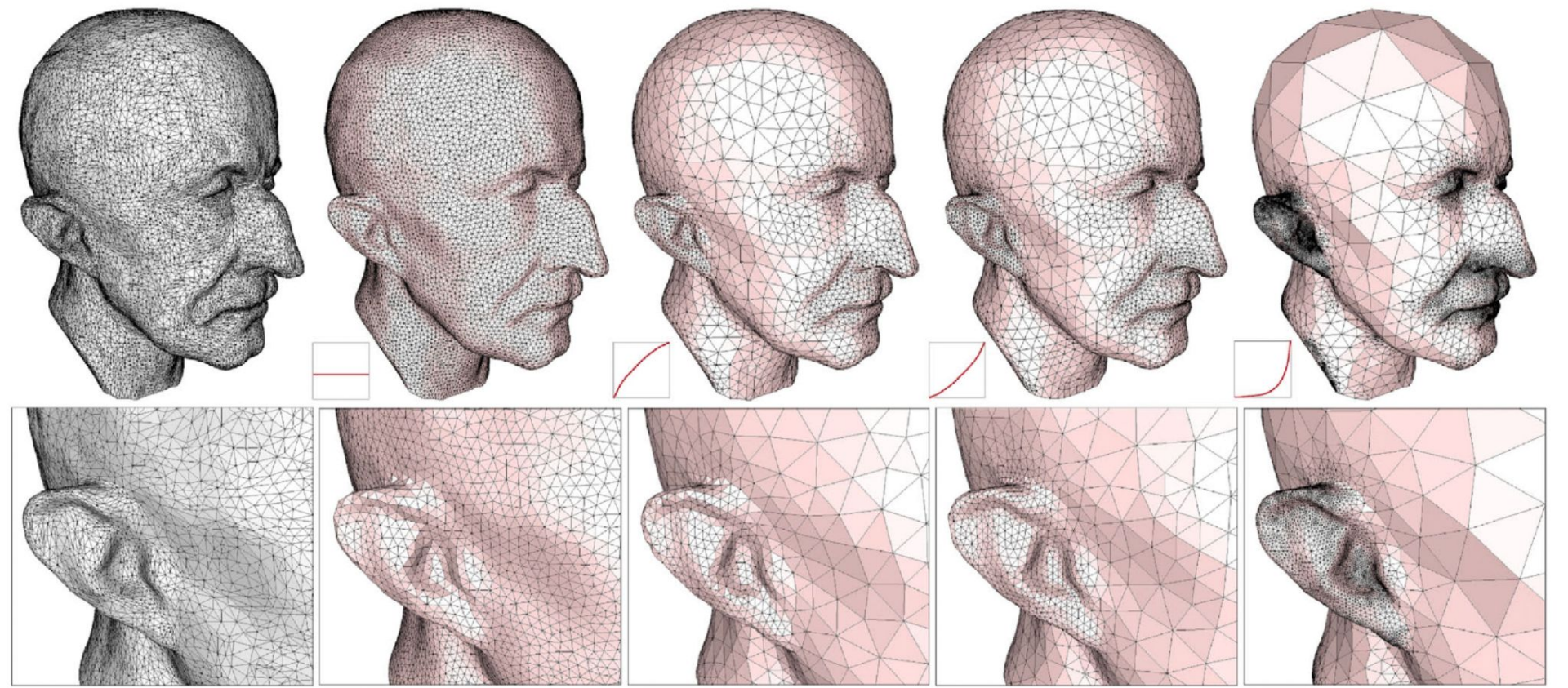


Figure 11: *Remeshing of the MaxPlanck model with various distribution of the sampling with respect to the curvature. The original model (left) is remeshed uniformly and with an increasing importance placed on highly curved areas (left to right) as the magnified area shows.*

"Interactive Geometry Remeshing"
Alliez, Meyer, & Desbrun, SIGGRAPH 2002

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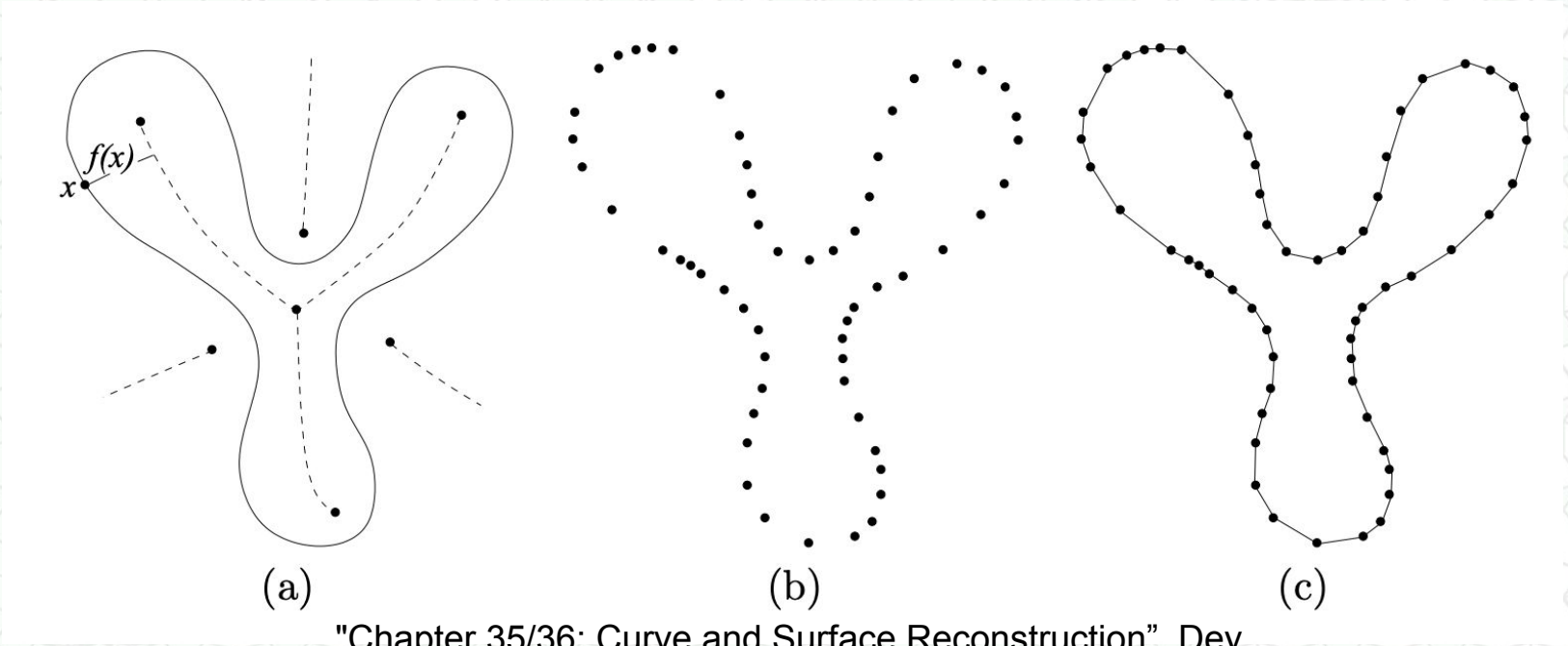
Curve Reconstruction

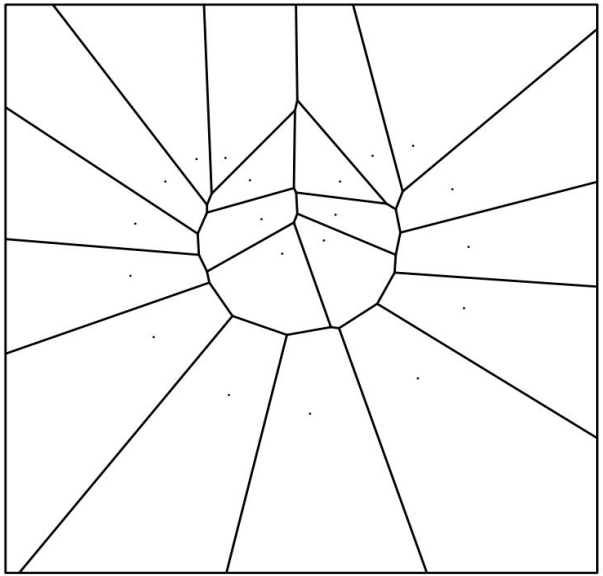
- Guaranteed reconstruction if sufficient sampling requirements are met.

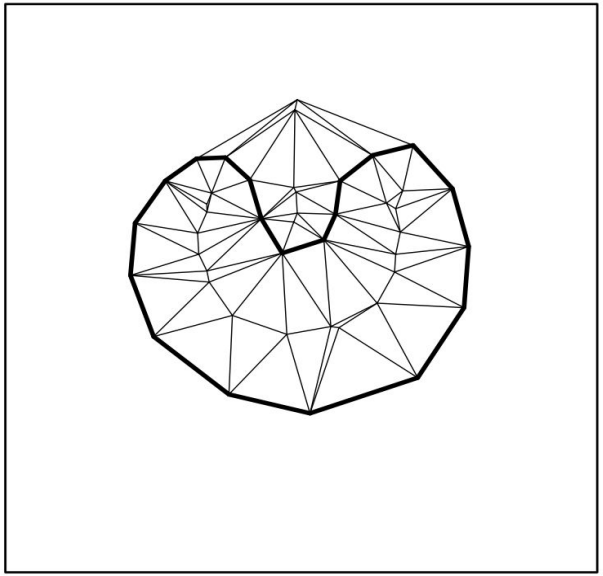


"Chapter 35/36: Curve and Surface Reconstruction", Dey, Handbook of Discrete and Computational Geometry, 2018

“The Crust and the β-Skeleton Combinatorial Curve Reconstruction”
Amenta, Bern, & Eppstein, 1997

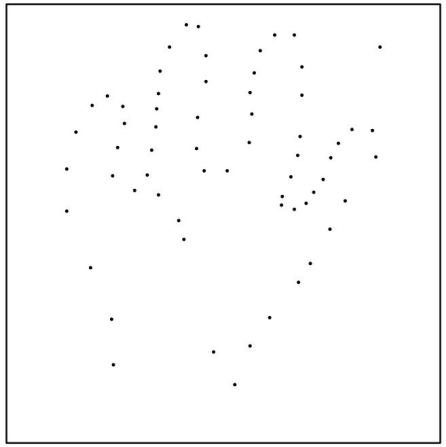
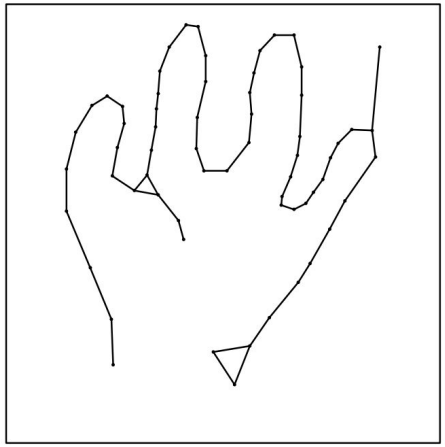
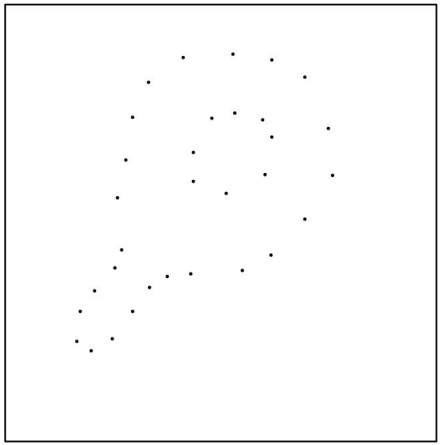
- Compute Voronoi Diagram of *sample points*
- Compute Delaunay Triangulation of *sample points* + the *Voronoi vertices*
- “Crust” is the Delaunay edges that do not connect 2 *sample points*

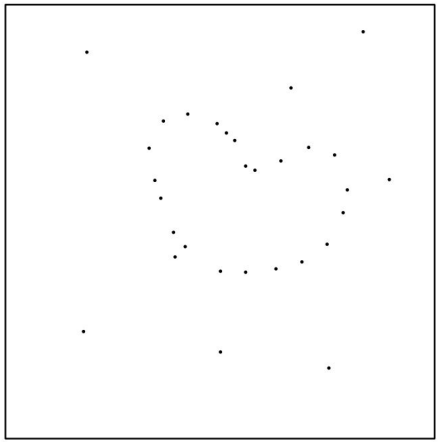
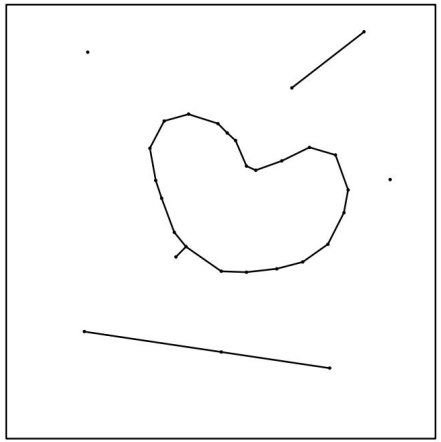


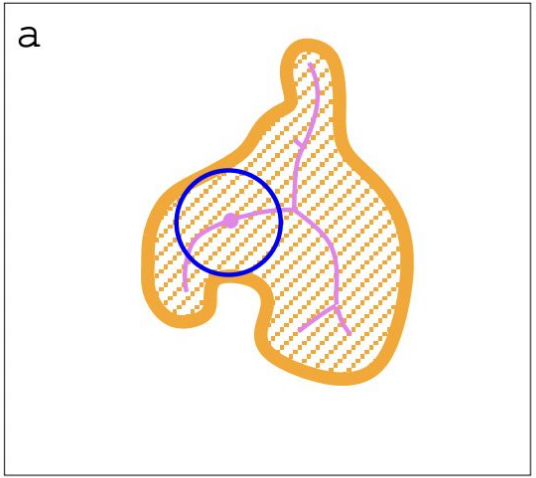


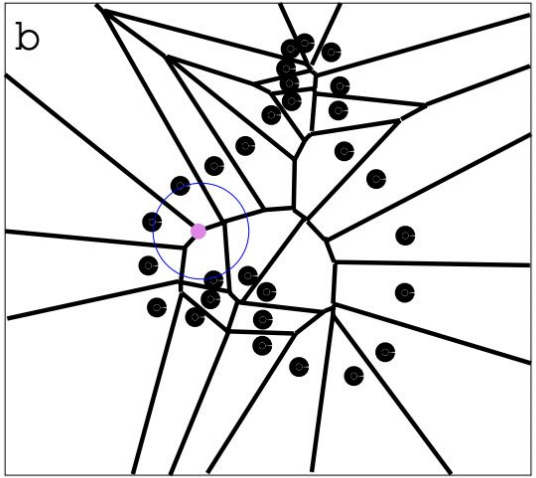
“The Crust and the β-Skeleton Combinatorial Curve Reconstruction”
Amenta, Bern, & Eppstein, 1997

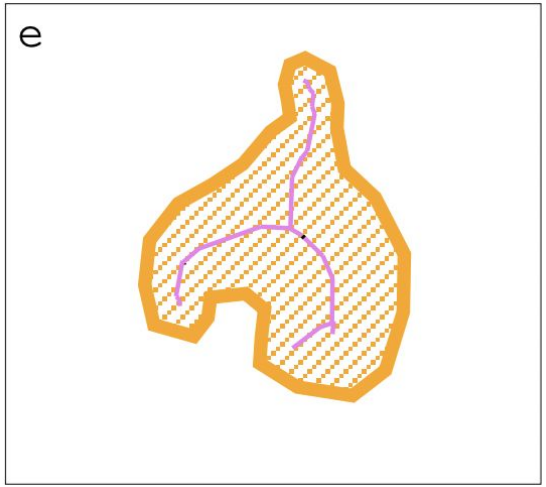
Works quite well!
Even for sparse or noisy data!

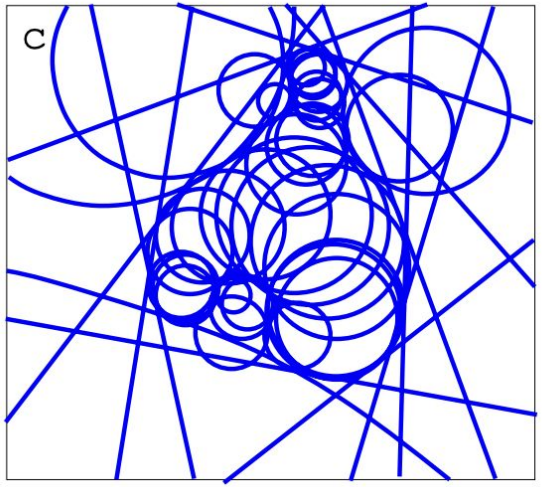












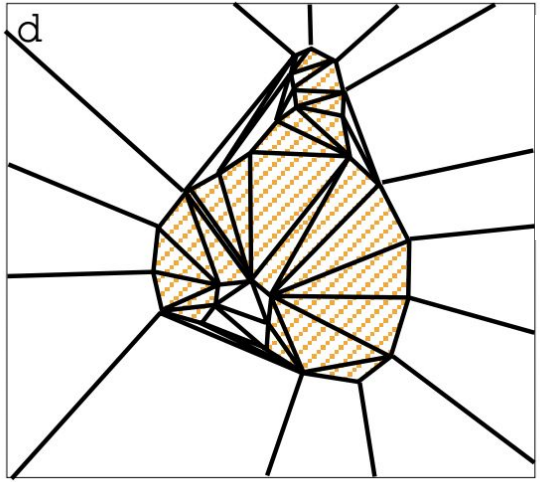
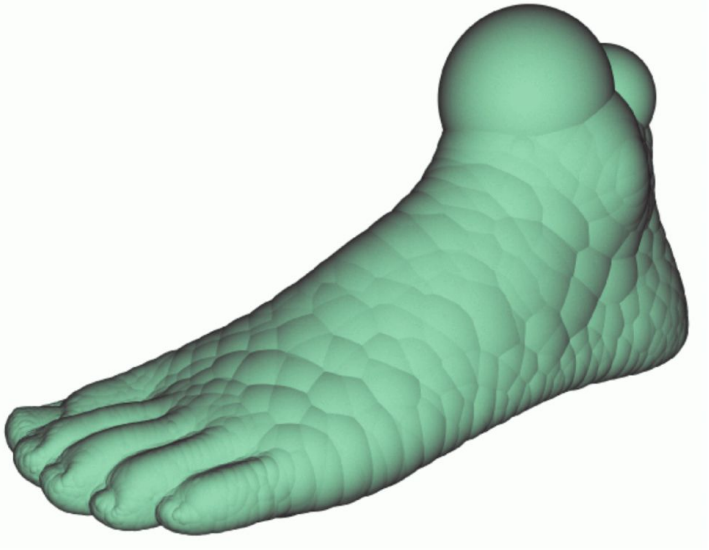
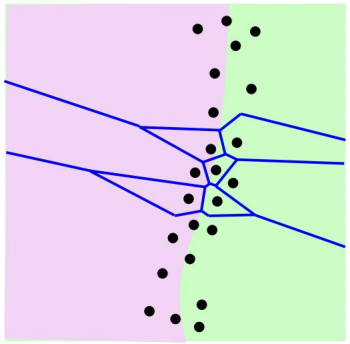
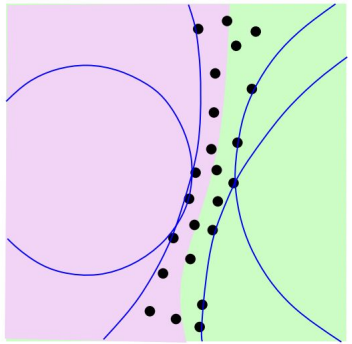


Figure 2. Two-dimensional example of power crust construction. a) An object with its medial axis; one maximal interior ball is shown. b) The Voronoi diagram of *S*, with the Voronoi ball surrounding one pole shown. In 2D, we can select all Voronoi vertices as poles, but not in 3D. c) The inner and outer polar balls. Outer polar balls with centers at infinity degenerate to halfspaces on the convex hull. d) The power diagram cells of the poles, labeled inner and outer. e) The power crust and the power shape of its interior solid.

"The Power Crust",
Amenta, Choi, Kolluri,
2001







"The Power Crust",
Amenta, Choi, Kolluri,
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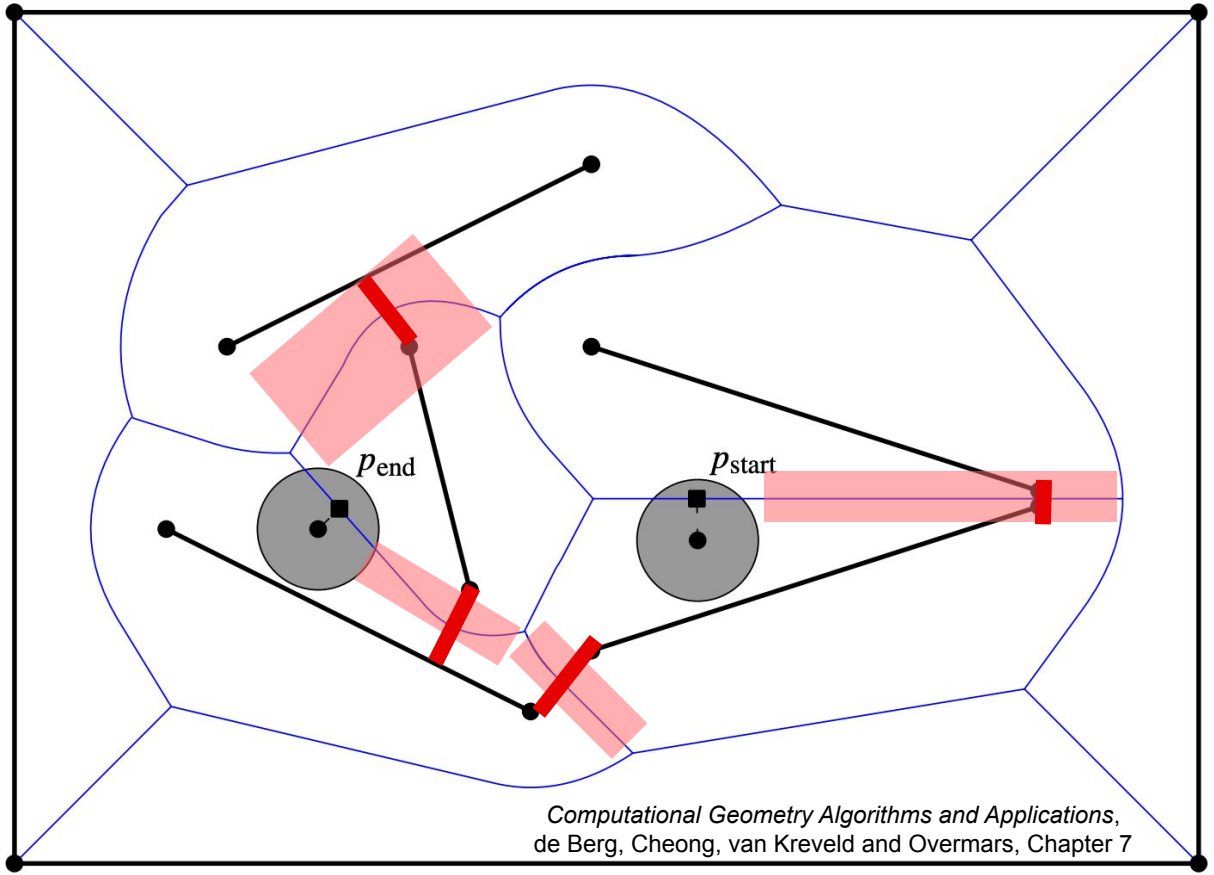
Application: Robotics & Motion Planning

- Step 1: Project the robot center to the closest Voronoi edge (line segment or parabolic curve)

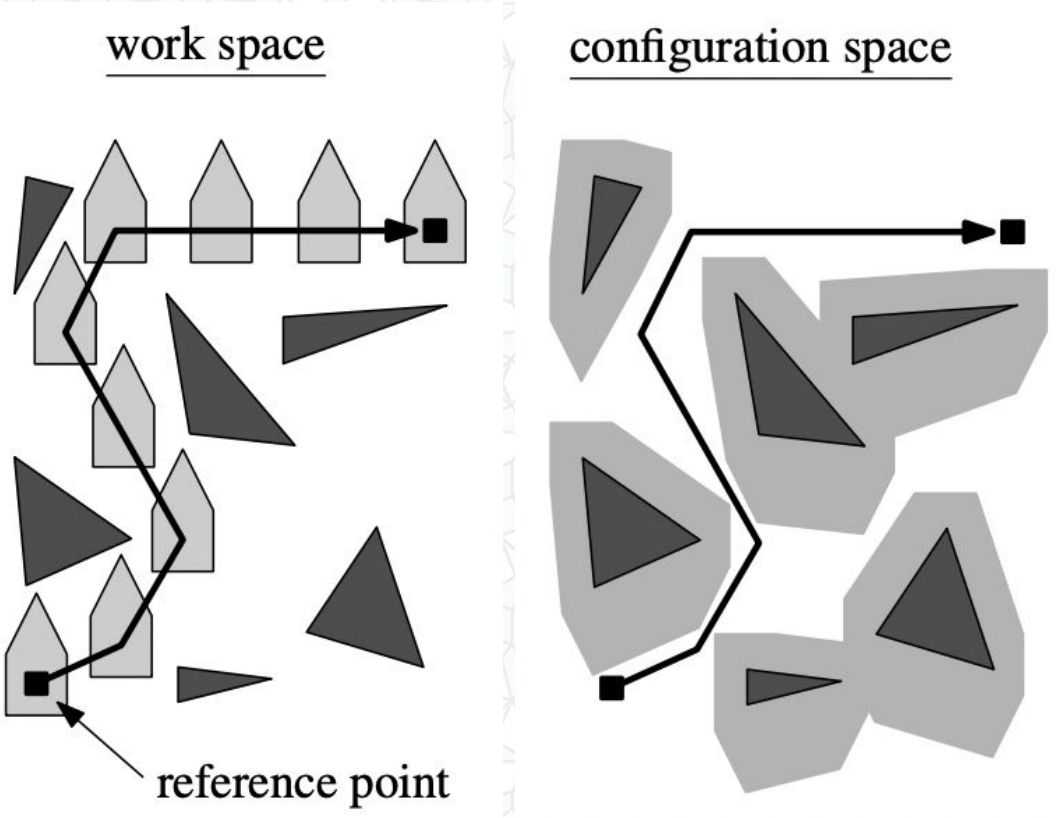
- Step 2: Remove edges from the diagram graph with smallest distance to segment < radius.



Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 7

Configuration Space

- The dimensions of configuration space match the DOF of the robot
- Usually configuration space is higher dimensional than the environment/workspace
- It is often useful to construct, visualize, and even solve the problem in “configuration space”



Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 13