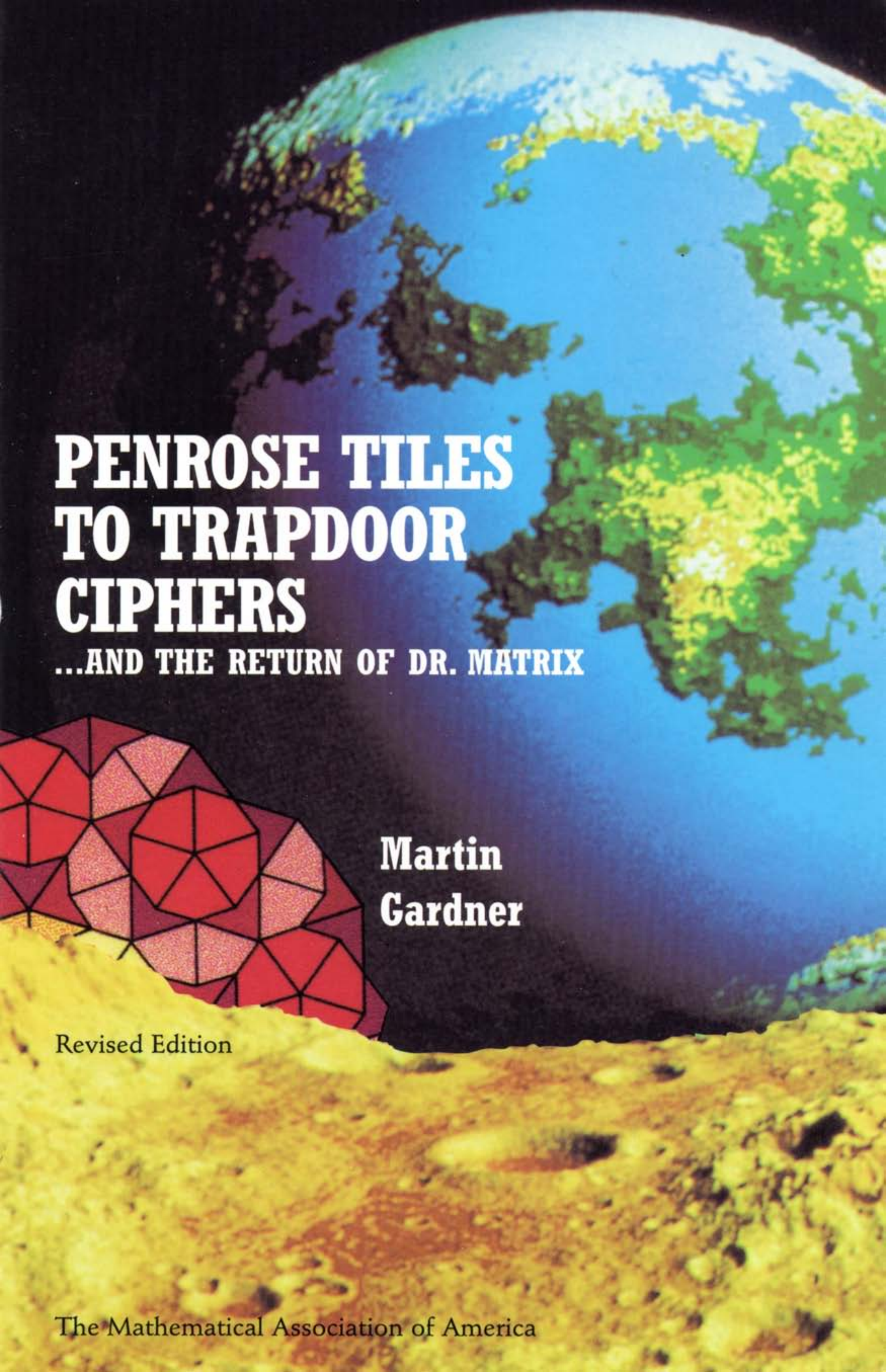

PENROSE TILES TO TRAPDOOR CIPHERS

...AND THE RETURN OF DR. MATRIX

Martin Gardner

Revised Edition

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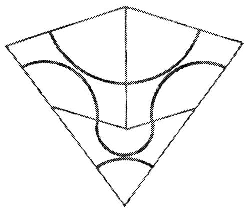
CHAPTER 1

Penrose Tiling



At the end of a 1975 *Scientific American* column on tiling the plane periodically with congruent convex polygons (reprinted in my *Time Travel and Other Mathematical Bewilderments*) I promised a later column on nonperiodic tiling. This chapter reprints my fulfillment of that promise—a 1977 column that reported for the first time a remarkable nonperiodic tiling discovered by Roger Penrose, the noted British mathematical physicist and cosmologist. First, let me give some definitions and background.

A periodic tiling is one on which you can outline a region that tiles the plane by translation, that is, by shifting the position of the region without rotating or reflecting it. M. C. Escher, the Dutch artist, is famous for his many pictures of periodic tilings with shapes that resemble living things. Figure 1 is typical. An adjacent black and white bird constitute a fundamental region that tiles by translation. Think of the plane as being covered with transparent paper on which each tile is outlined. Only if the tiling is periodic can you shift the paper, without rotation, to a new position where all outlines again exactly fit.

An infinity of shapes—for instance the regular hexagon—tile only periodically. An infinity of other shapes tile both periodically and nonperiodically. A checkerboard is easily converted to a nonperiodic tiling by identical isosceles right triangles or by quadrilaterals. Simply bisect each square as shown in Figure 2A, left, altering the orientations to prevent periodicity. It is also easy to tile nonperiodically with dominoes.

Isoceles triangles also tile in the radial fashion shown in the center of Figure 2A. Although the tiling is highly ordered, it is obviously not periodic. As Michael Goldberg pointed out in a 1955 paper titled "Central Tessellations," such a tiling can be sliced in half, and then the half planes can be shifted one step or more to make a spiral form of nonperiodic tiling, as shown in Figure 2A, right. The triangle can be distorted in an infinity of ways by replacing its two equal sides with congruent lines, as shown at the left in Figure 2B. If the new sides have straight edges, the result is a polygon of 5, 7, 9, 11 . . . edges that tiles spirally. Figure 3 shows a striking pattern obtained in this way from a nine-sided polygon. It was first found by Heinz Voderberg in a complicated procedure. Goldberg's method of obtaining it makes it almost trivial.

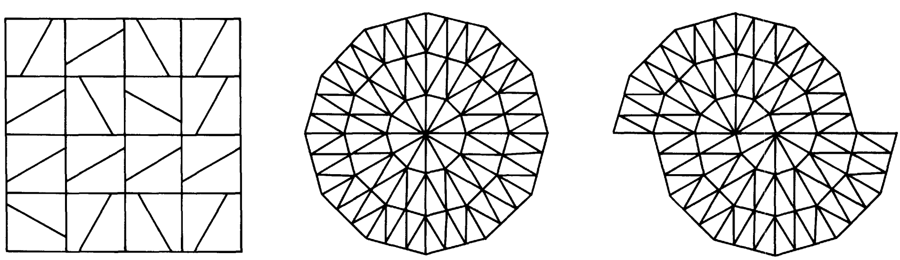
In all known cases of nonperiodic tiling by congruent figures the figure also tiles periodically. Figure 2B, right, shows how two of the Voderberg enneagons go together to make an octagon that tiles periodically in an obvious way.

Another kind of nonperiodic tiling is obtained by tiles that group together to form larger replicas of themselves. Solomon W. Golomb calls them "reptiles." (See Chapter 19 of my book *Unexpected Hanging*.) Figure 4 shows how a shape called the "sphinx" tiles nonperiodically by giving rise to ever larger sphinxes. Again, two sphinxes (with one sphinx rotated 180 degrees) tile periodically in an obvious way.

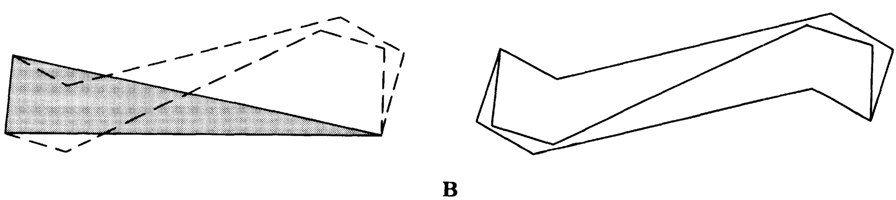
Are there sets of tiles that tile only nonperiodically? By "only" we mean that neither a single shape or subset nor the entire set tiles periodically, but that by using all of them a nonperiodic tiling is possible. Rotating and reflecting tiles are allowed.

For many decades experts believed no such set exists, but the supposition proved to be untrue. In 1961 Hao Wang became interested in tiling the plane with sets of unit squares whose edges were colored in various ways. They are called Wang dominoes, and Wang wrote a splendid article about them for *Scientific American* in 1965. Wang's problem was to find a procedure for deciding whether any given set of dominoes will tile by placing them so that abutting edges are the same color. Rotations and reflections are not allowed. The problem is important because it relates to decision questions in symbolic logic. Wang conjectured that any set of tiles which can tile the plane can tile it periodically and showed that if this is the case, there is a decision procedure for such tiling.

In 1964 Robert Berger, in his thesis for a doctorate from Harvard University in applied mathematics, showed that Wang's conjecture is



A



B

Figure 2 *(A) Nonperiodic tiling with congruent shapes (B) An enneagon* (dotted at left) *and a pair of enneagons* (right) *forming an octagon that tiles periodically*

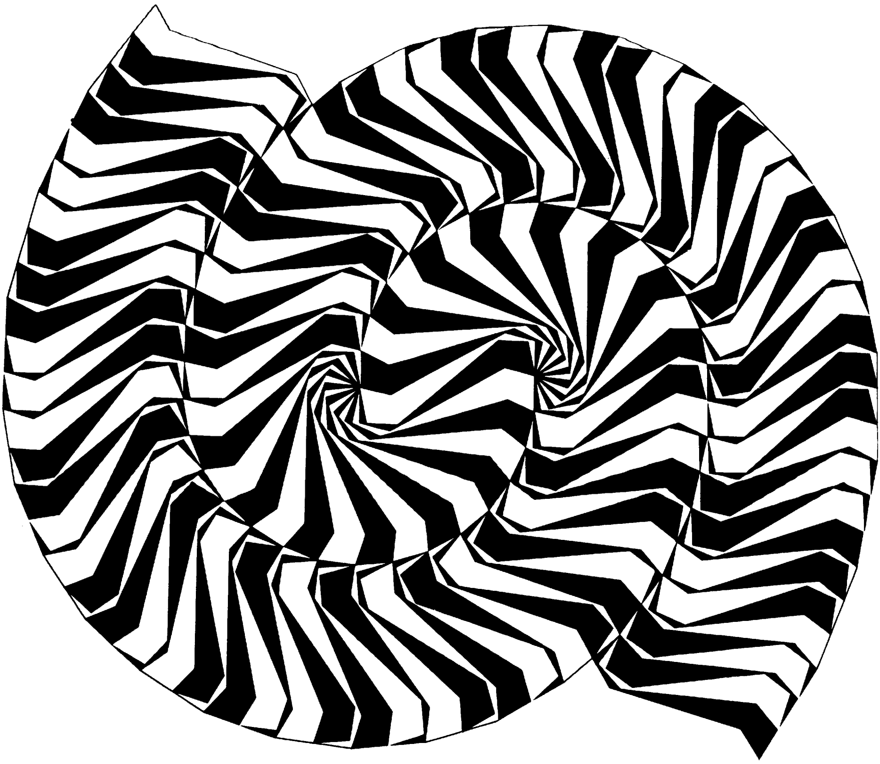


***Figure 3** A spiral tiling by Heinz Voderberg*

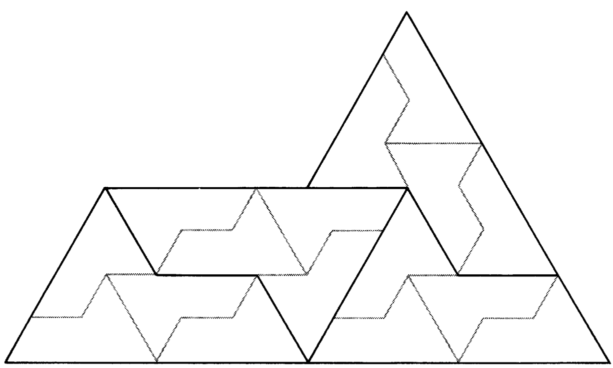


***Figure 4** Three generations of sphinxes in a nonperiodic tiling*

false. There is no general procedure. Therefore there is a set of Wang dominoes that tiles only nonperiodically. Berger constructed such a set, using more than 20,000 dominoes. Later he found a much smaller set of 104, and Donald Knuth was able to reduce the number to 92.

It is easy to change such a set of Wang dominoes into polygonal tiles that tile only nonperiodically. You simply put projections and slots on the edges to make jigsaw pieces that fit in the manner formerly prescribed by colors. An edge formerly one color fits only another formerly the same color, and a similar relation obtains for the other colors. By allowing such tiles to rotate and reflect Robinson constructed six tiles (see Figure 5) that force nonperiodicity in the sense explained above. In 1977 Robert Ammann found a different set of six tiles that also force nonperiodicity. Whether tiles of this square type can be reduced to less than six is not known, though there are strong grounds for believing six to be the minimum.

At the University of Oxford, where he is Rouse Ball Professor of Mathematics, Penrose found small sets of tiles, not of the square type,

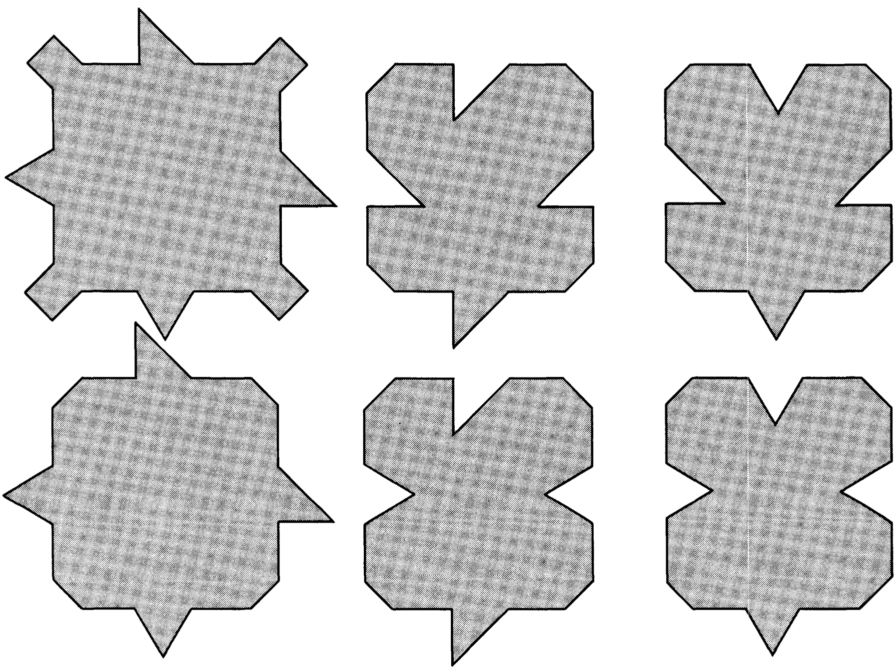


Figure 5 *Raphael M. Robinson's six tiles that force a nonperiodic tiling*

that force nonperiodicity. Although most of his work is in relativity theory and quantum mechanics, he continues the active interest in recreational mathematics he shared with his geneticist father, the late L. S. Penrose. (They are the inventors of the famous "Penrose staircase" that goes round and round without getting higher; Escher depicted it in his lithograph "Ascending and Descending.") In 1973 Penrose found a set of six tiles that force nonperiodicity. In 1974 he found a way to reduce them to four. Soon afterward he lowered them to two.

Because the tiles lend themselves to commercial puzzles, Penrose was reluctant to disclose them until he had applied for patents in the United Kingdom, the United States and Japan. The patents are now in force. I am equally indebted to John Horton Conway for many of the results of his study of the Penrose tiles.

The shapes of a pair of Penrose tiles can vary, but the most interesting pair have shapes that Conway calls "darts" and "kites." Figure 6A shows how they are derived from a rhombus with angles of 72 and 108 degrees. Divide the long diagonal in the familiar golden ratio of (1 +

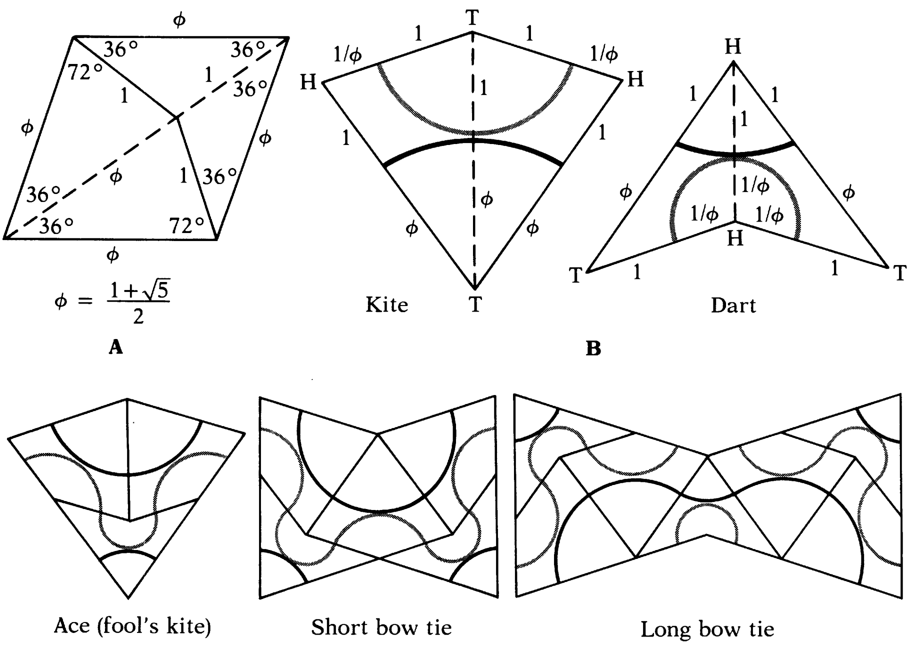


Figure 6 *(A) Construction of dart and kite (B) A coloring* (black and gray) *of dart and kite to force nonperiodicity (C) Aces and bow ties that speed constructions*

$\sqrt{5})/2 = 1.61803398 \ldots$, then join the point to the obtuse corners. That is all. Let phi stand for the golden ratio. Each line segment is either 1 or phi as indicated. The smallest angle is 36 degrees, and the other angles are multiples of it.

The rhombus of course tiles periodically, but we are not allowed to join the pieces in this manner. Forbidden ways of joining sides of equal length can be enforced by bumps and dents, but there are simpler ways. For example, we can label the corners H and T (heads and tails) as is shown in Figure 6B, and then give the rule that in fitting edges only corners of the same letter may meet. Dots of two colors could be placed in the corners to aid in conforming to this rule, but a prettier method, proposed by Conway, is to draw circular arcs of two colors on each tile, shown in the illustration as black and gray. Each arc cuts the sides as well as the axis of symmetry in the golden ratio. Our rule is that abutting edges must join arcs of the same color.

To appreciate the full beauty and mystery of Penrose tiling one should make at least 100 kites and 60 darts. The pieces need be colored on one side only. The number of pieces of the two shapes are (like their areas) in the golden ratio. You might suppose you need more of the smaller darts, but it is the other way around. You need 1.618 . . . as many kites as darts. In an infinite tiling this proportion is exact. The irrationality of the ratio underlies a proof by Penrose that the tiling is nonperiodic because if it were periodic, the ratio clearly would have to be rational.

A good plan is to draw as many darts and kites as you can on one sheet, with a ratio of about five kites to three darts, using a thin line for the curves. The sheet can be photocopied many times. The curves can then be colored with, say, red and green felt-tip pens. Conway has found that it speeds constructions and keeps patterns stabler if you make many copies of the three larger shapes as is shown in Figure 6C. As you expand a pattern, you can continually replace darts and kites with aces and bow ties. Actually an infinity of arbitrarily large *pairs* of shapes, made up of darts and kites, will serve for tiling any infinite pattern.

A Penrose pattern is made by starting with darts and kites around one vertex and then expanding radially. Each time you add a piece to an edge, you must choose between a dart and a kite. Sometimes the choice is forced, sometimes it is not. Sometimes either piece fits, but later you may encounter a contradiction (a spot where no piece can be legally added) and be forced to go back and make the other choice. It is a good plan to go around a boundary, placing all the forced pieces first. They cannot lead to a contradiction. You can then experiment with unforced pieces. It is always possible to continue forever. The more you play with the pieces, the more you will become aware of "forcing rules" that increase efficiency. For example, a dart forces two kites in its concavity, creating the ubiquitous ace.

There are many ways to prove that the number of Penrose tilings is uncountable, just as the number of points on a line is. These proofs rest on a surprising phenomenon discovered by Penrose. Conway calls it "inflation" and "deflation." Figure 7 shows the beginning of inflation. Imagine that every dart is cut in half and then all short edges of the original pieces are glued together. The result: a new tiling (shown in heavy black lines) by larger darts and kites.

Inflation can be continued to infinity, with each new "generation" of pieces larger than the last. Note that the second-generation kite, although it is the same size and shape as a first-generation ace, is formed differently. For this reason the ace is also called a fool's kite. It should never be mistaken for a second-generation kite. Deflation is the same process carried the other way. On every Penrose tiling we can draw smaller and smaller generations of darts and kites. This pattern too goes to infinity, creating a structure that is a fractal (see Chapter 3).

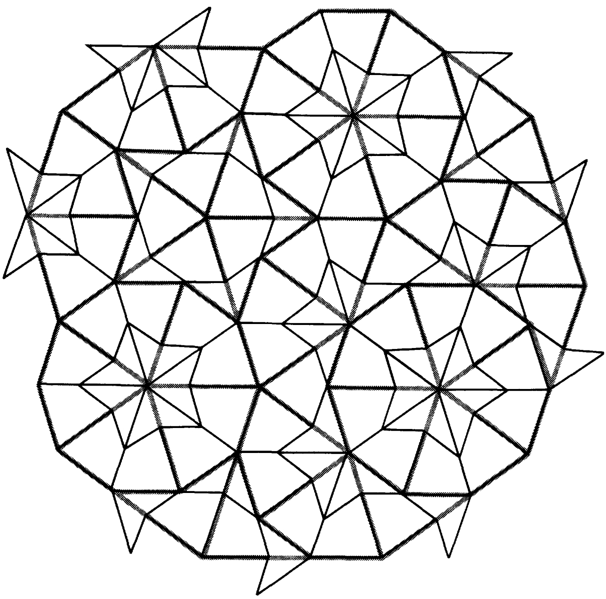


Figure 7 *How a pattern is inflated*

Conway's proof of the uncountability of Penrose patterns (Penrose had earlier proved it in a different way) can be outlined as follows. On the kite label one side of the axis of symmetry *L*, the other *R* (for left and right). Do the same on the dart, using *l* and *r*. Now pick a random point on the tiling. Record the letter that gives its location on the tile. Inflate the pattern one step, note the location of the same point in a second-generation tile and again record the letter. Continuing through higher inflations, you generate an infinite sequence of symbols that is a unique labeling of the original pattern seen, so to speak, from the selected point.

Pick another point on the original pattern. The procedure may give a sequence that starts differently, but it will reach a letter beyond which it agrees to infinity with the former sequence. If there is no such agreement beyond a certain point, the two sequences label distinct patterns. Not all possible sequences of the four symbols can be produced this way, but those that label different patterns can be shown to correspond in number with the number of points on a line.

We have omitted the colored curves on our pictures of tilings because they make it difficult to see the tiles. If you work with colored tiles, however, you will be struck by the beautiful designs created by these curves. Penrose and Conway independently proved that whenever a curve closes, it has a pentagonal symmetry, and the entire region within the curve has a fivefold symmetry. At the most a pattern can have two curves of each color that do not close. In most patterns all curves close.

Although it is possible to construct Penrose patterns with a high degree of symmetry (an infinity of patterns have bilateral symmetry), most patterns, like the universe, are a mystifying mixture of order and unexpected deviations from order. As the patterns expand, they seem to be always striving to repeat themselves but never quite managing it. G. K. Chesterton once suggested that an extraterrestrial being, observing how many features of a human body are duplicated on the left and the right, would reasonably deduce that we have a heart on each side. The world, he said, "looks just a little more mathematical and regular than it is; its exactitude is obvious, but its inexactitude is hidden; its wildness lies in wait." Everywhere there is a "silent swerving from accuracy by an inch that is the uncanny element in everything . . . a sort of secret treason in the universe." The passage is a nice description of Penrose's planar worlds.

There is something even more surprising about Penrose universes. In a curious finite sense, given by the "local isomorphism theorem," all Penrose patterns are alike. Penrose was able to show that every finite region in any pattern is contained somewhere inside every other pattern. Moreover, it appears infinitely many times in every pattern.

To understand how crazy this situation is, imagine you are living on an infinite plane tessellated by one tiling of the uncountable infinity of Penrose tilings. You can examine your pattern, piece by piece, in ever expanding areas. No matter how much of it you explore you can never determine which tiling you are on. It is no help to travel far out and examine disconnected regions, because all the regions belong to one large finite region that is exactly duplicated infinitely many times on all patterns. Of course, this is trivially true of any periodic tessellation, but Penrose universes are not periodic. They differ from one another in infinitely many ways, and yet it is only at the unobtainable limit that one can be distinguished from another.

Suppose you have explored a circular region of diameter *d*. Call it the "town" where you live. Suddenly you are transported to a randomly chosen parallel Penrose world. How far are you from a circular region that exactly matches the streets of your home town? Conway answers with a truly remarkable theorem. The distance from the perimeter of the home town to the perimeter of the duplicate town is never more than *d* times half of the cube of the golden ratio, or 2.11+ times *d*. (This is an upper bound, not an average.) If you walk in the right direction, you need not go more than that distance to find yourself inside an exact copy of your home town. The theorem also applies to the universe in which you live. Every large circular pattern (there is an infinity of different ones) can be reached by walking a distance in some direction that is certainly less than about twice the diameter of the pattern and more likely about the same distance as the diameter.

The theorem is quite unexpected. Consider an analogous isomorphism exhibited by a sequence of unpatterned digits such as pi. If you pick a finite sequence of 10 digits and then start from a random spot in pi, you are pretty sure to encounter the same sequence if you move far enough along pi, but the distance you must go has no known upper bound, and the expected distance is enormously longer than 10 digits. The longer the finite sequence is, the farther you can expect to walk to find it again. On a Penrose pattern you are always very close to a duplicate of home.

There are just seven ways that darts and kites will fit around a vertex. Let us consider first, using Conway's nomenclature, the two ways with pentagonal symmetry.

The sun (shown in white in Figure 8) does not force the placing of any other piece around it. If you add pieces so that pentagonal symmetry

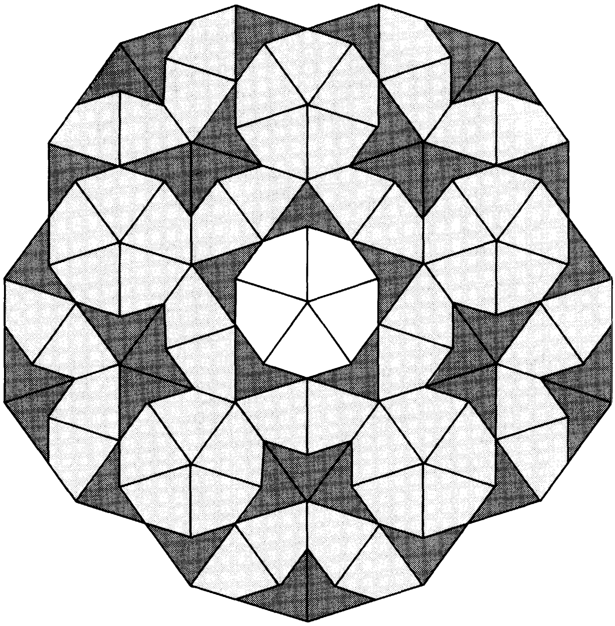


Figure 8 *The infinite sun pattern*

is always preserved, however, you will be forced to construct the beautiful pattern shown. It is uniquely determined to infinity.

The star, shown in white in Figure 9, forces the 10 light gray kites around it. Enlarge this pattern, always preserving the fivefold symmetry, and you will create another flowery design that is infinite and unique. The star and sun patterns are the only Penrose universes with perfect pentagonal symmetry, and there is a lovely sense in which they are equivalent. Inflate or deflate either of the patterns and you get the other.

The ace is a third way to tile around a vertex. It forces no more pieces. The deuce, the jack and the queen are shown in white in Figure 10, surrounded by the tiles they immediately force. As Penrose discovered (it was later found independently by Clive Bach), some of the seven vertex figures force the placing of tiles that are not joined to the immediately forced region. Plate 1 shows in deep color the central portion of the king's "empire." (The king is the dark gray area.) All the deep colored tiles are forced by the king. (Two aces, just outside the left and right borders, are also forced but are not shown.)

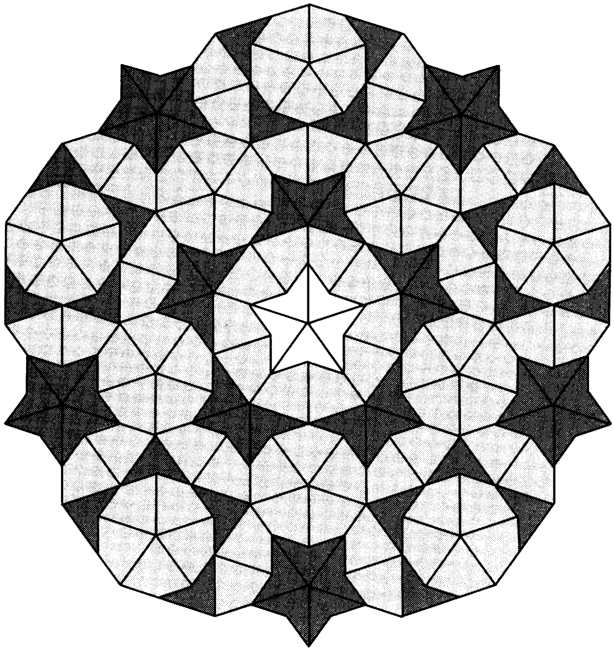


Figure 9 *The infinite star pattern*

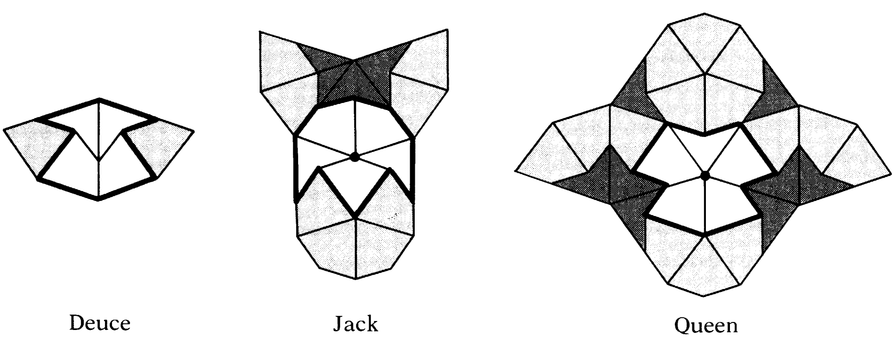


Figure 10 *The "empires" of deuce, jack and queen*

This picture of the king's empire was drawn by a computer program written by Eric Regener of Concordia University in Montreal. His program deflates any Penrose pattern any number of steps. The heavy black lines show the domain immediately forced by the king. The thin black lines are a third-generation deflation in which the king and almost all of his empire are replicated.

The most extraordinary of all Penrose universes, essential for understanding the tiles, is the infinite cartwheel pattern, the center of which is shown in Figure 11. The regular decagon at the center, outlined in heavy black (each side is a pair of long and short edges), is what Conway calls a "cartwheel." Every point on any pattern is inside a cartwheel exactly like

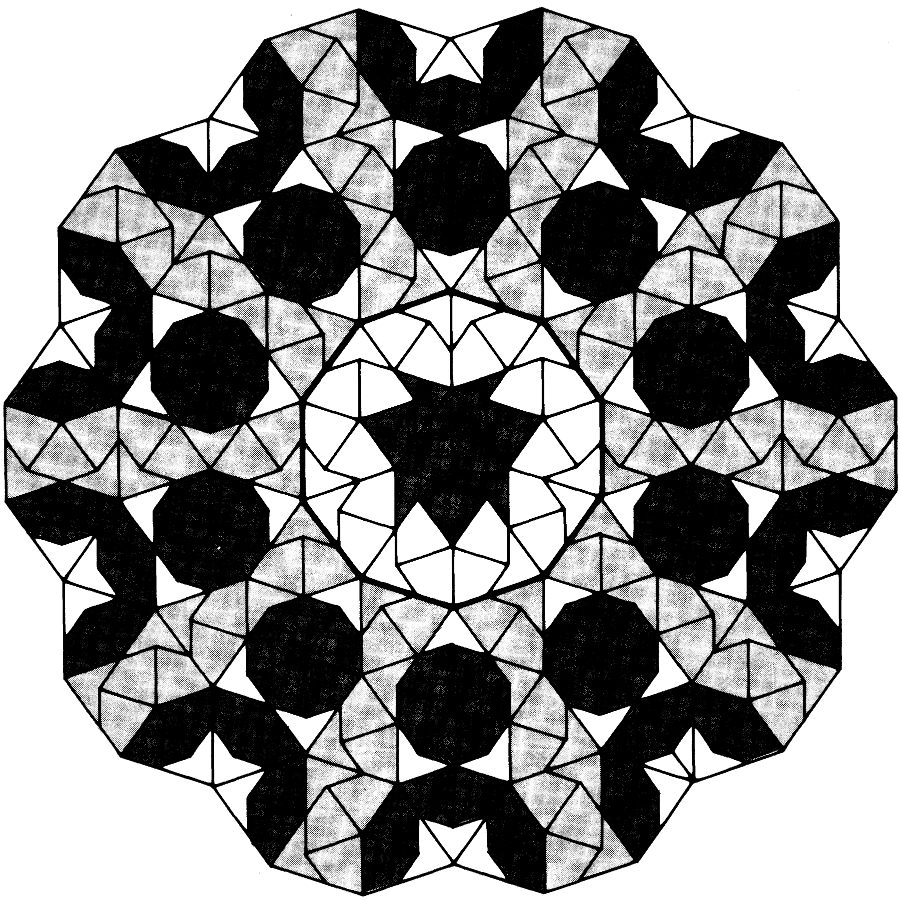


Figure 11 *The cartwheel pattern surrounding Batman*

this one. By one-step inflation we see that every point will be inside a larger cartwheel. Similarly, every point is inside a cartwheel of every generation, although the wheels need not be concentric.

Note the 10 light gray spokes that radiate to infinity. Conway calls them "worms." They are made of long and short bow ties, the number of long ones being in the golden ratio to the number of short ones. Every Penrose universe contains an infinite number of arbitrarily long worms. Inflate or deflate a worm and you get another worm along the same axis. Observe that two full worms extend across the central cartwheel in the infinite cartwheel pattern. (Inside it they are not gray.) The remaining spokes are half-infinite worms. Aside from spokes and the interior of the central cartwheel, the pattern has perfect tenfold symmetry. Between any two spokes we see an alternating display of increasingly large portions of the sun and star patterns.

Any spoke of the infinite cartwheel pattern can be turned side to side (or, what amounts to the same thing, each of its bow ties can be rotated end for end), and the spoke will still fit all surrounding tiles except for those inside the central cartwheel. There are 10 spokes; thus there are $2^{10} = 1024$ combinations of states. After eliminating rotations and reflections, however, there are only 62 distinct combinations. Each combination leaves inside the cartwheel a region that Conway has named a "decapod."

Decapods are made up of 10 identical isosceles triangles with the shapes of enlarged half darts. The decapods with maximum symmetry are the buzzsaw and the starfish shown in Figure 12. Like a worm, each triangle can be turned. As before, ignoring rotations and reflections, we get 62 decapods. Imagine the convex vertexes on the perimeter of each decapod to be labeled *T* and the concave vertexes labeled *H*. To continue tiling, these *H*'s and *T*'s must be matched to the heads and tails of the tiles in the usual manner.

When the spokes are arranged the way they are in the infinite cartwheel pattern shown, a decapod called Batman is formed at the center. Batman (shown in dark gray) is the only decapod that can legally be tiled. (No finite region can have more than one legal tiling.) Batman does not, however, force the infinite cartwheel pattern. It merely allows it. Indeed, no finite portion of a legal tiling can force an entire pattern, because the finite portion is contained in *every* tiling.

Note that the infinite cartwheel pattern is bilaterally symmetrical, its axis of symmetry going vertically through Batman. Inflate the pattern and it remains unchanged except for mirror reflection in a line perpen-

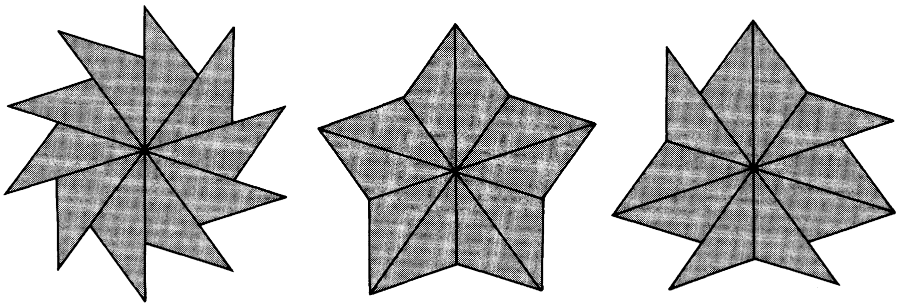


Figure 12 *Three decapods*

dicular to the symmetry axis. The five darts in Batman and its two central kites are the only tiles in any Penrose universe that are not inside a region of fivefold symmetry. All other pieces in this pattern or any other one are in infinitely many regions of fivefold symmetry.

The other 61 decapods are produced inside the central cartwheel by the other 61 combinations of worm turns in the spokes. All 61 are "holes" in the following sense. A hole is any finite empty region, surrounded by an infinite tiling, that cannot be legally tiled. You might suppose each decapod is the center of infinitely many tilings, but here Penrose's universes play another joke on us. Surprisingly, 60 decapods force a unique tiling that differs from the one shown only in the composition of the spokes. Only Batman and one other decapod, called Asterix* after a French cartoon character, do not. Like Batman, Asterix allows an infinite cartwheel pattern, but it also allows patterns of other kinds.

Now for a startling conjecture. Conway believes, although he has not completed the proof, that every possible hole, of whatever size or shape, is equivalent to a decapod hole in the following sense. By rearranging tiles around the hole, taking away or adding a finite number of pieces if necessary, you can transform every hole into a decapod. If this is true, any finite number of holes in a pattern can also be reduced to one decapod. We have only to remove enough tiles to join the holes into one big hole, then reduce the big hole until an untileable decapod results.

*Asterix the Gaul is featured in a popular series of French picture books for children. The stories are fantasies taking place at the time of Julius Caesar. Asterix is also an intended pun on "asterisk."

Think of a decapod as being a solid tile. Except for Batman and Asterix, each of the 62 decapods is like an imperfection that solidifies a crystal. It forces a unique infinite cartwheel pattern, spokes and all, that goes on forever. If Conway's conjecture holds, any "foreign piece" (Penrose's term) that forces a unique tiling, no matter how large the piece is, has an outline that transforms into one of 60 decapod holes.

Kites and darts can be changed to other shapes by the same technique described earlier for changing isosceles triangles into spiral-tiling polygons. It is the same technique that Escher employed for transforming polygonal tiles into animal shapes. Figure 13 shows how Penrose changed his darts and kites into chickens that tile only nonperiodically. Note that although the chickens are asymmetrical, it is never necessary to turn any of them over to tile the plane. Alas, Escher died before he could know of Penrose's tiles. How he would have reveled in their possibilities!

By dissecting darts and kites into smaller pieces and putting them together in other ways you can make other pairs of tiles with properties similar to those of darts and kites. Penrose found an unusually simple pair: the two rhombuses in the sample pattern of Figure 14. All edges are the same length. The larger piece has angles of 72 and 108 degrees and

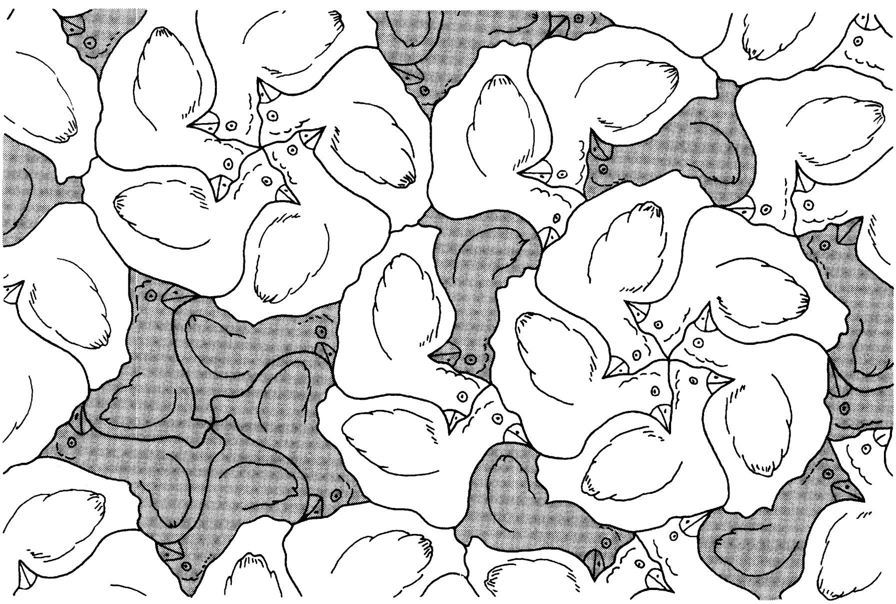


Figure 13 *Penrose's nonperiodic chickens*

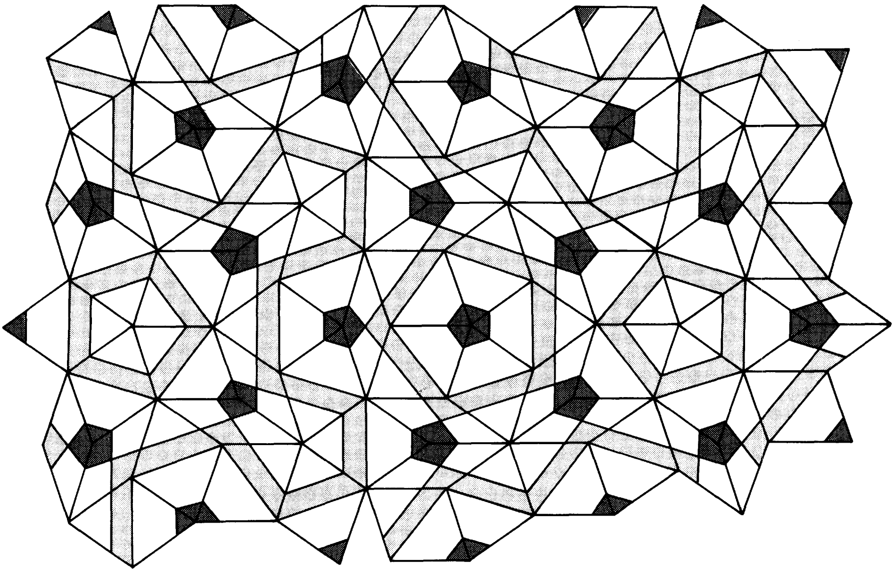


Figure 14 *A nonperiodic tiling with Roger Penrose's rhombuses*

the smaller one has angles of 36 and 144 degrees. As before, both the areas and the number of pieces needed for each type are in the golden ratio. Tiling patterns inflate and deflate and tile the plane in an uncountable infinity of nonperiodic ways. The nonperiodicity can be forced by bumps and dents or by a coloring such as the one suggested by Penrose and shown in the illustration by the light and dark gray areas.

We see how closely the two sets of tiles are related to each other and to the golden ratio by examining the pentagram in Figure 15. This was the mystic symbol of the ancient Greek Pythagorean brotherhood and the diagram with which Goethe's Faust trapped Mephistopheles. The construction can continue forever, outward and inward, and every line segment is in the golden ratio to the next smaller one. Note how all four Penrose tiles are embedded in the diagram. The kite is *ABCD*, and the dart is *AECB*. The rhombuses, although they are not in the proper relative sizes, are *AECD* and *ABCF*. As Conway likes to put it, the two sets of tiles are based on the same underlying "golden stuff." Any theorem about kites and darts can be translated into a theorem about the Penrose rhombuses or any other pair of Penrose tiles and vice versa. Conway prefers to work with darts and kites, but other mathematicians prefer working with the simpler rhombuses. Robert Ammann has found a

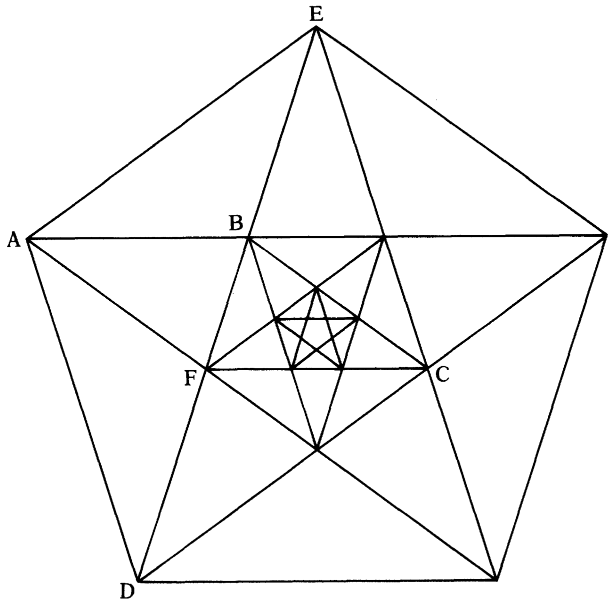


Figure 15 *The Pythagorean pentagram*

bewildering variety of other sets of nonperiodic tiles. One set, consisting of two convex pentagons and a convex hexagon, forces nonperiodicity without any edge markings. He found several pairs, each a hexagon with five interior angles of 90 degrees and one of 270 degrees. You'll find these sets depicted and their remarkable properties discussed in the book by Branko Grünbaum and G. C. Shephard listed in the next chapter's bibliography.

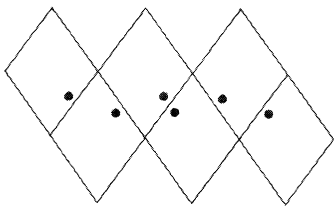
Are there pairs of tiles not related to the golden ratio that force nonperiodicity? Is there a pair of *similar* tiles that force nonperiodicity? Is there a pair of convex tiles that will force nonperiodicity without edge markings?

Of course, the major unsolved problem is whether there is a *single* shape that will tile the plane only nonperiodically. Most experts think not, but no one is anywhere near proving it. It has not even been shown that if such a tile exists, it must be nonconvex.

CHAPTER

2

Penrose Tiling II



In the decade since my column on Penrose tiling ran in *Scientific American* (January 1977), Roger Penrose, John Conway, Robert Ammann and others have made enormous strides in exploring nonperiodic tiling. (I will continue here to use the term "nonperiodic," although Branko Grünbaum and G. C. Shephard in their monumental work *Tilings and Patterns* prefer to call a set of tiles "aperiodic" if it tiles only nonperiodically.) The discovery of what are now called Ammann bars or lines and of 3-space analogues of Penrose tiling has led to an amazing development in crystallography, but first let me summarize in this previously unpublished chapter some of the developments that preceded this breakthrough.

Robert Ammann, a brilliant young mathematician working at low-level computer jobs in Massachusetts, independently discovered Penrose's rhomb tiles in 1976, about eight months before my column on Penrose tiling appeared. In correspondence I informed him of the darts and kites, as well as Penrose's earlier discovery of the rhombs. Ammann soon realized that both pairs of tiles formed patterns that were determined by five families of parallel lines that cross the plane in five different directions, intersecting one another at 360/5 = 72-degree angles. One family of such lines, now called Ammann bars, is shown in Figure 16.

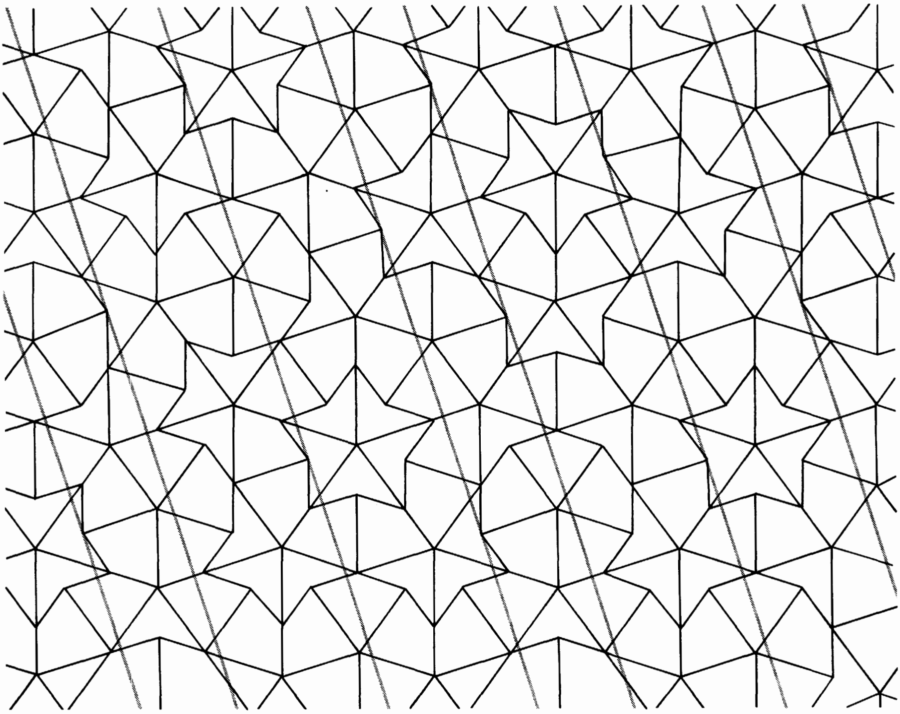


Figure 16 *A family of Ammann bars displaying* (left to right) *a* SLLSLLS *sequence*

Observe that the lines cross the concave corners of darts that point in the same or opposite direction. This is not strictly accurate, but for our purposes lines ruled in this easy way are adequate. For a precise positioning of the lines, see the Grünbaum/Shephard book. When accurately placed, each line is a trifle outside a dart's concave corner. Inside each regular decagon (ten-sided polygon) on the pattern, the Ammann bars form a perfect pentagram (five-pointed star).

Note that the spacings between bars are of two lengths which we will call *L* (for long) and *S* (for short). When the lines are properly drawn, these two lengths are in golden ratio. Moreover, on the infinite plane the number of *L*'s in a family of bars and the number of *S*'s in that same family are in golden ratio. Moving in either direction perpendicular to a family of bars, we can record the sequence of spacings as a sequence of *L*'s and *S*'s. This sequence is nonperiodic, and constitutes a remarkable 1-dimensional analogue of Penrose tiling. The local isomorphism theorem applies. If you select any finite portion of the sequence, you can always find it duplicated not far away. Start anywhere and write down the letters to any finite length, say a billion. If you start at any other spot in the sequence, you are certain to reach this identical billion-letter sequence. Only when the sequence is taken as infinite is it unique.

Conway discovered that this sequence can be obtained from the golden ratio in the following way. Write down in ascending order the multiples of the golden ratio $(1 + \sqrt{5})/2$ and round them down to the nearest integer. The result is the sequence that begins 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, 29, 30, 32, 33, 35, 37, 38, 40, 42, 43, 45, 46, 48, 50. . . . It is sequence 917 in N. J. A. Sloane's *Handbook of Integer Sequences*. If you round down multiples of the square of the golden ratio, you get the sequence 2, 5, 7, 10, 13, 15, 18, 20, 23. . . . The two sequences are called "complementary." Together they display every positive integer once and only once. Successive multiples of any real number a, rounded down to the nearest integer, form a sequence called the spectrum of a. If a is irrational, the sequence is called a Beatty sequence after Samuel Beatty, a Canadian mathematician who called attention to such sequences in 1926. As we shall see in Chapter 8, the complementary Beatty sequences based on the golden ratio provide the winning strategy for a famous variant of Nim known as Wythoff's game. References on Beatty sequences are given in that chapter's bibliography.

Adjacent numbers in the golden Beatty sequence differ by either 1 or 2. Put down this first row of differences, then change each 1 to 0 and each 2 to 1. You get an endless binary sequence that starts 101101011011010. . . . This is a portion of the sequence of S's and L's in any infinite family of Ammann bars. Conway uses the term "musical sequence" for any finite segment of the golden ratio sequence. Following Penrose, I shall call them Fibonacci sequences.

Such sequences have many curious properties. For example, put a decimal point in front of the Fibonacci sequence given above in binary notation. The result is an irrational binary fraction that is generated by the following continued fraction:

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2^2+\cfrac{1}{2^3+\cfrac{1}{2^5+\cfrac{1}{2^8+\cfrac{1}{2^{13}+\cfrac{1}{2^{21}+\cfrac{1}{2^{34}+\cdots}}}}}}}}}}$$

The exponents of this continued fraction are none other than the Fibonacci numbers. Conway has many unpublished results on the way Penrose tilings are related to Fibonacci numbers, which are in turn related to the growth patterns of plants.

Penrose tilings are, as we have seen, self-similar in the sense that inflating or deflating them produces another tiling. Fibonacci sequences have the same self-similar property. There are many techniques for inflating and deflating them to produce another such sequence, but the simplest is as follows. To deflate, replace each *S* by an *L*, each *LL* by *S*, and drop all single *L*'s. For example, the sequence *LSLLSLSLLSLLSLS* deflates by these rules to *LSLLSLSLL*. To inflate, replace each *L* by *S*, each *S* by *LL*, then add an *L* between each pair of *S*'s.

A Fibonacci sequence cannot contain *SS* or *LLL*. This provides a simple way to tell if a sequence of *S*'s and *L*'s is Fibonacci. Apply the deflation rules until you reach either a sequence that contains an *SS* or an *LLL* (in which case the sequence is not Fibonacci) or a single letter that proves it is. If you inflate or deflate a Penrose tiling, the sequence in each family of Ammann bars also inflates or deflates. The sequence of long and short bow ties in any worm, such as the worms in the ten spokes of the cartwheel pattern, is also a Fibonacci sequence.

Two families of Ammann bars tessellate the plane with nonperiodic parallelograms that form a grid into which the tiles fit. As Grünbaum and Shephard put it, instead of thinking of the tiles as determining Ammann bars, "it is the system of bars which are fundamental and the only function of the tiles is to give a practical realization to them." The bars are something vaguely like the quantum fields that determine the positions and paths of particles. Ammann was the first to perceive, early in 1977, that his grid of bars leads into "forcing theorems"—theorems that tell how a small set of tiles will force the positions of infinite sets of other tiles.

As Ammann expressed it in a letter to me: "Whenever a set of tiles forces two parallel lines to occupy certain positions, it forces an infinite number of nonadjacent parallel lines also to occupy certain positions. Whenever three lines cross at the proper angles, a tile is forced." This property of a finite set of tiles forcing the positions of tiles at arbitrarily long distances belongs also to the Penrose rhombs and to Robinson squares, even though they have no connection with the golden ratio.

Taking off from Ammann's discoveries, Conway went on to develop many remarkable forcing theorems. I will say here only that two Penrose tiles (each can be of either type), suitably placed and arbitrarily far apart, will determine two infinite families (not complete families) of bars.

Intersections of the two families in turn determine the positions of an infinite set of tiles. The king, queen, jack, deuce and star, for example, force an infinite set of tiles in their empires. (The ace and sun do not force any tiles.) The king's empire is unusually dense. You might expect the density of such forced tiles to thin out as you get farther from the center, but this is not the case. The density remains constant for the entire plane.

Ammann's other great discovery, also made in 1976, was a set of two rhombohedra (parallelepipeds with six congruent rhomb faces) which, with suitable face-matching rules, force a nonperiodic tiling of space. Nets for the two solids are shown in Figure 17. If you cut these two nets out of cardboard, fold along the lines and tape the edges, you will obtain the two solids shown at the bottom of the illustration. One can be thought of as a cube that has been squashed along a space diagonal and

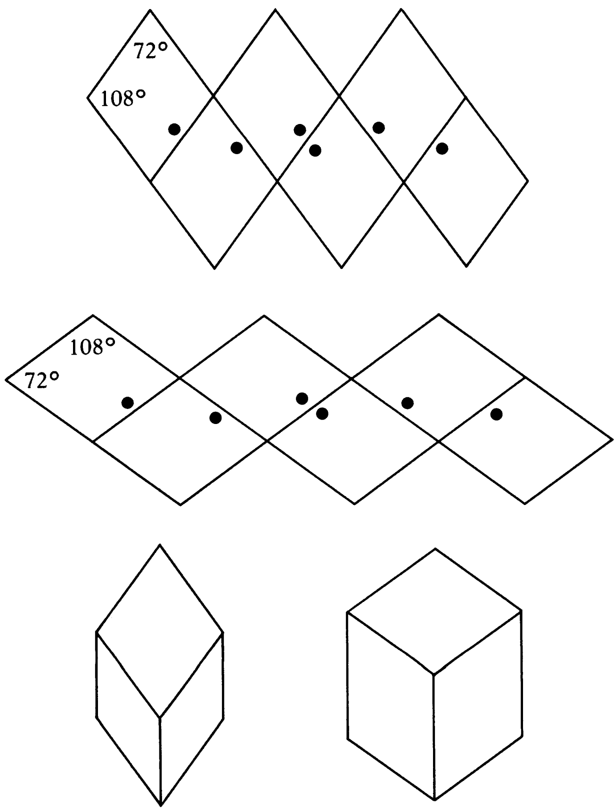


Figure 17 *Nets for the obtuse and acute golden rhombohedra*

the other as a cube that has been stretched along a space diagonal. All twelve faces are congruent, with their diagonals in golden ratio. The geometer H. S. M. Coxeter, in a note added on page 161 to the thirteenth edition of W. W. Rouse Ball's classic *Mathematical Recreations and Essays* (Dover, 1987), which he edited, calls a rhombohedron of this type a "golden rhombohedron." There are just two kinds, both of which had been studied by Kepler. The acute golden rhombohedron has two opposite corners where three equal acute angles meet. The obtuse golden rhombohedron has two opposite corners where equal obtuse angles meet. Other corners on both solids are mixtures of acute and obtuse angles.

Ammann's two rhombohedra are the two golden types. The faces of the acute solid meet along edges at angles of 72 and 108 degrees. Those on the obtuse solid meet at 36 and 144 degrees. (The four dihedral angles are multiples of $360/10 = 36$ degrees.) The face angles are close to 64 and 116 degrees. Periodic tiling is ruled out by suitably placed holes and projections. Note the spots on the unfolded faces in the illustration. Imagine each solid with a duplicate that has its spots in a pattern that is a mirror image of the other. This forms a set of four solids that force nonperiodicity if you put them together so every spot touches another spot. It is not known if there is a way to avoid this mirror-image marking so that just two solids, suitably marked, will force nonperiodicity. If a plane is passed through the space tiling at a suitable angle, the plane displays a tiling very close to a tiling by Penrose rhombs.

I sent Ammann's results to Penrose. In a letter dated May 4, 1976, Penrose asked me to convey his congratulations to Ammann on two counts: for his independent discovery of the rhomb tiles and for the space tiling by the two golden rhombohedra. He continued:

> It is just possible that these things may have some significance in biology. You will recall that some viruses grow in the shapes of regular dodecahedra and icosahedra. It has always seemed puzzling how they do this. But with Ammann's non-periodic solids as basic units, one would arrive at quasi-periodic 'crystals' involving such seemingly impossible (crystallographically) cleavage directions along dodecahedral or icosahedral planes. Is it possible that the viruses might grow in some such way involving non-periodic basic units—or is the idea too fanciful?

A year after Ammann's discovery of his nonperiodic space tiling, it was rediscovered in Japan by Koji Miyazaki, an architect at Kobe University. He also discovered another way that the two golden rhombohedra can tile space nonperiodically, although the tiling is not forced. Five acute and five obtuse golden rhombohedra will fit together to form a rhombic triacontrahedron. Two such solids, joined by a common obtuse vertex, can be surrounded with 60 more golden rhombohedra (30 of each type) to make a larger rhombic triacontrahedron. This enlargement can be continued to infinity, tiling space in a honeycomb that has a center of icosahedral symmetry.

Penrose's conjectures about crystals, even his terminology, proved to be amazingly prophetic. In the early 1980's a number of scientists and mathematicians began to speculate cautiously about the possibility that the atomic structure of crystals might be based on a nonperiodic lattice. Then in 1984 Dany Schechtman and his colleagues at the National Bureau of Standards made a dramatic announcement. They had found a nonperiodic structure in the electron micrographs of a rapidly cooled aluminum-manganese alloy that some chemists immediately dubbed Schechtmanite. The micrographs displayed a clear fivefold symmetry which strongly suggested a nonperiodic space tiling analogous to Penrose tiling.

Nothing like this had been seen before. It was, as science writer Ivars Peterson put it, as if someone had observed a five-sided snowflake. It had long been a dogma in crystallography that crystals could exhibit rotational symmetry of only 2, 3, 4 and 6 rotations, but never 5, 7 or 8. Another dogma was that solid matter took only two forms: either with atoms in a periodic arrangement or with disordered atoms in such amorphous material as glass.

The ordered lattices of all crystals then known derived from three Platonic solids: the tetrahedron, cube and octahedron. The dodecahedron and icosahedron were ruled out because their fivefold symmetry made periodic tiling impossible. Yet here was a material that seemed to exhibit icosahedral symmetry. Like Penrose tiling, when the material was rotated by 72 degrees, or 1/5 of a circle, it remained essentially the same in an overall statistical way, but without long-range periodicity. It seemed to be a form of matter halfway between glass and ordinary crystals, suggesting that instead of a sharp demarcation between the two forms, there could be a continuum of in-between structures.

Among physicists, chemists and crystallographers the effect of this discovery was explosive. Similar nonperiodic structures were soon being induced in other alloys, and dozens of papers began to appear. It became clear that solid matter could exhibit nonperiodic lattices with any kind of rotational symmetry. Wide varieties of solid tiles in sets of two or more were proposed as models, some forcing nonperiodicity, some merely allowing it. A crystal structure was produced made of layers of sheets with two-dimensional Penrose rhomb tiling. N. G. de Bruijn in the Netherlands developed an algebraic theory of nonperiodic tiling based on what he calls "pentagrids," similar to Ammann bars. In a 1987 paper, he reported a surprising connection between nonperiodic tiling theory and a shuffling theorem known to card magicians as the Gilbreath principle. (On this principle see Chapter 9 of my *New Mathematical Diversions from Scientific American.*)

There is now enormous ferment in the ongoing empirical and theoretical investigations of "quasicrystals," as the new halfway crystals are called. There is also opposition to the view that their lattices are genuinely nonperiodic. The leading opponent is Linus Pauling, who argues that the micrographs should be interpreted as a spurious form of fivefold symmetry known to crystallographers as multiple twinning. "Crystallographers can now cease to worry that the validity of one of the accepted bases of their science has been questioned," Pauling concluded in a 1985 report in *Nature*. Another possibility is that quasicrystals are simply extremely large unit cells of a periodic pattern that will be found when larger samples are made. And there are other possibilities. Proponents of quasicrystals maintain that all these alternative interpretations of the micrographs have been eliminated and that true nonperiodicity is the simplest explanation. It could be that in a few years empirical studies will disconfirm this, and quasicrystals may go the ill-fated way of polywater; but if the nonperiodic interpretation holds, it will be a sensational turning point in crystallography.

Assuming quasicrystals are real, the next few years should see increasingly efficient techniques for producing them. Many questions cry out for answers. What physical forces are involved in the formation of these strange crystals? Penrose has suggested that perhaps nonlocal quantum field effects play a role because without an overall plan it is hard to see how such a crystal could grow in such a way as to preserve its long-range nonperiodic pattern. (In the passage quoted earlier from his 1976 letter, Penrose's speculations about viruses reflected his concern over how a quasicrystal could grow without guidance by nonlocal forces.) What are the elastic and electronic properties of quasicrystals? Will geologists ever find quasicrystals produced by nature?

If quasicrystals are what their defenders think they are, they provide a striking example of how work done in recreational mathematics, purely for fun and aesthetic satisfaction, can turn out to have significant practical applications to the physical world and to technology.

In 1980 I heard Conway lecture on Penrose tiling at Bell Laboratories. Discussing "hole theory," he said he liked to imagine a vast temple with a floor tesselated by Penrose tiles and a circular column exactly in the center. The tiles seem to go under the column. Actually, the column covers a hole that can't be tesselated. Incidentally, on such patterns the Ammann bars get broken out of alignment as they pass through the hole.

A Penrose tiling can, of course, always be colored with four colors so that no two tiles of the same color share a common edge. Can it always be colored with three? It can be shown, Conway said, from the local isomorphism theorem, that if any Penrose tiling is three-colorable, all are, but so far no one has proved that any infinite Penrose tiling is three-colorable.

Conway gave the following simple *reductio ad absurdum* proof (which he credited to Peter Barlow, a British mathematician who died in 1862, best known today for his books of tables) that no tiling pattern can have more than one center of fivefold symmetry. Assume it has more than one. Select the two, A and B, that are closest together. (See Figure 18.) Rotate the pattern $360/5 = 72$ degrees clockwise around B, carrying A to A' as shown. Return to the original position, and rotate the pattern 72 degrees counterclockwise around A, taking B to B'. Result: Both rotations (if our assumption is true) would leave the pattern unchanged, but now it has two centers of fivefold symmetry, A' and B', that are closer together than A and B. This contradicts our second assumption that A and B are the closest centers.

There are single tiles (and sets of tiles) that tile the plane periodically in only one way: the regular hexagon and the cross pentomino, for example. All triangles and all parallelograms tile in an uncountable

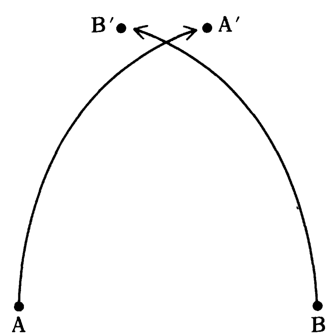


Figure 18 *Barlow's proof that no pattern can have two centers of fivefold symmetry*

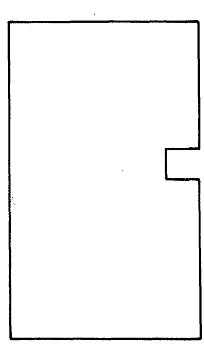


Figure 19 *The Conway tile that tiles in zero ways*

infinity of ways. Grünbaum and Shephard conjecture that no tile exists that tiles periodically in a *countable* infinity of ways. They also conjecture that given any positive integer *r*, there are single tiles that tile the plane in just *r* ways. Such tiles have been found for $r = 1$ through 10. In his lecture Conway exhibited what he calls the "Conway tile" (Figure 19) for $r = 0$. He concluded by saying it was the first lecture he had ever given on Penrose tiling in which he didn't inadvertently say "karts and dites."

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