Chapter 36 Sprouts and Brussels Sprouts

I made sprouts fontaneously. . . .
—James Joyce, *Finnegans Wake,* p. 542

"A friend of mine, a classics student at Cambridge, introduced me recently to a game called 'Sprouts' which became a craze at Cambridge last term. The game has a curious topological flavor."

So began a letter I received in 1967 from David Hartshorne, then a mathematics student at the University of Leeds. Soon other British readers were writing to me about this amusing pencil-and-paper game that had sprouted suddenly on the Cambridge grounds.

I am pleased to report that I successfully traced the origin of this game to its source: the joint creative efforts of John Horton Conway, then a teacher of mathematics at Sidney Sussex College, Cambridge, and Michael Stewart Paterson, then a graduate student working at Cambridge on abstract computer programming theory.

The game begins with n spots on a sheet of paper. Even with as few as three spots, Sprouts is more difficult to analyze than ticktacktoe, so that it is best for beginners to play with no more than three or four initial spots. A move consists of drawing a line that joins one spot to another or to itself and then placing a new spot anywhere along the line. These restrictions must be observed:

1. The line may have any shape but it must not cross itself, cross a previously drawn line, or pass through a previously made spot.
2. No spot may have more than three lines emanating from it.

Players take turns drawing curves. In normal Sprouts, the recommended form of play, the winner is the last person able to play. As in Nim and other games of the "take-away" type, the game can also be played in "misère" form, a French term that applies to a variety of card games in the whist family in which one tries to *avoid* taking tricks. In misère Sprouts the first person unable to play is the winner.

The typical three-spot normal game shown in Figure 36.1 was won on the seventh move by the first player. It is easy to see how the game got its name, for it sprouts into fantastic patterns as the game progresses. The most delightful feature is that it is not merely a combinatorial game, as so many connect-the-dots games are, but one that actually exploits the topological properties of the plane. In more technical language, it makes use of the Jordan-curve theorem, which asserts that simple closed curves divide the plane into outside and inside regions.

One might guess at first that a Sprouts game could keep sprouting for-

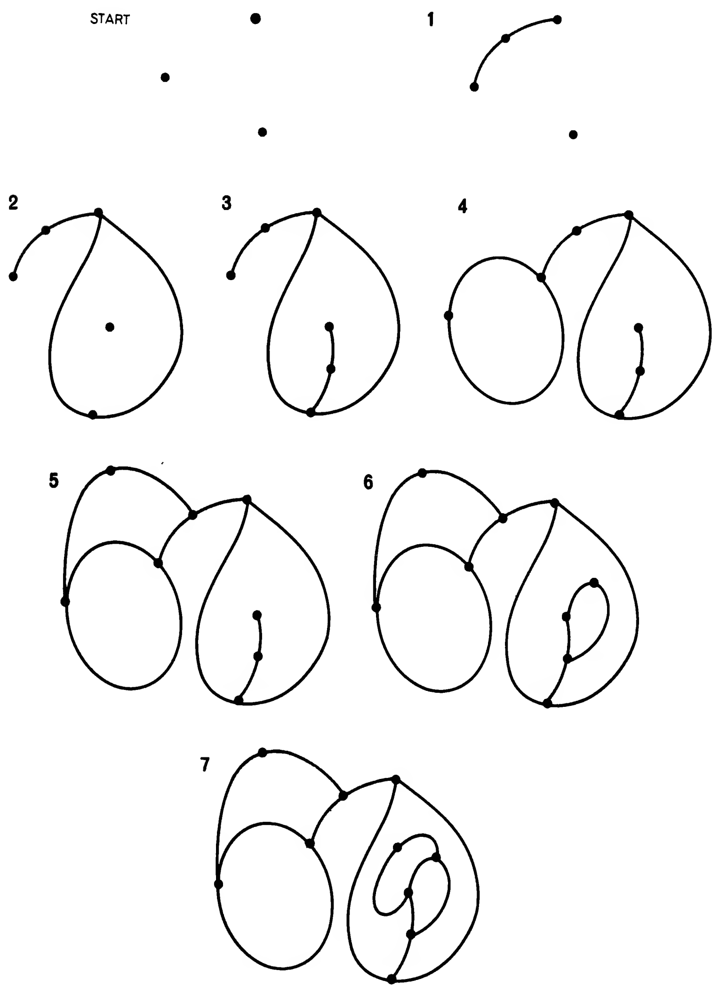


Figure 36.1. A typical game of three-spot Sprouts

ever, but Conway offers a simple proof that it must end in at most $3n - 1$ moves. Each spot has three "lives"—the three lines that may meet at that point. A spot that acquires three lines is called a "dead spot" because no more lines can be drawn to it. A game that begins with n spots has a starting life of $3n$. Each move kills two lives, at the beginning and at the end of the curve, but adds a new spot with a life of 1. Each move therefore decreases the total life of the game by 1. A game obviously cannot continue when only one life remains, since it requires at least two lives to make a move. Accordingly no game can last beyond $3n - 1$ moves. It is also easy to show that every game must last at least $2n$ moves. The three-spot game starts with nine lives, must end on or before the eighth move, and must last at least six moves.

The one-spot game is trivial. The first player has only one possible move: connecting the spot to itself. The second player wins in the normal game (loses in misère) by joining the two spots, either inside or outside the closed curve. These two second moves are equivalent, as far as playing the game is concerned, because before they are made there is nothing to distinguish the inside from the outside of the closed curve. Think of the game as being played on the surface of a sphere. If we puncture the surface by a hole inside a closed curve, we can stretch the surface into a plane so that all points previously outside the curve become inside, and vice versa. This topological equivalence of inside and outside is important to bear in mind because it greatly simplifies the analysis of games that begin with more than two spots.

With two initial spots, Sprouts immediately takes on interest. The first player seems to have a choice of five opening moves (see Figure 36.2), but the second and third openings are equivalent for reasons of symmetry. The same holds true of the fourth and fifth, and in light of the inside–outside equivalence just explained, all four of these moves can be considered identical. Only two topologically distinct moves, therefore, require exploring. It is not difficult to diagram a complete

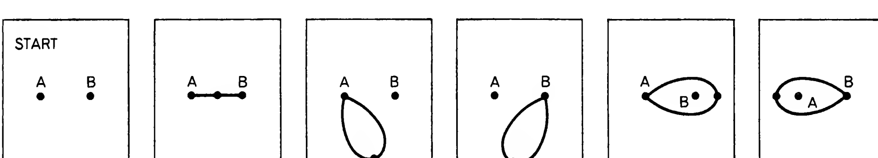


Figure 36.2. Initial spots *(A* and *B)* and first player's possible opening moves in two-spot game

tree chart of all possible moves, inspection of which shows that in both normal and misère forms of the two-spot game the second player can always win.

Conway found that the first player can always win the normal three-spot game and the second player can always win the misère version. Denis P. Mollison has shown that the first player has the win in normal four- and five-spot games. In response to a 10-shilling bet made with Conway that he could not complete his analysis within a month, Mollison produced a 49-page proof that the second player wins the normal form of the six-spot game. The second player wins the misère four-spot game.

Although no strategy for perfect play has been formulated, one can often see toward the end of a game how to draw closed curves that will divide the plane into regions in such a way as to lead to a win. It is the possibility of this kind of planning that makes Sprouts an intellectual challenge and enables a player to improve his skill at the game. But Sprouts is filled with unexpected growth patterns, and there seems to be no general strategy that one can adopt to make sure of winning.

Sprouts was invented on the afternoon of Tuesday, February 21, 1967, when Conway and Paterson had finished having tea in the mathematics department's common room and were doodling on paper in an effort to devise a new pencil-and-paper game. Conway had been working on a game invented by Paterson that originally involved the folding of attached stamps, and Paterson had put it into pencil-and-paper form. They were thinking of various ways of modifying the rules when Paterson remarked, "Why not put a new dot on the line?"

"As soon as this rule was adopted," Conway has written me, "all the other rules were discarded, the starting position was simplified to just n points, and Sprouts sprouted." The importance of adding the new spot was so great that all parties concerned agree that credit for the game should be on a basis of 3/5 to Paterson and 2/5 to Conway. "And there are complicated rules," Conway adds, "by which we intend to share any monies which might accrue from the game."

"The day after Sprouts sprouted," Conway continues, "it seemed that everyone was playing it. At coffee or tea times there were little groups of people peering over ridiculous to fantastic Sprout positions. Some people were already attacking Sprouts on toruses, Klein bottles, and the like, while at least one man was thinking of higher-dimensional versions. The secretarial staff was not immune; one found the remains of

Sprout games in the most unlikely places. Whenever I try to acquaint somebody new to the game nowadays, it seems he's already heard of it by some devious route. Even my three- and four-year-old daughters play it, though I can usually beat them."

The name "Sprouts" was given the game by Conway. An alternative name, "measles," was proposed by a graduate student because the game is catching and it breaks out in spots, but Sprouts was the name by which it quickly became known. Conway later invented a superficially similar game that he calls "Brussels Sprouts" to suggest that it is a joke. I shall describe this game but leave to the reader the fun of discovering why it is a joke before the explanation is given in the answer section.

Brussels Sprouts begins with *n* crosses instead of spots. A move consists of extending any arm of any cross into a curve that ends at the free arm of any other cross or the same cross; then a crossbar is drawn anywhere along the curve to create a new cross. Two arms of the new cross will, of course, be dead, since no arm may be used twice. As in Sprouts, no curve may cross itself or cross a previously drawn curve, nor may it go through a previously made cross. As in Sprouts, the winner of the normal game is the last person to play and the winner of the misère game is the first person who cannot play.

After playing Sprouts, Brussels Sprouts seems at first to be a more complicated and more sophisticated version. Since each move kills two crossarms and adds two live crossarms, presumably a game might never end. Nevertheless, all games do end and there is a concealed joke that the reader will discover if he succeeds in analyzing the game. To make the rules clear, a typical normal game of two-cross Brussels Sprouts is shown that ends with victory for the second player on the eighth move (see Figure 36.3).

A letter from Conway reports several important breakthroughs in Sproutology. They involve a concept he calls the "order of moribundity" of a terminal position, and the classification of "zero order" positions into five basic types: louse, beetle, cockroach, earwig, and scorpion (see Figure 36.4). The larger insects and arachnids can be infested with lice, sometimes in nested form, and Conway draws one pattern he says is "merely an inside-out earwig inside an inside-out louse." Certain patterns, he points out, are much lousier than others. And there is the FTOZOM (fundamental theorem of zero-order moribundity), which is quite deep. Sproutology is sprouting so rapidly that I shall have to postpone my next report on it for some time.

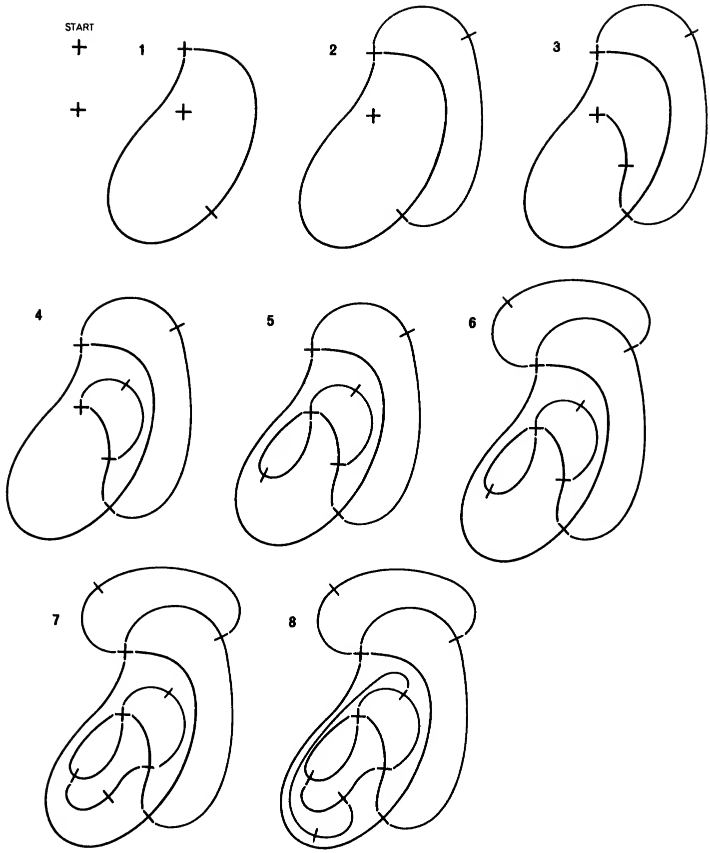
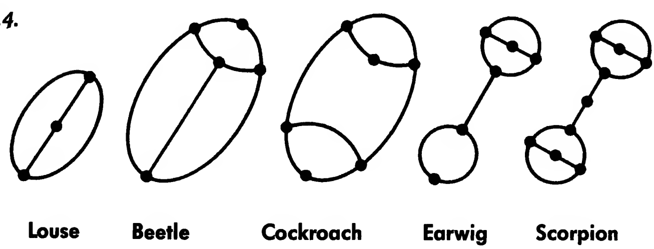


Figure 36.3. Typical game of two-cross Brussels Sprouts

Figure 36.4.



Addendum

Sprouts made an instant hit with *Scientific American* readers, many of whom suggested generalizations and variations of the game. Ralph J. Ryan III proposed replacing each spot with a tiny arrow, extending from one side of the line and allowing new lines to be drawn only to the arrow's point. Gilbert W. Kessler combined spots and crossbars in a game he called "succotash." George P. Richardson investigated Sprouts on the torus and other surfaces. Eric L. Gans considered a generalization of Brussels Sprouts (called "Belgian Sprouts") in which spots are replaced by "stars"—n crossbars crossing at the same point. Vladimir Ygnetovich suggested the rule that a player, on each turn, has a choice of adding one, two, or no spots to his line.

Several readers questioned the assertion that every game of normal Sprouts must last at least $2n$ moves. They sent what they believed to be counterexamples, but in each case failed to notice that every isolated spot permits two additional moves.

Since this chapter appeared in *Scientific American* in 1967, David Applegate, Guy Jacobson, and Danny Sleator wrote the first computer program for analyzing Sprouts (see bibliography). They found that in normal Sprouts the first player wins when n (the number of spots) is 3, 4, 5, 9, 10, and 11. Beyond $n = 11$ their program was unable to cope with sprouting complexity. They did not go beyond $n = 9$ for misère Sprouts. The program proved a first player win when $n = 1$, 5, and 6.

The authors conjecture that in normal Sprouts the first player wins if $n = 3$, 4, or 5 (modulo 6) and wins in misère Sprouts if $n = 0$ or 1 (modulo 5). They add that Sleator believes both conjectures, but Applegate disbelieves both. Details about their program can be obtained from Applegate or Jacobson at Bell Labs, in Murray Hill, NJ, or Sleator at Carnegie Mellon University.

Answers

Why is the game of Brussels Sprouts, which appears to be a more sophisticated version of Sprouts, considered a joke by its inventor, John Horton Conway? The answer is that it is impossible to play Brussels Sprouts either well or poorly because every game must end in exactly $5n - 2$ moves, where n is the number of initial crosses. If played in standard form (the last to play is the winner), the game is always won by the first player if it starts with an odd number of crosses, by the second player if it starts with an even number. (The reverse is true, of course, in misère play.) After introducing someone to Sprouts, which is a genuine contest, one can switch to the fake game of Brussels Sprouts, make bets, and always know in advance who will win every game. I leave to the reader the task of proving that each game must end in $5n - 2$ moves.

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