CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/S22/

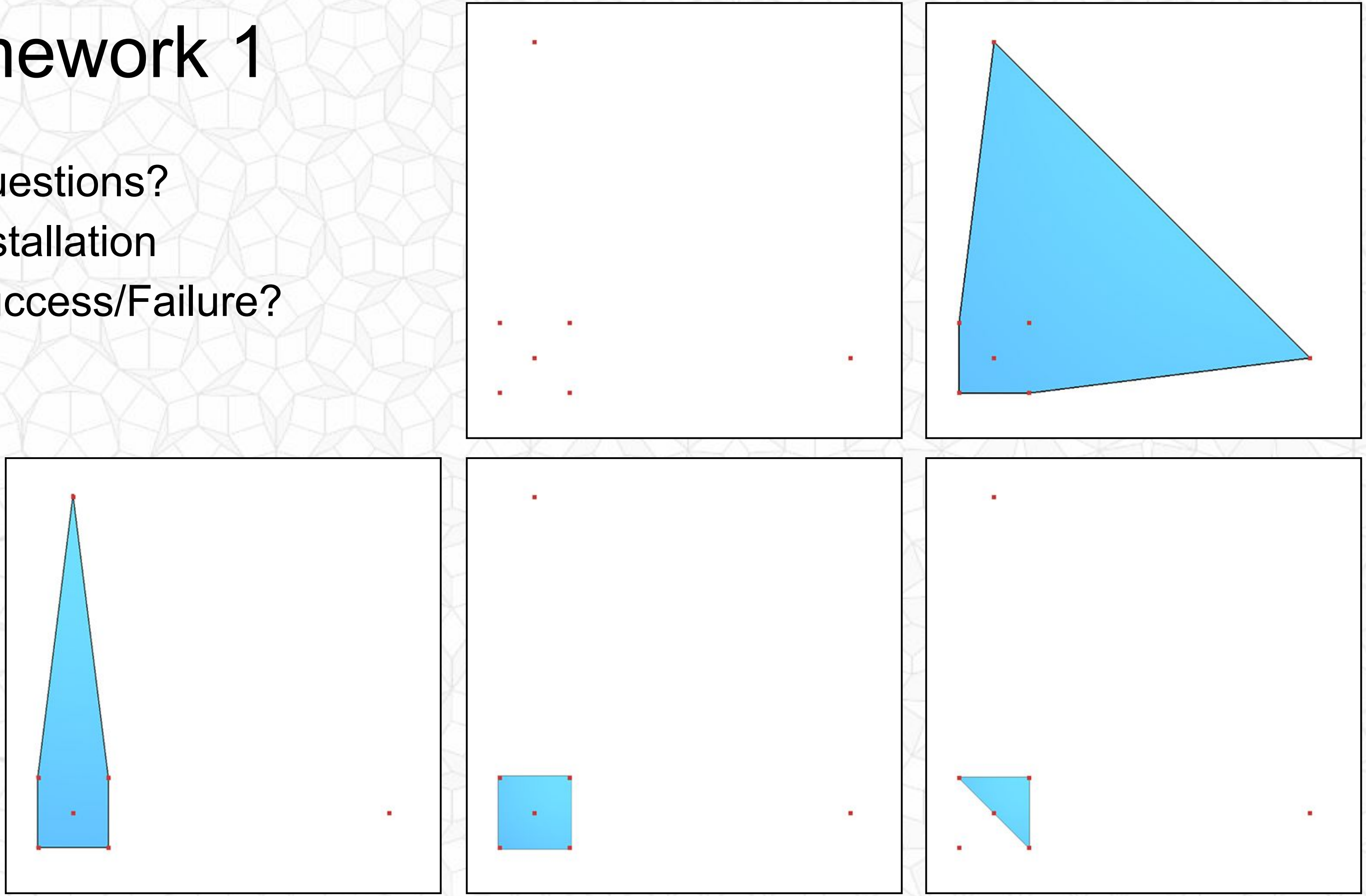
Lecture 2: Line Segment Intersections

Outline for Today

- Questions about Homework 1?
  Questions about CGAL/Qt installation?
- Motivating Applications for Line Segment Intersection Problem
- Line/Segment Intersection Math
- Naive vs. Output Sensitive Algorithms
- A Plane/Line Sweep Algorithm
- Specific Choices for Data Structures
- Analysis
- Corner Cases / Degeneracies
- Next Time

Homework 1

- Questions?
- Installation Success/Failure?



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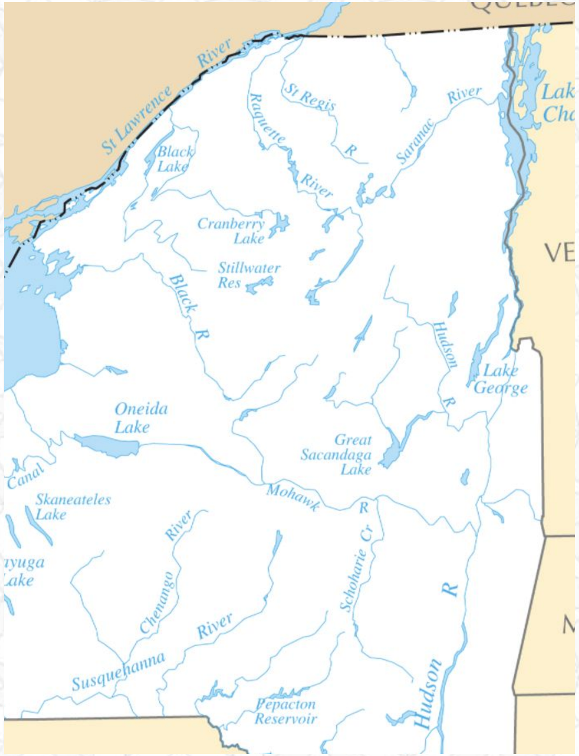
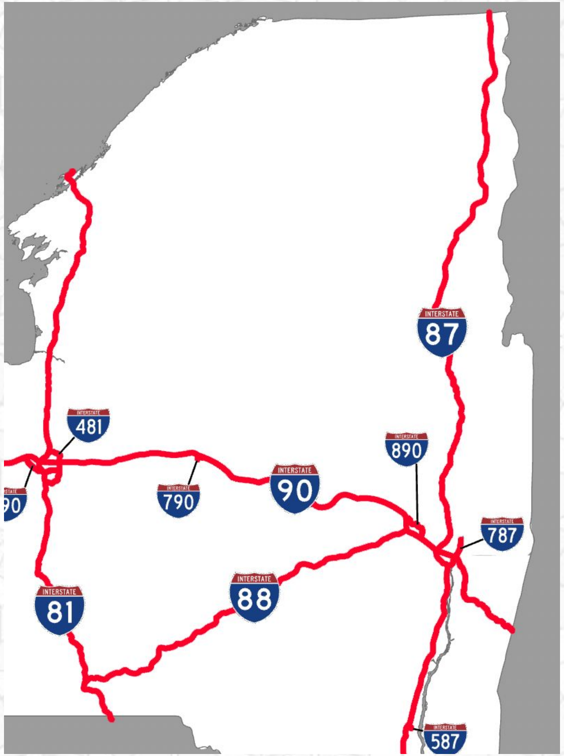
Motivating Application: Cartography Map Overlay

- 2 map layers storing the rivers & roads in NYS
- Each road/river stored as a *polyline - sequence of line segments*
- Find all intersections between a road segment and a river segment
- These are the bridges we need to build, inspect, repair, etc.



https://www.mapsof.net/new-york/new-york-rivers-and-lakes



https://upload.wikimedia.org/wikipedia/commons/1/17/NYInterstates.svg

Application: Machine Learning

- Is my data classifiable? Is my data separable?

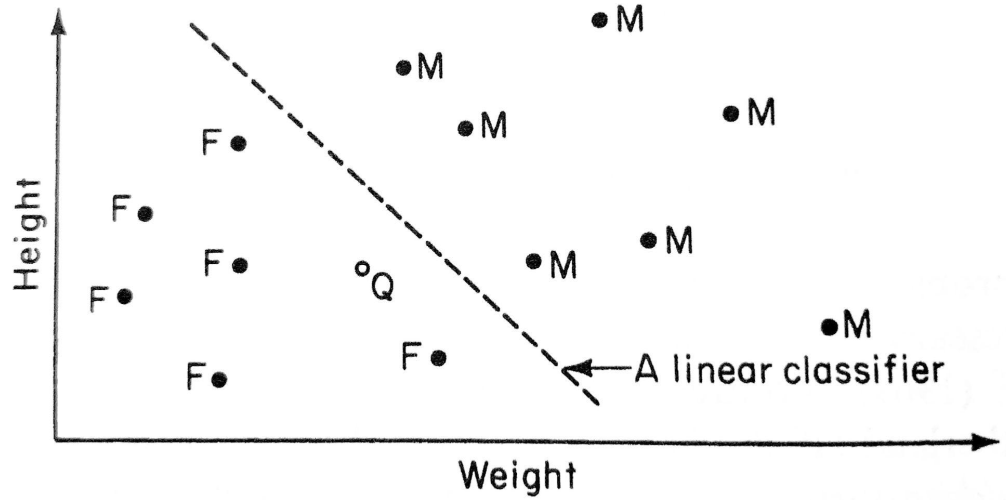
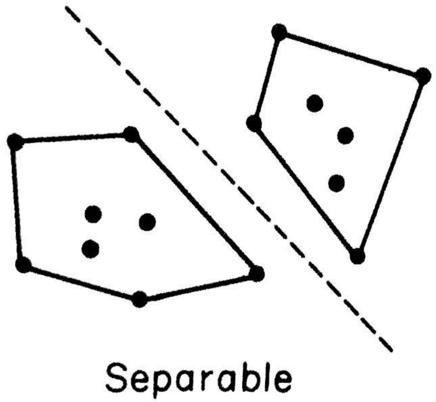
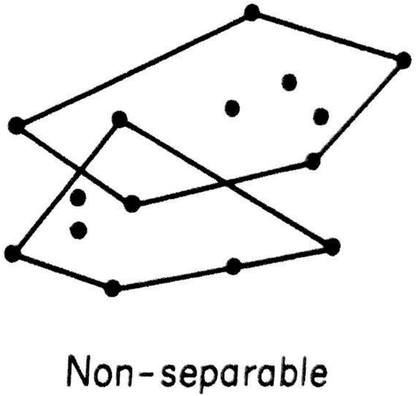


Figure 7.2 A two-variable classification problem.





Computational Geometry: An Introduction
Preparata & Shamos, Springer 1985

Figure 7.3 Two sets are separable if and only if their convex hulls are disjoint.

Self-Intersection of Non Convex Polygons

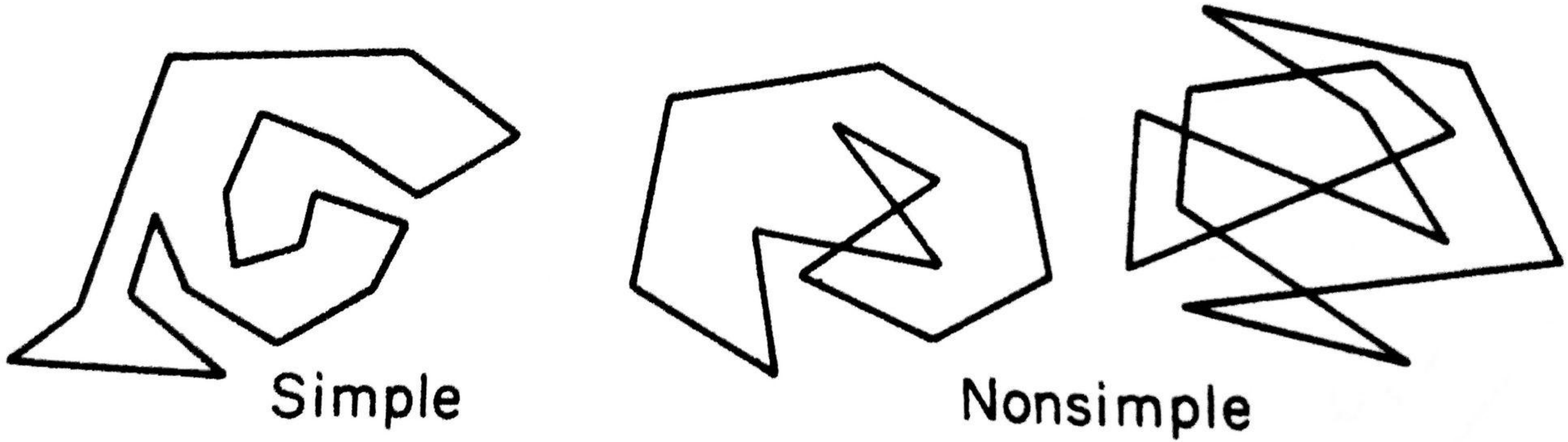


Figure 7.13 Simple and nonsimple polygons.

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Hidden Line (Hidden Surface) Removal

- A classic problem from the early days of Computer Graphics
- Identify and remove portions of the object that are not visible from a particular viewing angle

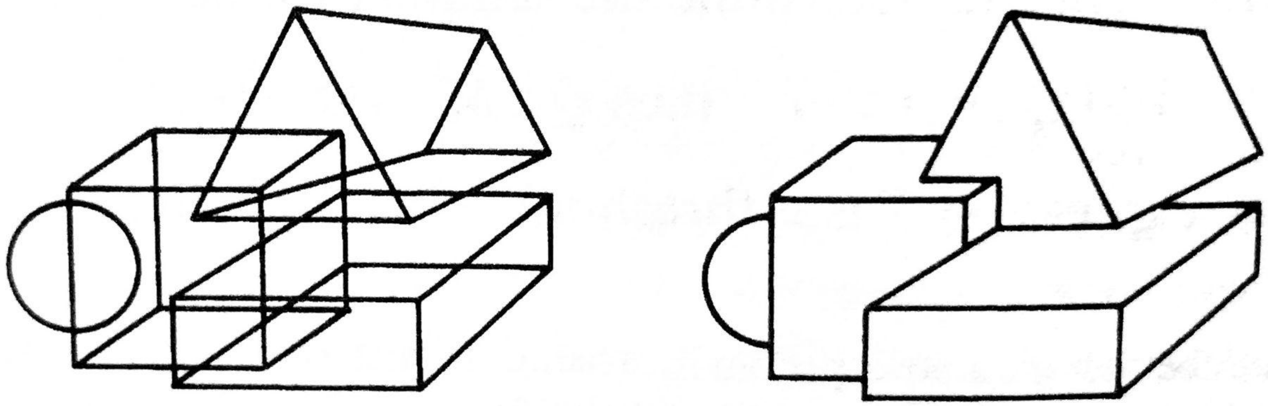

Figure 7.1 Elimination of hidden lines.

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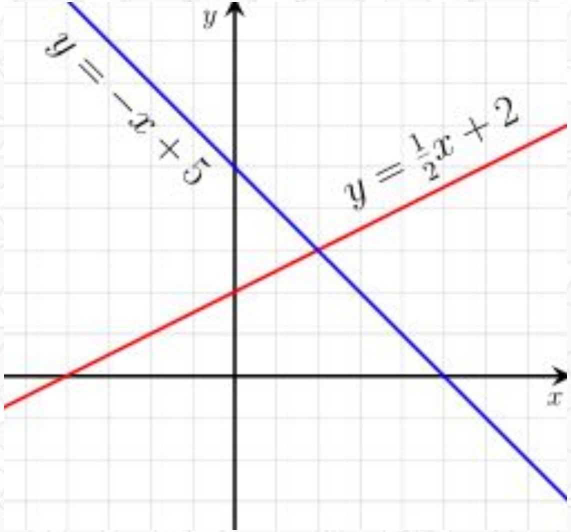
Intersection of 2 Lines in a Plane



https://en.wikipedia.org/wiki/Linear_equation

Intersection of 2 Lines in a Plane

- Using line slope equations:

$$y = ax + c \text{ and } y = bx + d$$

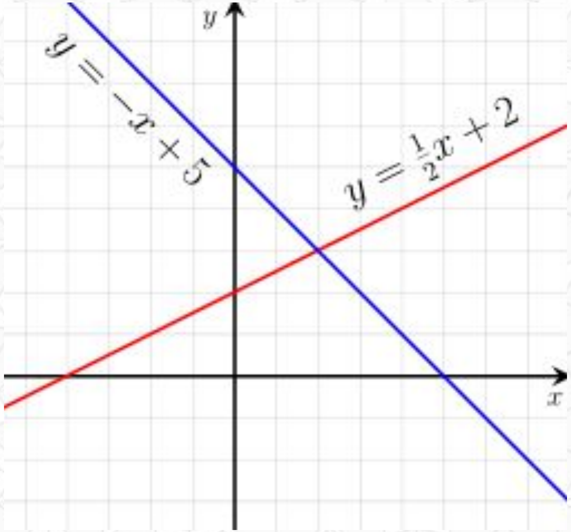
- Set them equal to each other:

$$ax + c = bx + d.$$

- Solve for x and y:

$$x = \frac{d-c}{a-b} \qquad y = a\frac{d-c}{a-b} + c.$$

- Concerns?



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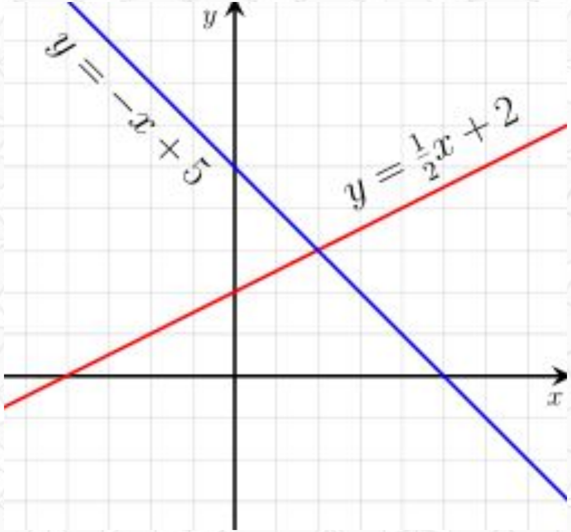
Intersection of 2 Lines in a Plane



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- Solve for x and y:

$$x = \frac{d - c}{a - b} \qquad y = a\frac{d - c}{a - b} + c.$$

- Concerns?
  - Does it handle vertical lines?
  - How do we detect parallel (non-intersecting) lines?
  - How do we determine if *line segments* intersect (between endpoints)?

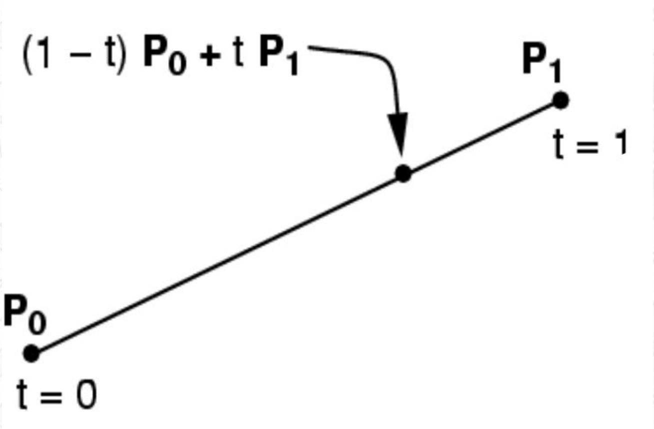
Intersection of 2 Line Segments in a Plane

- Let's use the *Parametric Equation* for a line segment:

$$L_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

- For every value of *t* from in the interval [0,1],
  Plug *t* into this equation, and you'll get a point on the line segment

- Linearly interpolating between the endpoints
- A weighted average of the endpoints



$P_1 = (x_2, y_2)$

$P_0 = (x_1, y_1)$

Intersection of 2 Line Segments in a Plane

- Two parametric equations:

$$L_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + u \begin{bmatrix} x_4 - x_3 \\ y_4 - y_3 \end{bmatrix}$$

- Solve for *t* and *u:*

$$t = \frac{\begin{vmatrix} x_1 - x_3 & x_3 - x_4 \\ y_1 - y_3 & y_3 - y_4 \end{vmatrix}}{\begin{vmatrix} x_1 - x_2 & x_3 - x_4 \\ y_1 - y_2 & y_3 - y_4 \end{vmatrix}} = \frac{(x_1 - x_3)(y_3 - y_4) - (y_1 - y_3)(x_3 - x_4)}{(x_1 - x_2)(y_3 - y_4) - (y_1 - y_2)(x_3 - x_4)}$$

$$u = \frac{\begin{vmatrix} x_1 - x_3 & x_1 - x_2 \\ y_1 - y_3 & y_1 - y_2 \end{vmatrix}}{\begin{vmatrix} x_1 - x_2 & x_3 - x_4 \\ y_1 - y_2 & y_3 - y_4 \end{vmatrix}} = \frac{(x_1 - x_3)(y_1 - y_2) - (y_1 - y_3)(x_1 - x_2)}{(x_1 - x_2)(y_3 - y_4) - (y_1 - y_2)(x_3 - x_4)}$$

- *Concerns?*
  - *Vertical lines?*
  - *Parallel lines?*
  - *Line vs. segment intersection?*

https://en.wikipedia.org/wiki/Line%E2%80%93line_intersection

Intersection of 2 Line Segments in a Plane

- Two parametric equations:

$$L_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + u \begin{bmatrix} x_4 - x_3 \\ y_4 - y_3 \end{bmatrix}$$

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- *Concerns?*
  - *Vertical lines?*
  - *Parallel lines?*
  - *Line vs. segment intersection?*

$$(P_x, P_y) = (x_1 + t(x_2 - x_1),\ y_1 + t(y_2 - y_1))$$

$0.0 \le t \le 1.0$ and $0.0 \le u \le 1.0$.

https://en.wikipedia.org/wiki/Line%E2%80%93line_intersection

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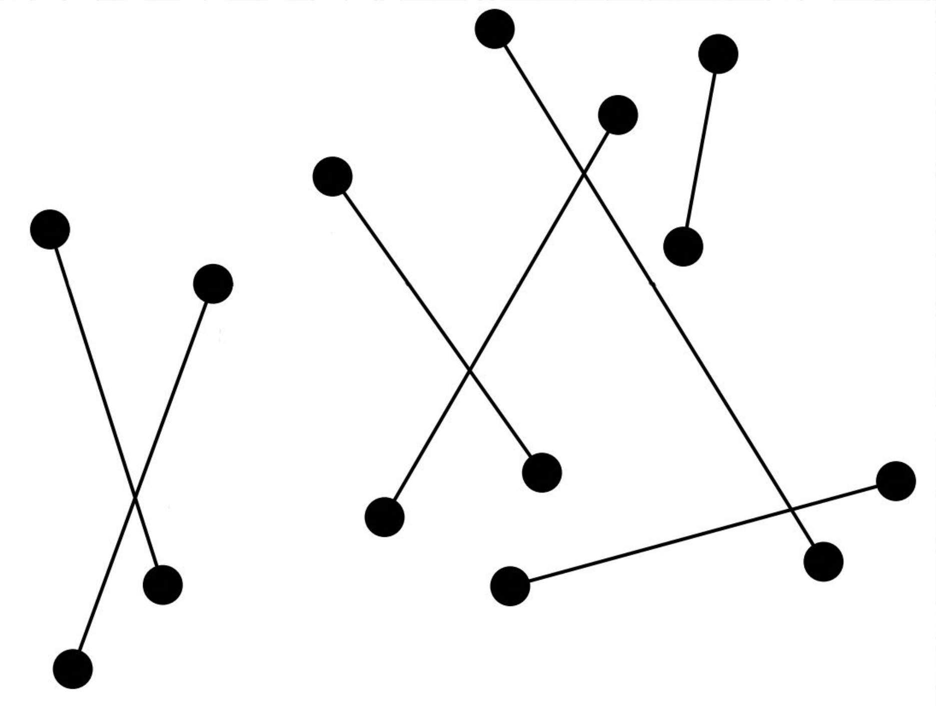
Line Segment Intersection - Brute Force Solution

- Ignore labeling of road vs. river (just compare everything)

- Analysis?



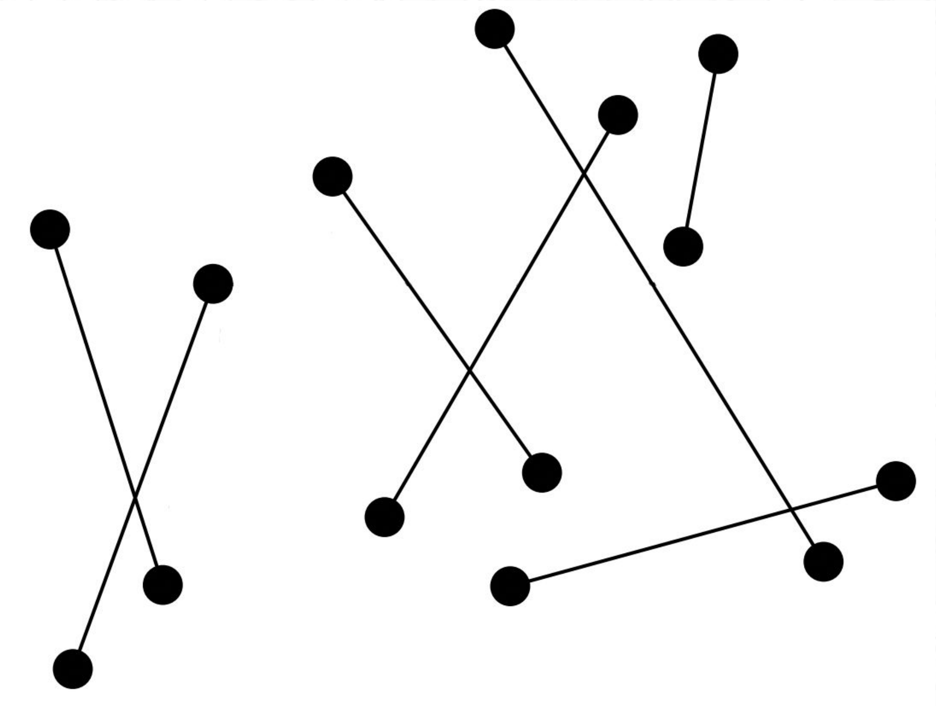
Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 2

Line Segment Intersection - Brute Force Solution

- Ignore labeling of road vs. river (just compare everything)
- Nested for loop: Intersect each segment with every other segment

- Analysis?
  $O(n^2)$

- Can we do better?



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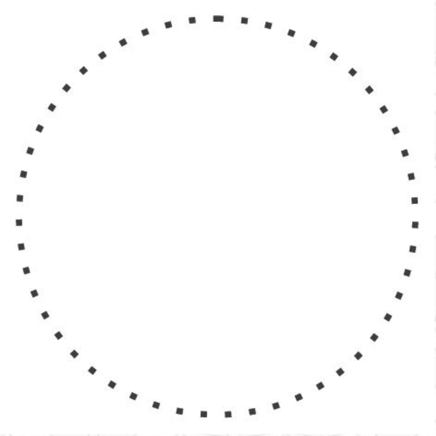
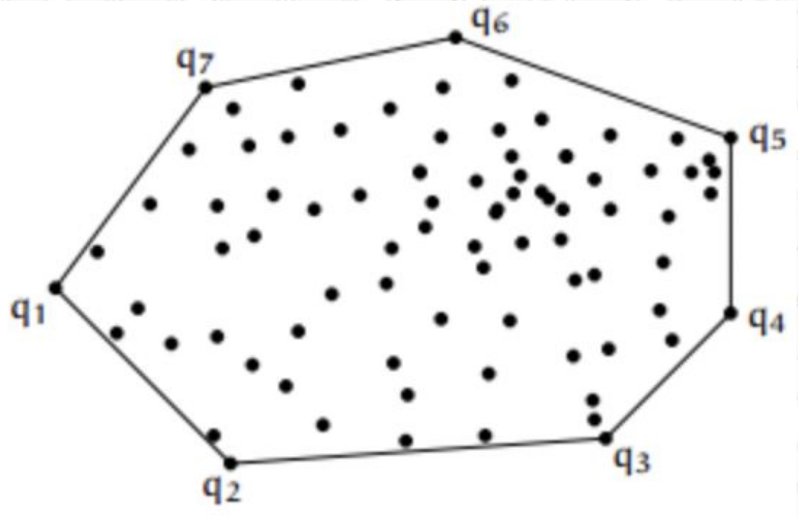
Definition: *Output Sensitive*

- When algorithm running time depends on the size of the output for that specific input

- The Convex Hull Algorithms from last
  n = # of input points
  h = # of points on final convex hull
  - Naive: $O(n^3)$
  - Compute upper hull: $O(n \log n)$
  - Gift Wrapping:
    $O(n * h)$ ← *output sensitive!*

  - *… there are also O(n log h) convex hull algorithms!*

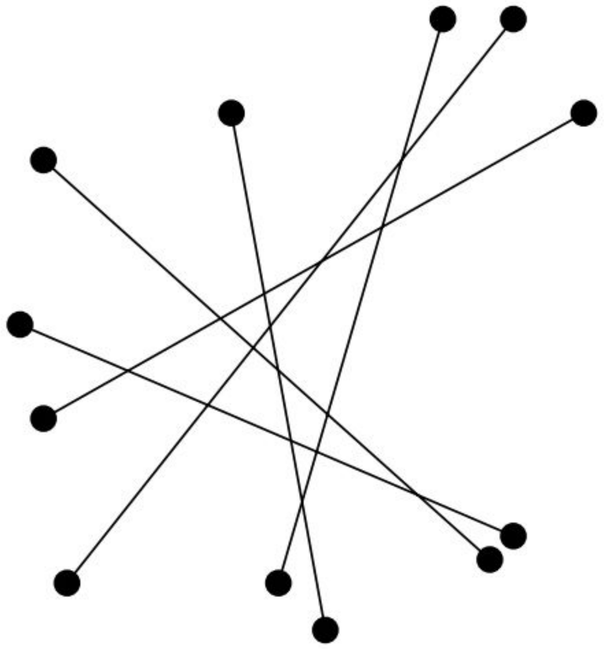




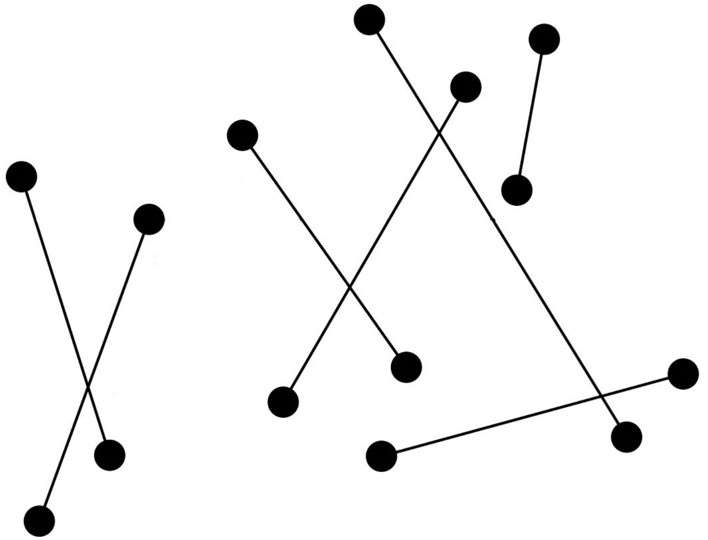
https://medium.com/@harshitsikchi/convex-hulls-explained-baab662c4e94

Output Sensitive Line Segment Intersection

- For specific worst case inputs, $O(n^2)$ is the best we can do…



But most problems aren't worst case!



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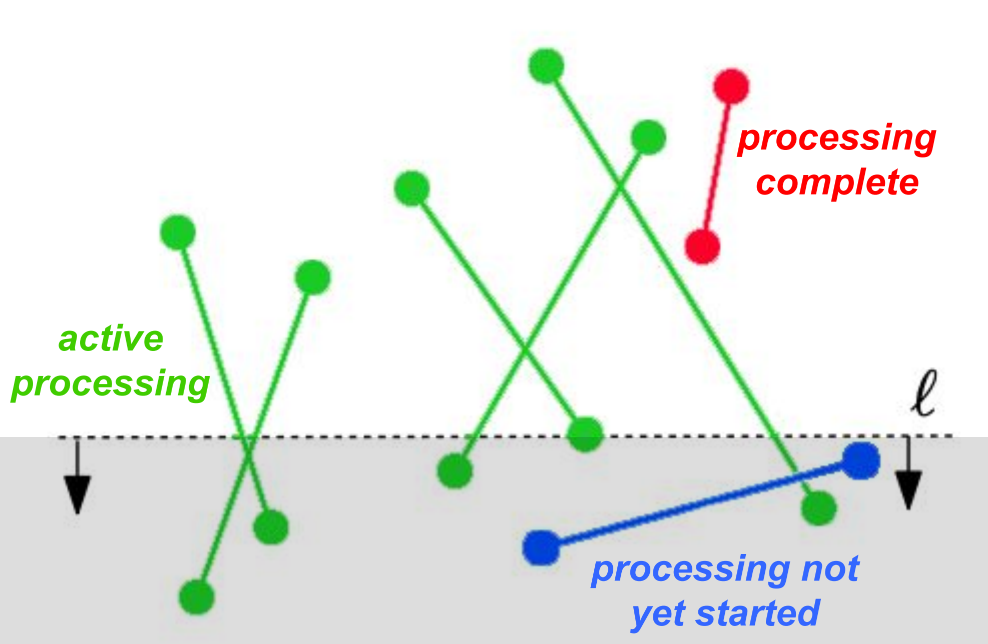
A Classic Computational Geometry Tool: The Line-Sweep *(or Plane-Sweep)* Algorithm

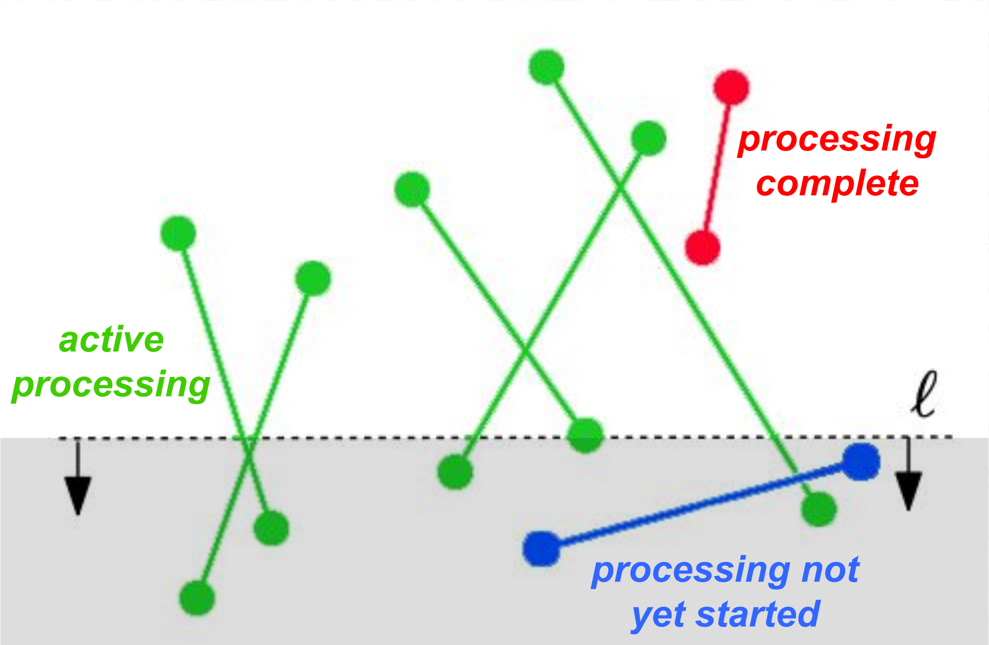
- Incrementally focus on a subset of the data at a time
- Sweep line will move from top to bottom across our dataset
- Sweep line/plane is used to define the current status
- **Active segments** = those that touch/intersect the sweep line's current position



A Classic Computational Geometry Tool: The Line-Sweep *(or Plane-Sweep)* Algorithm

- We will only look for intersections between **green segments**
- We will never check for intersections between a **red line** and a **blue line**

- Why is this ok?



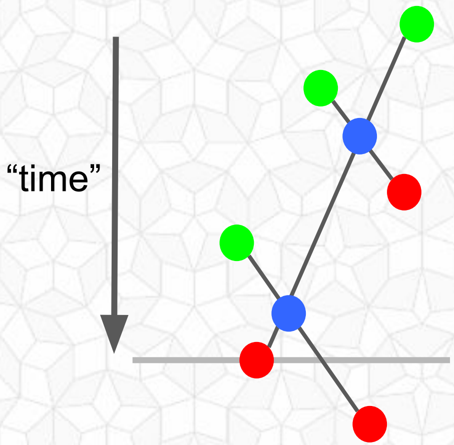
As line sweeps down, handle Events in **Event Queue**

- Line segment **added** to active set
- Line segment **removed** from active set

We know "when" (vertical position) these events will happen and can pre-schedule them. Simply sort the y coordinates of all of the input line segments.

- Line segment **intersection**

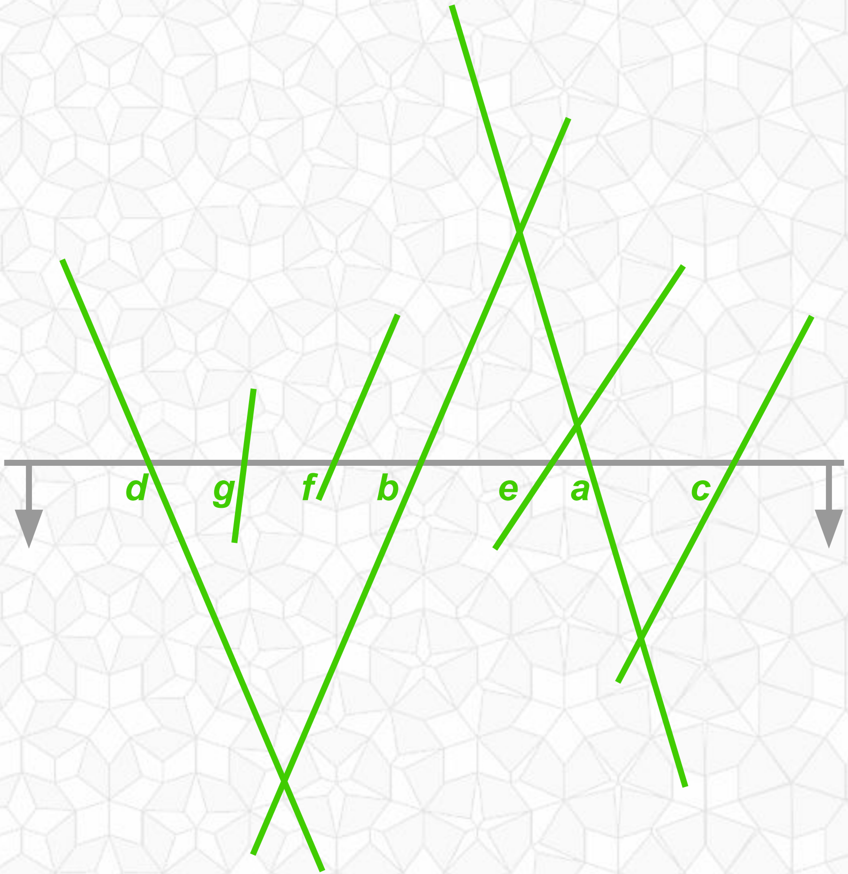
We don't know when these will happen! This is what we're trying to solve for!



Add seg_A
Add seg_B
seg_A & seg_B intersect
Remove seg_B
Add seg_C
seg_A & seg_C intersect
Remove seg_A
Remove seg_C

Intersections between Active Segments

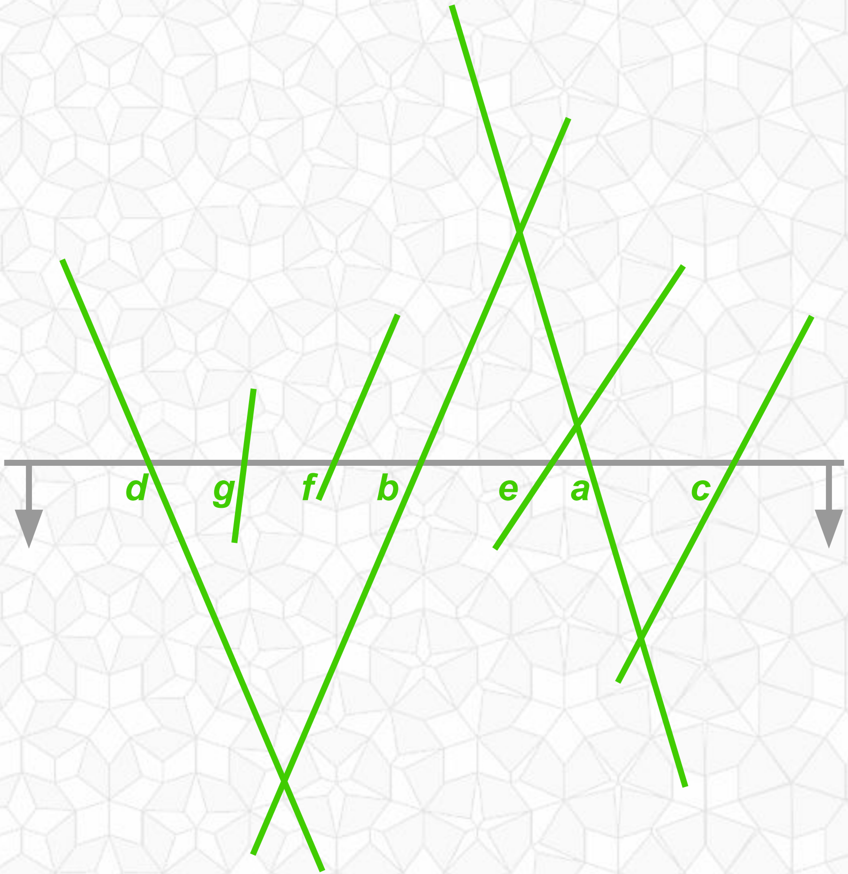
- Must we intersect every active segment to every other active segment?



Intersections between Active Segments

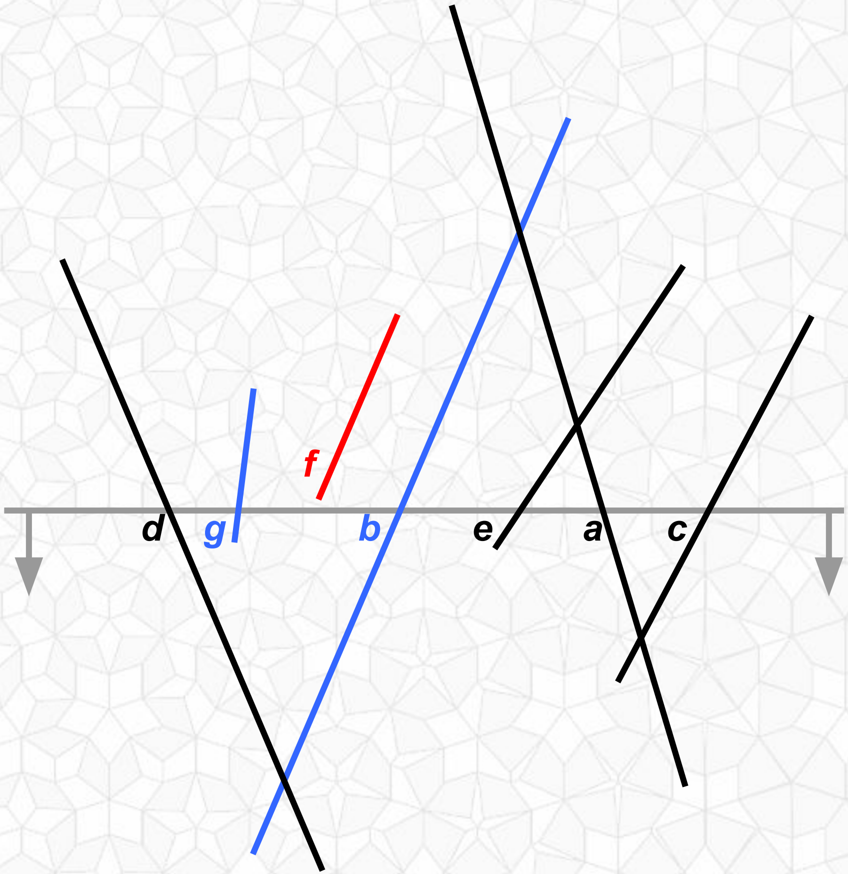
- Must we intersect every active segment to every other active segment?

- No… We can do better!
  - Maintain the active segments ordered by the x position of intersection with the current sweep line
  - Only compare segments that are adjacent in this ordering



Intersections between Active Segments

- When a segment (f) is removed



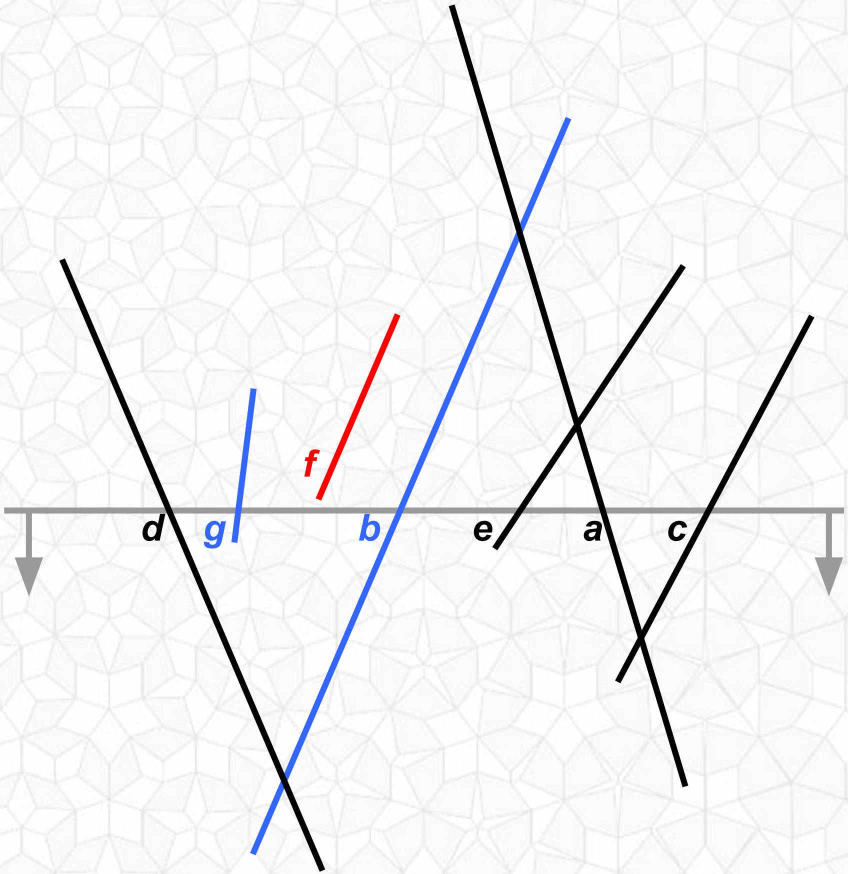
Intersections between Active Segments

- When a segment (f) is removed

  d g f b e a c

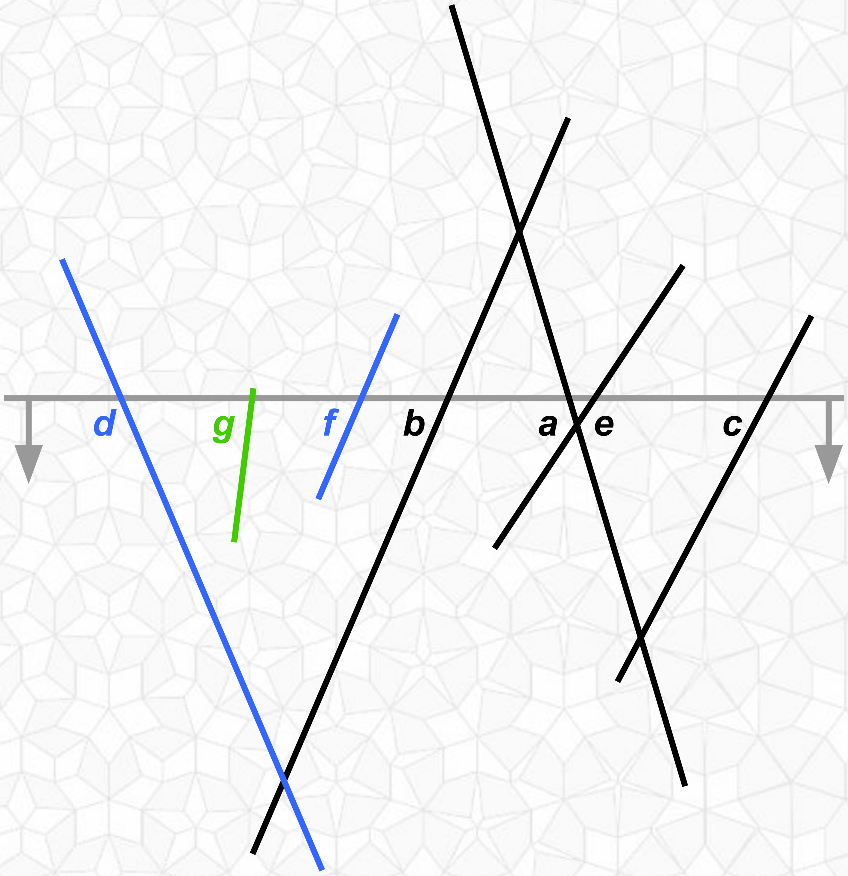
  d g b e a c

  The newly adjacent
  segments (g & b)
  are checked for intersection



Intersections between Active Segments

- When a segment (g) is added



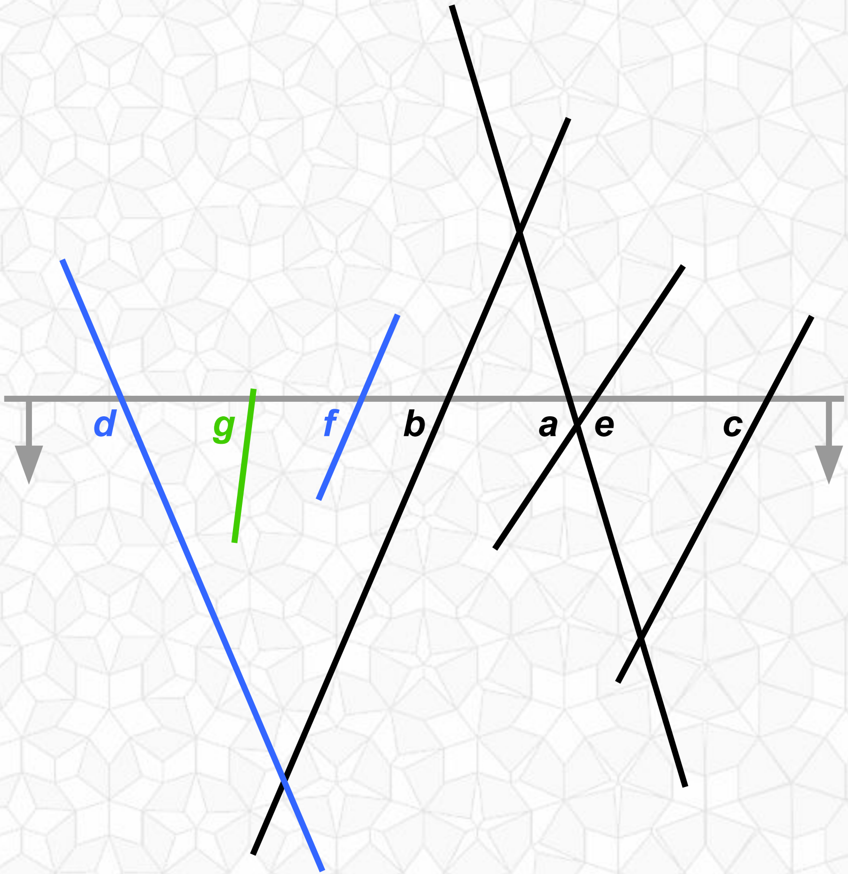
Intersections between Active Segments

- When a segment (g) is added

  d f b a e c

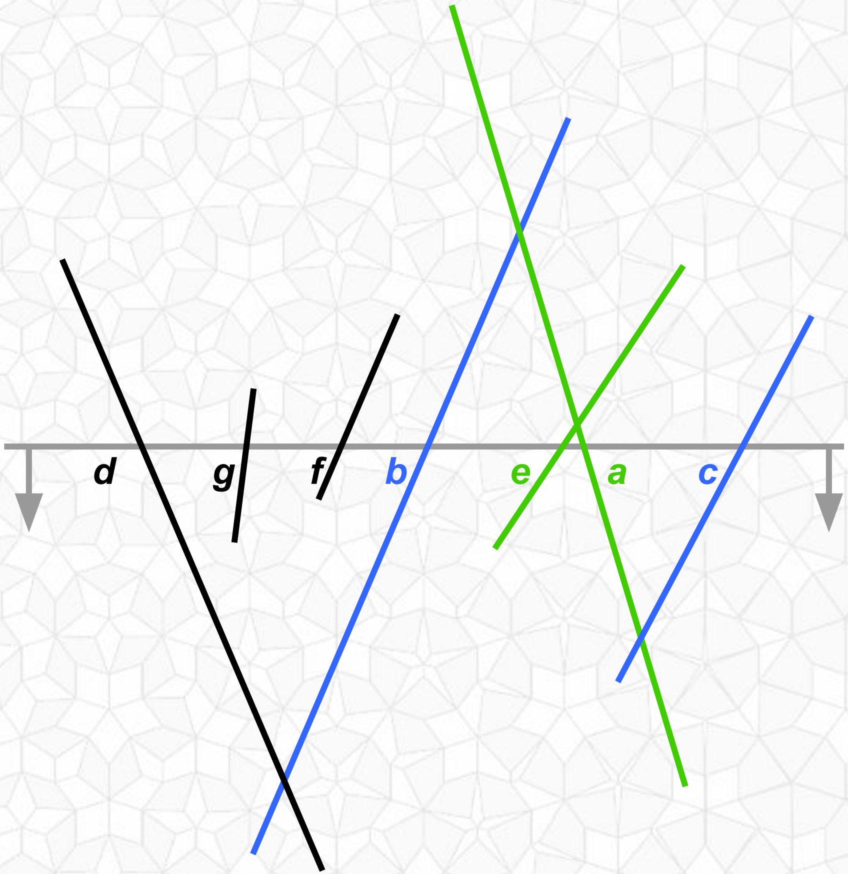
  d g f b a e c

  The newly adjacent segments
  (d & g, g & f)
  are checked for intersection



Intersections between Active Segments

- When the sweep line reaches an intersection (a&e)



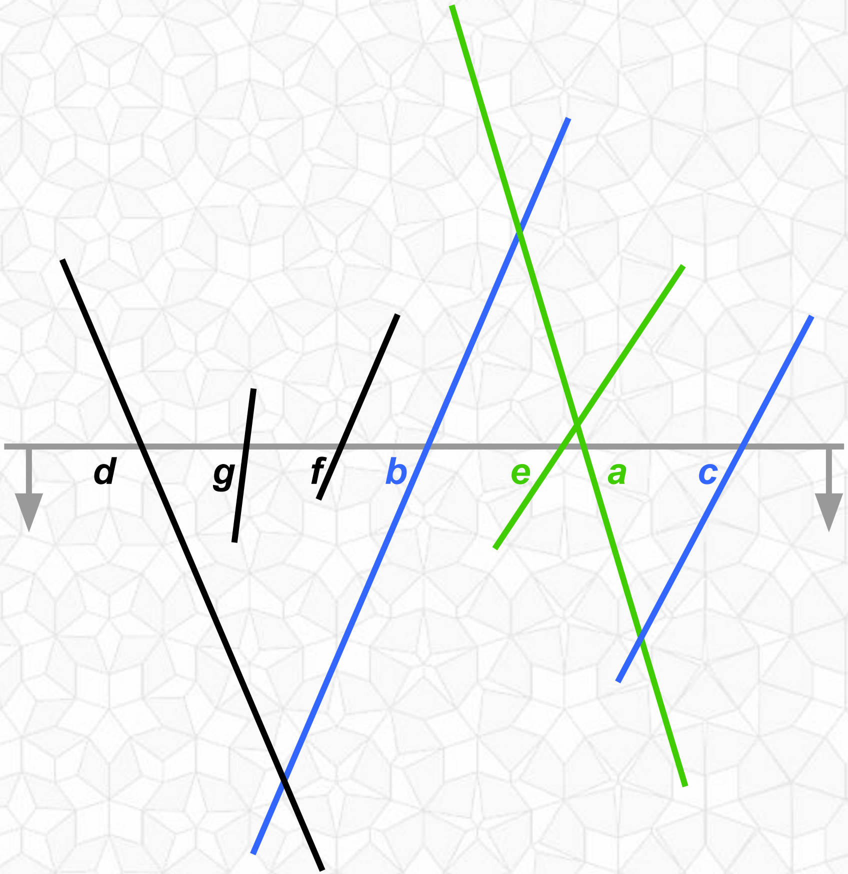
Intersections between Active Segments

- When the sweep line reaches an intersection (a&e)

  d g f b a e c
  d g f b e a c

  Swap the positions in the horizontal ordering

  And check for intersections with the new neighbors (b & e, a & c)

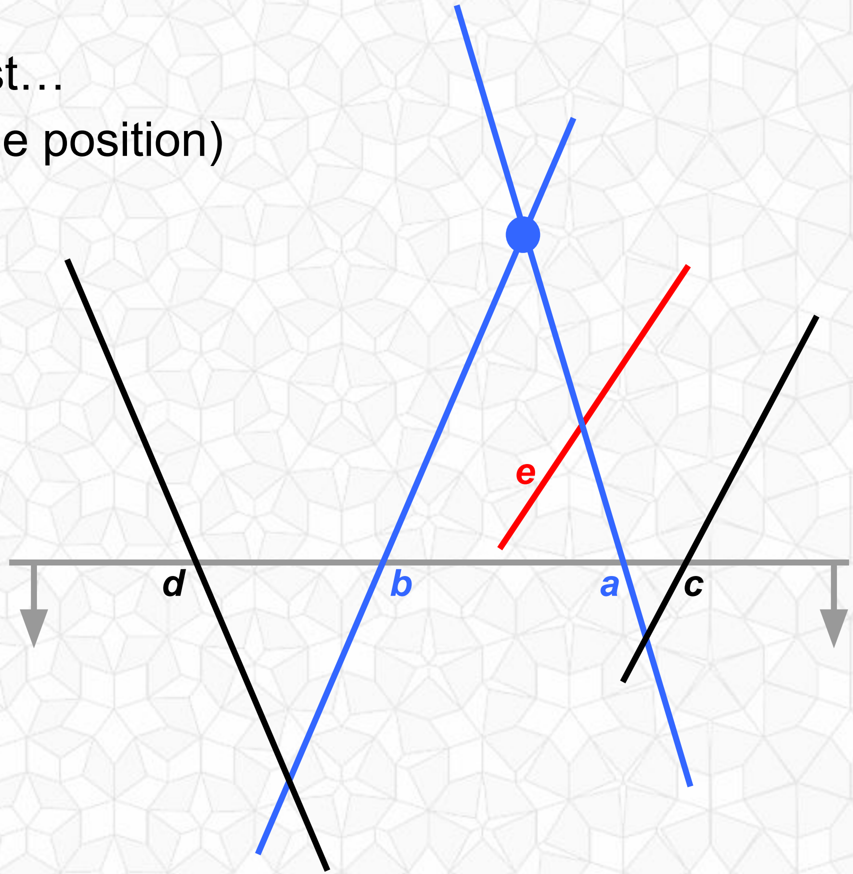


Intersections between Active Segments

- Sometimes the intersection is in the past…
  (y coordinate is above current sweep line position)

- We've already processed this intersection

  Do nothing

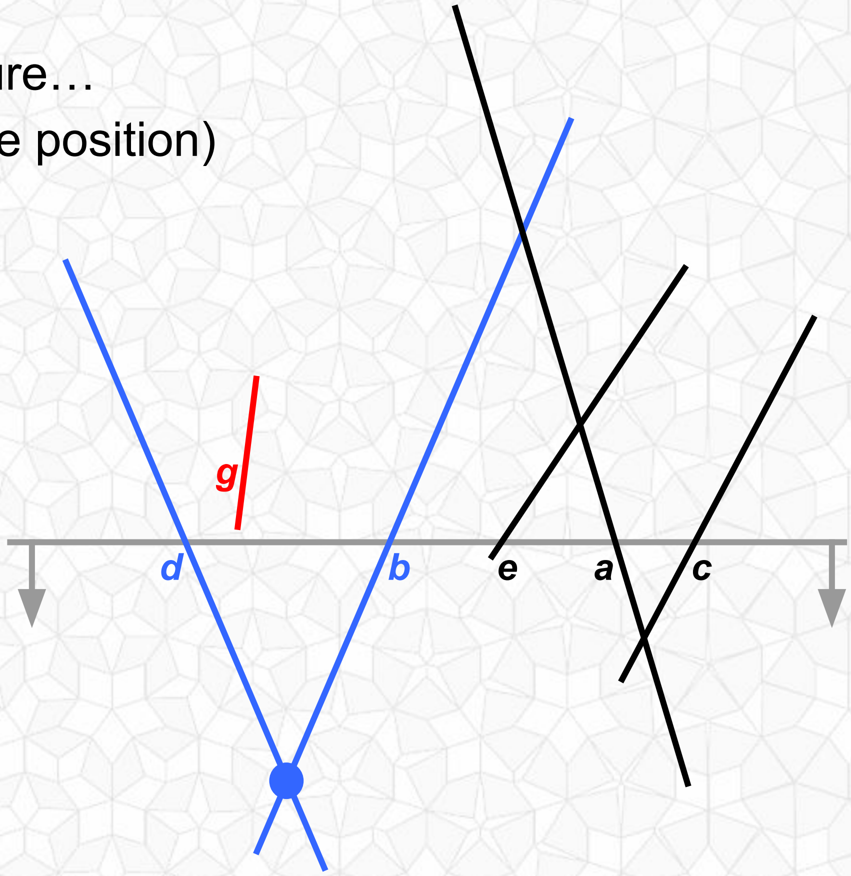


Intersections between Active Segments

- Sometimes the intersection is in the future…
  (y coordinate is below current sweep line position)

- We may or may not have already detected this intersection…

  It may or may not already be in the **Event Queue** (just make sure we don't add a duplicate!)



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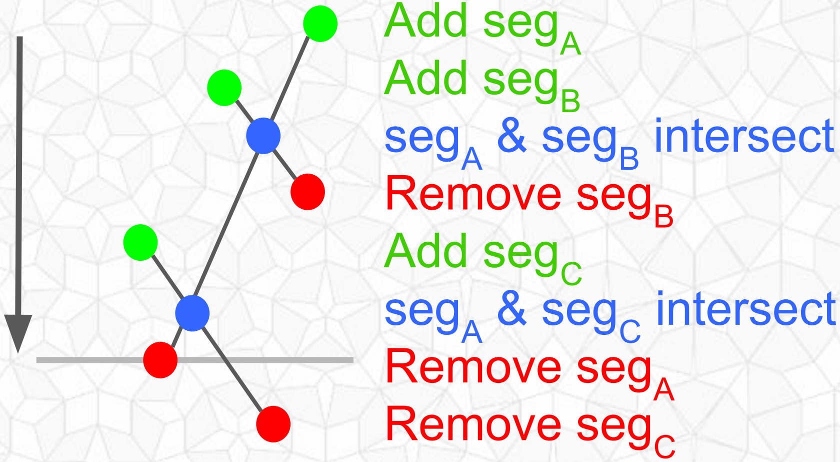
Data / Data Structures maintained during Sweep

- What data structure do we use for the vertically-ordered **Event Queue?**
  - Is it an array?
  - Is it a linked list?
  - Is it a priority queue?
  - Is it a binary search tree?
  - Is it a hash table?



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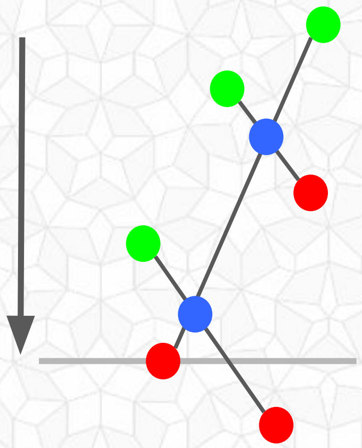
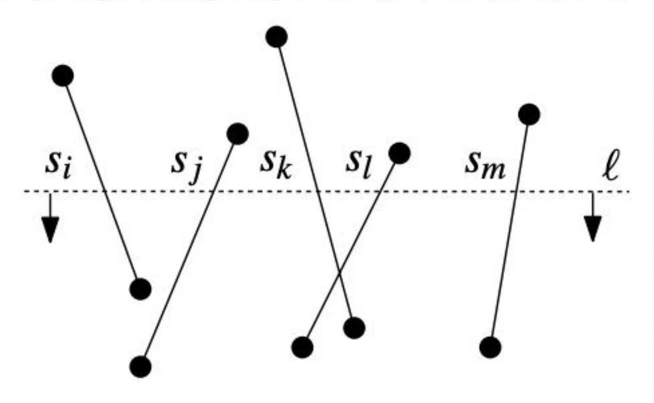


Add seg_A
Add seg_B
seg_A & seg_B intersect
Remove seg_B
Add seg_C
seg_A & seg_C intersect
Remove seg_A
Remove seg_C

- We start with a vertically-sorted collection of all of the end points
- We remove events one at a time in order
- We insert in intersection points as they are detected, one at a time, not necessarily in a particular order
- We need to check for existence before adding a duplicate

Data / Data Structures maintained during Sweep

- What data structure do we use for the horizontally-ordered **Active Segment Status Structure?**
  - Is it an array?
  - Is it a linked list?
  - Is it a priority queue?
  - Is it a binary search tree?
  - Is it a hash table?



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  - Is it an array?
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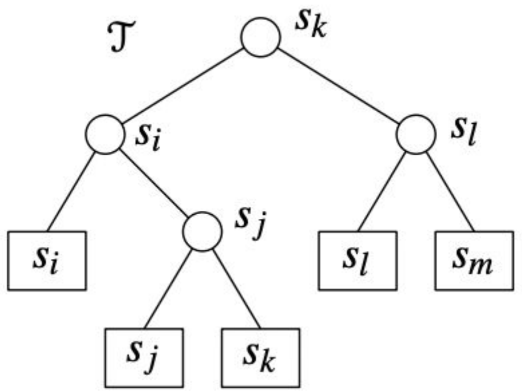
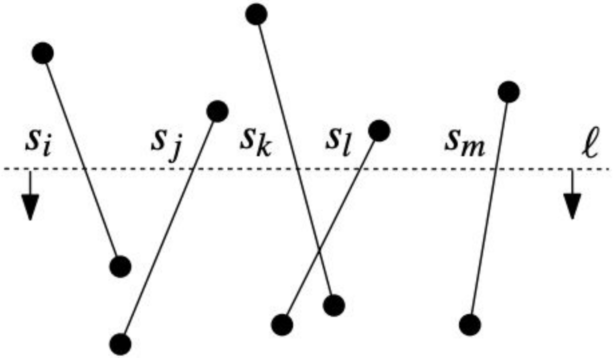
  - Is it a binary search tree?
  - Is it a hash table?

- Initially empty
- Segments are added, removed, and swapped
- Adjacent neighbors are queried often





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Analysis - Running Time

- For *n* = # *of input* segments,
  k = # of output intersections
  s = max # of items items on sweep line / in status structure at one time

- Step 1: Create add segment and remove segment events,
  sort and initialize the **Event Queue**

- Step 2: For each entry in the **Event Queue**
  - Update the **Active Segment Status Structure**
  - Compute intersections between newly adjacent segments
  - Add new intersections to the **Event Queue**

- Overall:

Analysis - Running Time

- For n = # *of input* segments,
  k = # of output intersections → $k \leq n(n-1)/2$
  s = max # of items items on sweep line / in status structure at one time → $s \leq n$
- Step 1: Create add segment and remove segment events,
  sort and initialize the **Event Queue** → $O(n \log n)$
- Step 2: For each entry in the **Event Queue** → $O(n + k)$
  - Update the **Active Segment Status Structure** → $O(\log s)$
  - Compute intersections between newly adjacent segments → $O(1)$
  - Add new intersections to the **Event Queue**
    → $O(\log (n+k)) \rightarrow O(\log (n+n^2)) \rightarrow O(2 * \log n) \rightarrow O(\log n)$
- Overall: $O(n * \log n + (n+k)*(\log n))$ → $O((k+n) * \log n)$
- *Algorithm & result has been improved… lower bound is: $\Omega(n \log n + k)$*

Analysis - Storage / Memory

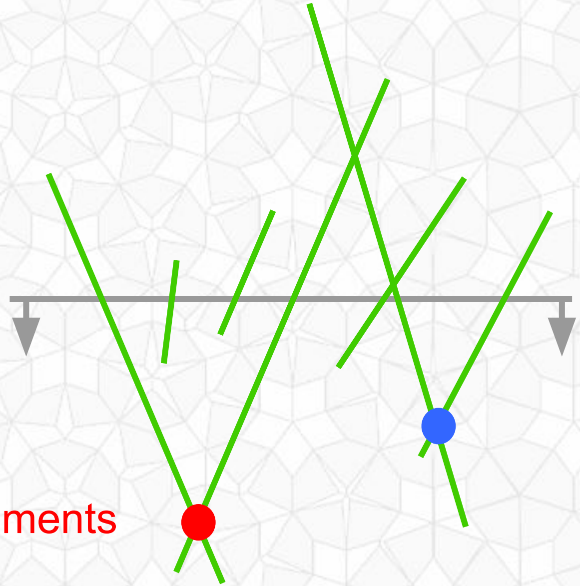
- For n = # *of input* segments,
  k = # of output intersections → $k \leq n(n-1)/2$
  s = max # of items items on sweep line / in status structure at one time → $s \leq n$
- Step 1: Create add segment and remove segment events,
  sort and initialize the **Event Queue**
- Step 2: For each entry in the **Event Queue**
  Update the **Active Segment Status Structure**
- Overall:

Analysis - Storage / Memory

- For n = *# of input* segments,
  k = # of output intersections → $k \leq n(n-1)/2$
  s = max # of items items on sweep line / in status structure at one time → $s \leq n$
- Step 1: Create add segment and remove segment events,
  sort and initialize the **Event Queue**
  → “in place” sorting algorithm, *O(1) add’l memory*
- Step 2: For each entry in the **Event Queue**
  → maximum size $O(n + k)$
  - Update the **Active Segment Status Structure**
    → maximum size, $O(\log s)$
- Overall: → $O(n + k)$ *extra memory!*

Analysis - Storage / Memory

- For n = *# of input* segments,
  k = # of output intersections → $k \leq n(n-1)/2$
  s = max # of items items on sweep line / in status structure at one time → $s \leq n$
- Step 1: Create add segment and remove segment events, sort and initialize the **Event Queue**
  → "in place" sorting algorithm, *O(1) add'l memory*
- Step 2: For each entry in the **Event Queue**
  → maximum size $O(n + k)$
  - Update the **Active Segment Status Structure**
    → maximum size, $O(\log s)$
- Overall: → $O(n + k)$ *extra memory!*
- Better: Don't store "future" intersection of non-adjacent segments
  → $O(n)$ *extra memory!*



Outline for Today

- Questions about Homework 1?
  Questions about CGAL/Qt installation?
- Motivating Applications for Line Segment Intersection Problem
- Line/Segment Intersection Math
- Naive vs. Output Sensitive Algorithms
- A Plane/Line Sweep Algorithm
- Specific Choices for Data Structures
- Analysis
- Corner Cases / Degeneracies
- Next Time

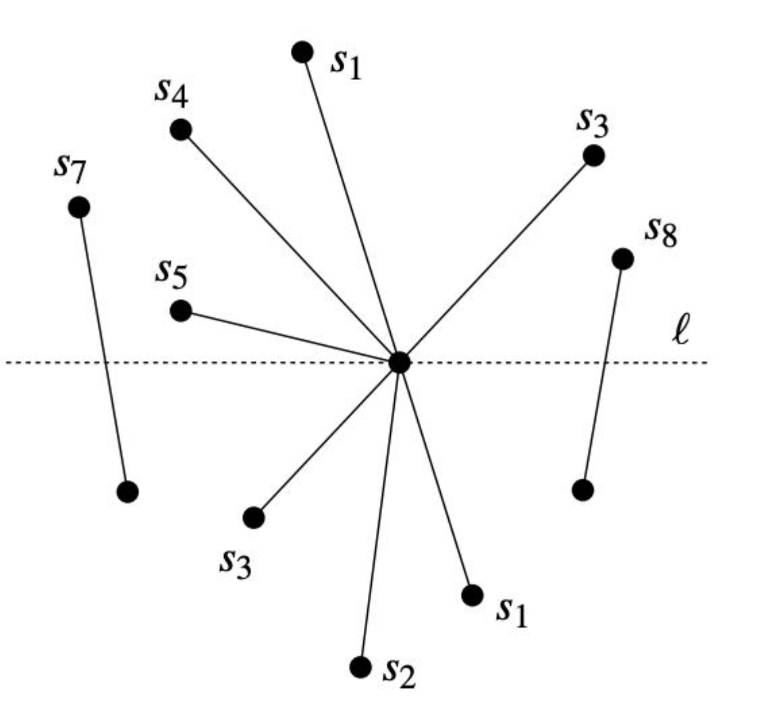
Corner cases / Degeneracies

- We assumed these situations don't occur:

Corner cases / Degeneracies

- We assumed these situations don't occur:
  - 3 or more segments intersect at a point
  - Intersection may be at the segment endpoint (rather than in the middle)
  - Segments may be perfectly horizontal (parallel to sweep line)
  - 2 or more simultaneous events (add segment, remove segment, intersection)
  - And general floating point rounding headaches…
- However, these situations can be handled properly in the algorithm without too much more fuss… see the textbook for details

Note: segments touching at endpoints is not a rare occurrence for this application. Our river & road polylines are connected at the endpoints!



Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 2

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Next Time

- Cartography (map making) is not just river and road polylines, it is also the areas or regions
- How do we describe and store a region?
- How do we overlay, intersect, & union map areas or regions?

https://www.natureconservancy.ca/assets/images/graphics/nat/maps/Forest-regions-map-NCC-1000px-custom.jpg

Next Time

- Complexity of the intersection of non-convex polygons…

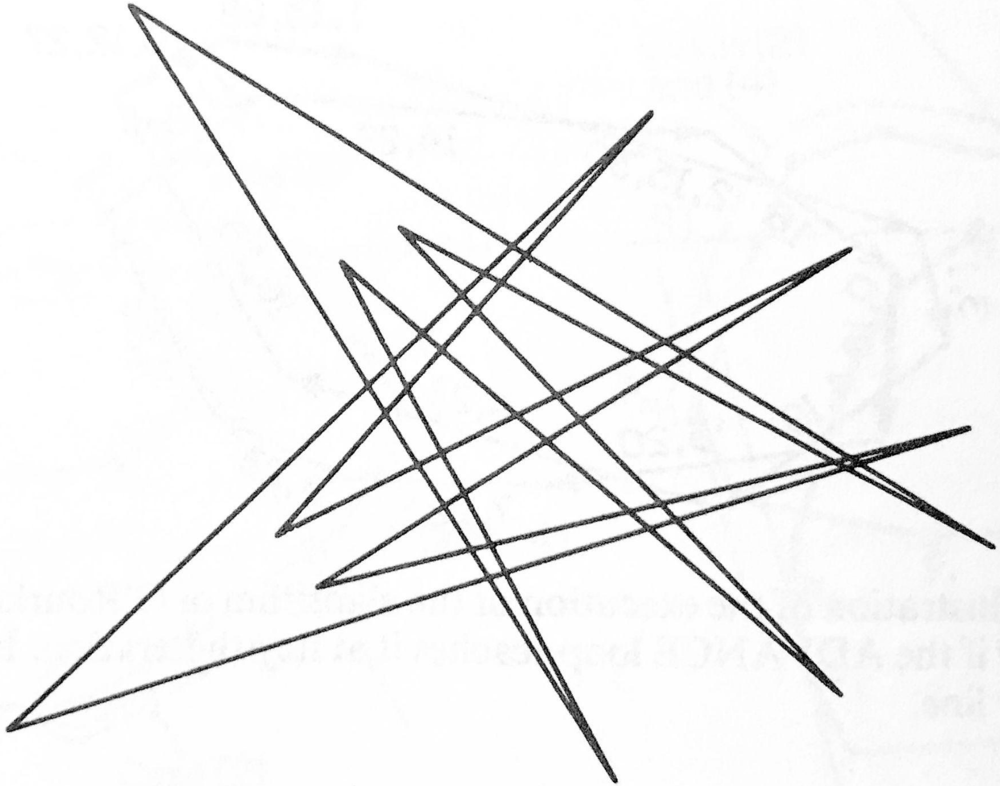

Figure 7.11 The intersection of two star-shaped polygons.

Computational Geometry: An Introduction
Preparata & Shamos, Springer 1985