CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/S22/

Lecture 4: Triangulation, part 1

Outline for Today

- Homework 1 Questions?
- Last Time: Line Intersection & Map Overlay
- Today's Motivation
  - Art gallery problem
  - Visibility for architectural walkthrough
- Triangulation
  - Proof of Existence & Size
  - Algorithm & Analysis
- Next Time: Improved Algorithm / Analysis
- Other Applications
  - Mesh Simplification
  - Hole filling for 3D Scanning

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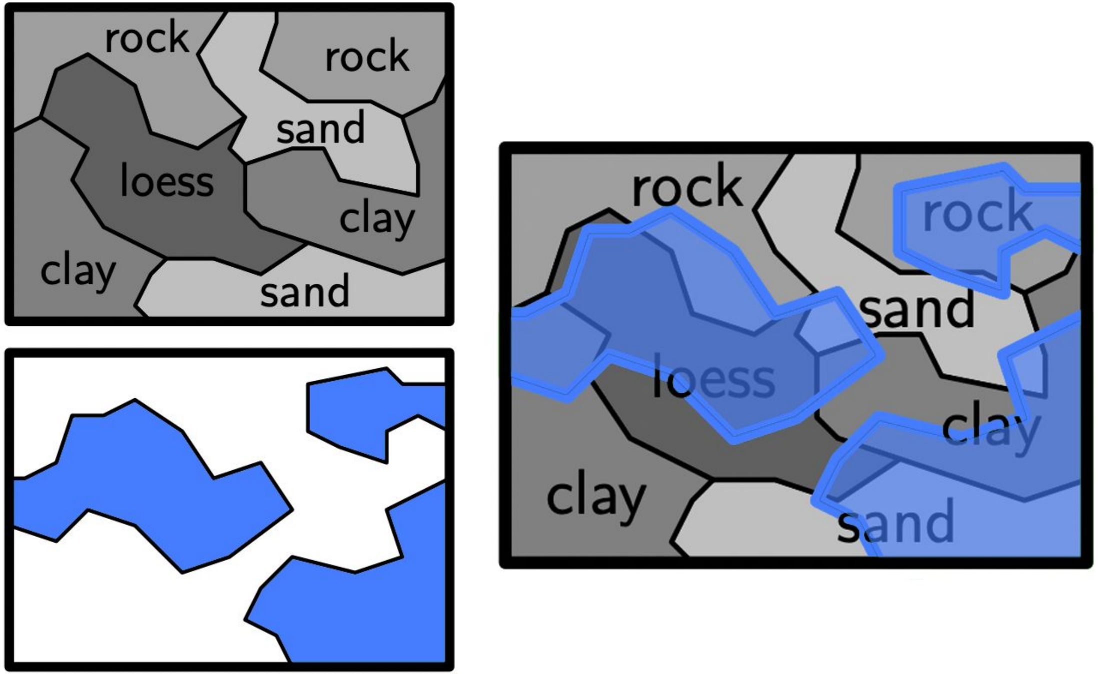
Last Time?

- “What is the total area of all lakes that occur over the geological soil type “rock”?

  → *Need to compute intersection* of areas/regions from two or more map layers



Frank Staals, http://www.cs.uu.nl/docs/vakken/ga/2021/

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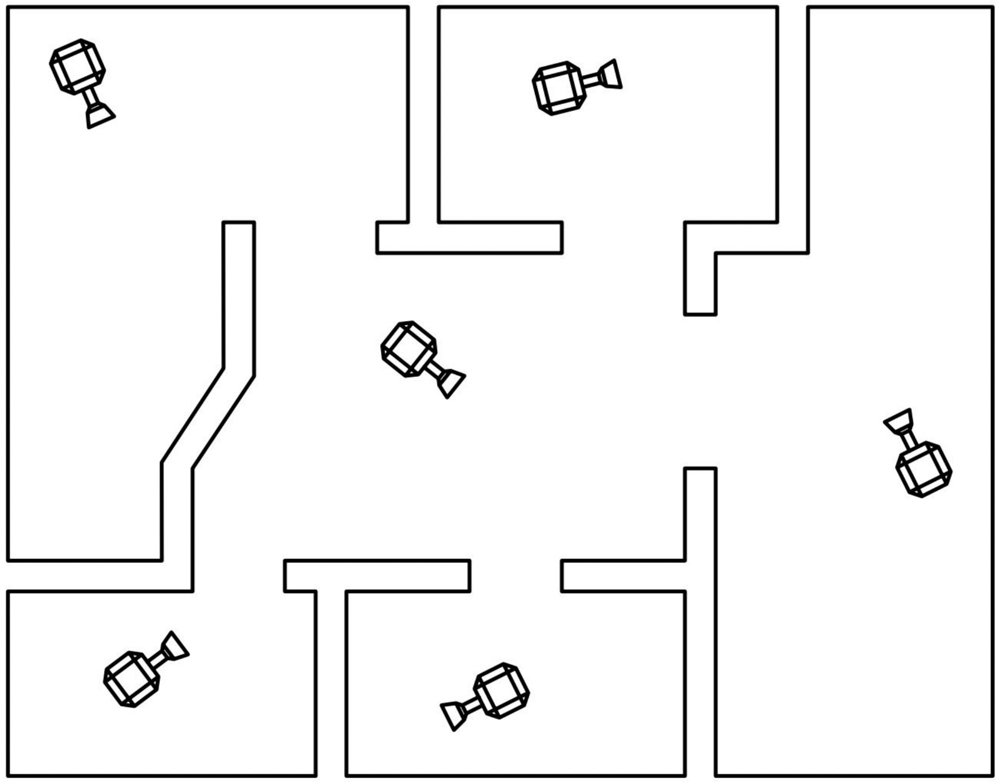
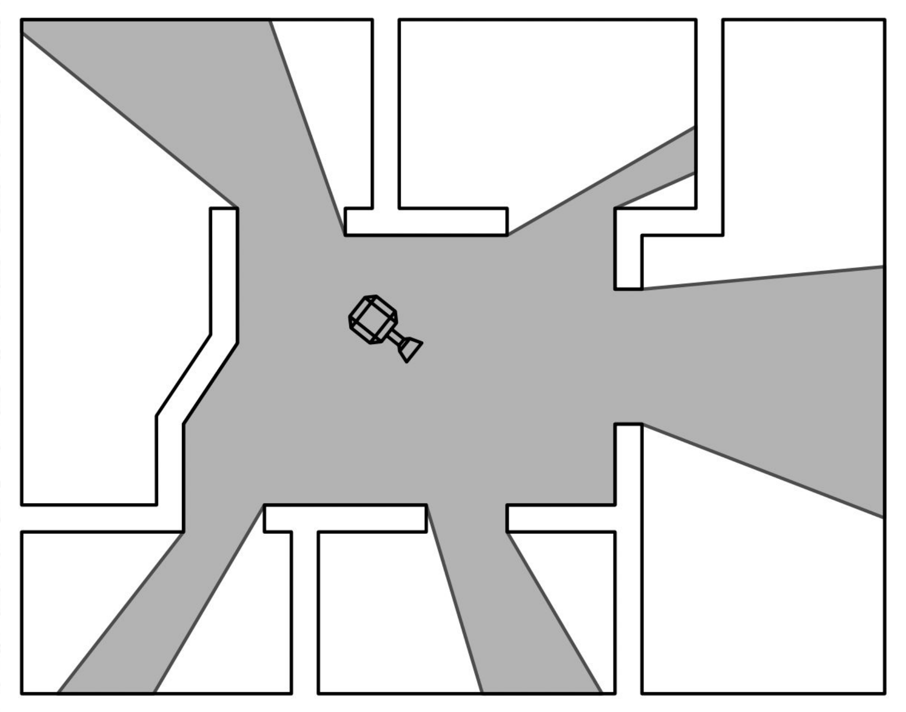
Motivation: Art Gallery Problem

- What is the minimum number of cameras (with 360° rotation) we need to place to get 100% coverage of a 2D floor plan?



Definition: Simple Polygon

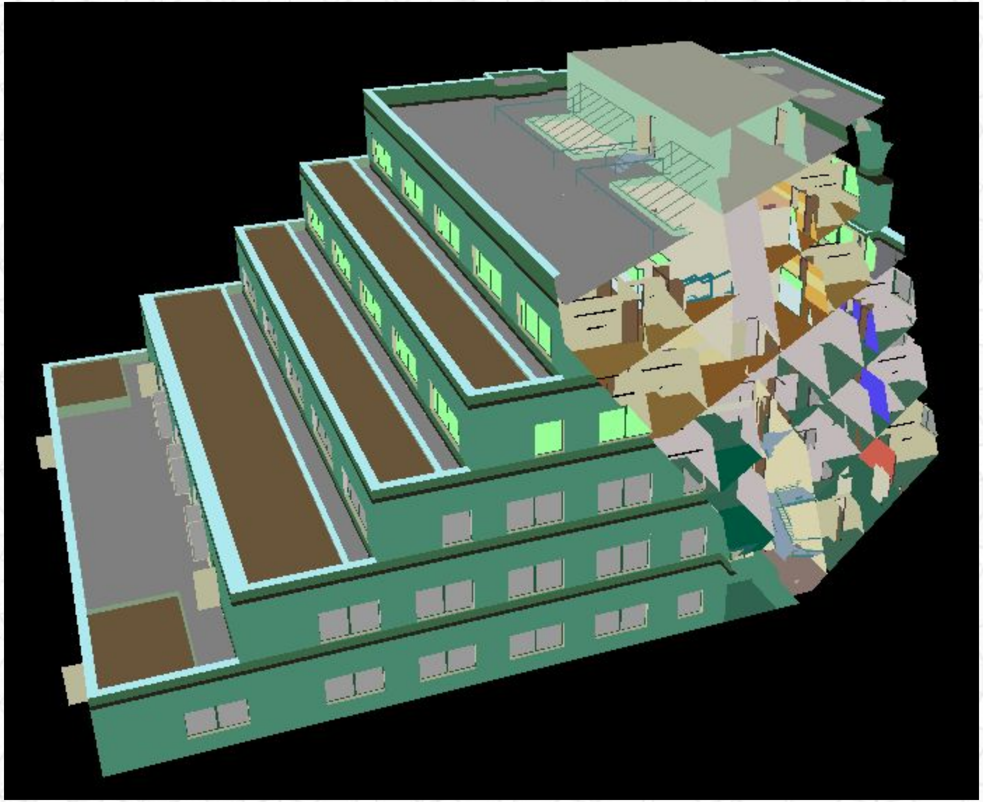
- The gallery will be a *simple polygon*.
- What can be viewed from a single camera is also a *simple polygon*.



- Single closed polygonal chain boundary
- Connected
- No interior holes
- Does not self intersect

Motivation: Architectural Walkthrough

- UC Berkeley’s new Computer Science Building
- Pre-construction visualization
- Very large dataset!
- Interactive/real-time camera motion!



Seth Teller, PhD thesis, 1992, Berkeley Soda Hall walkthrough

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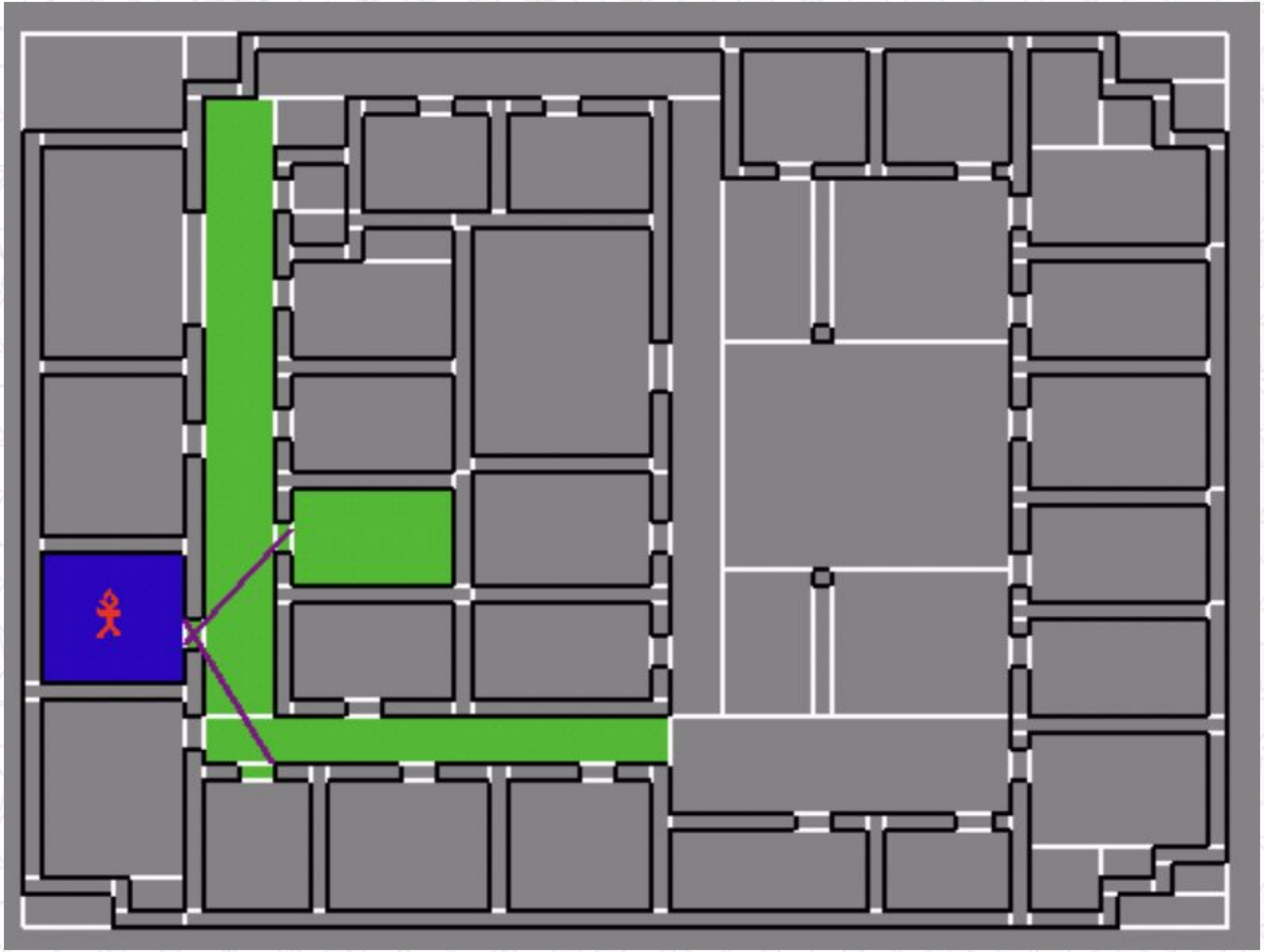
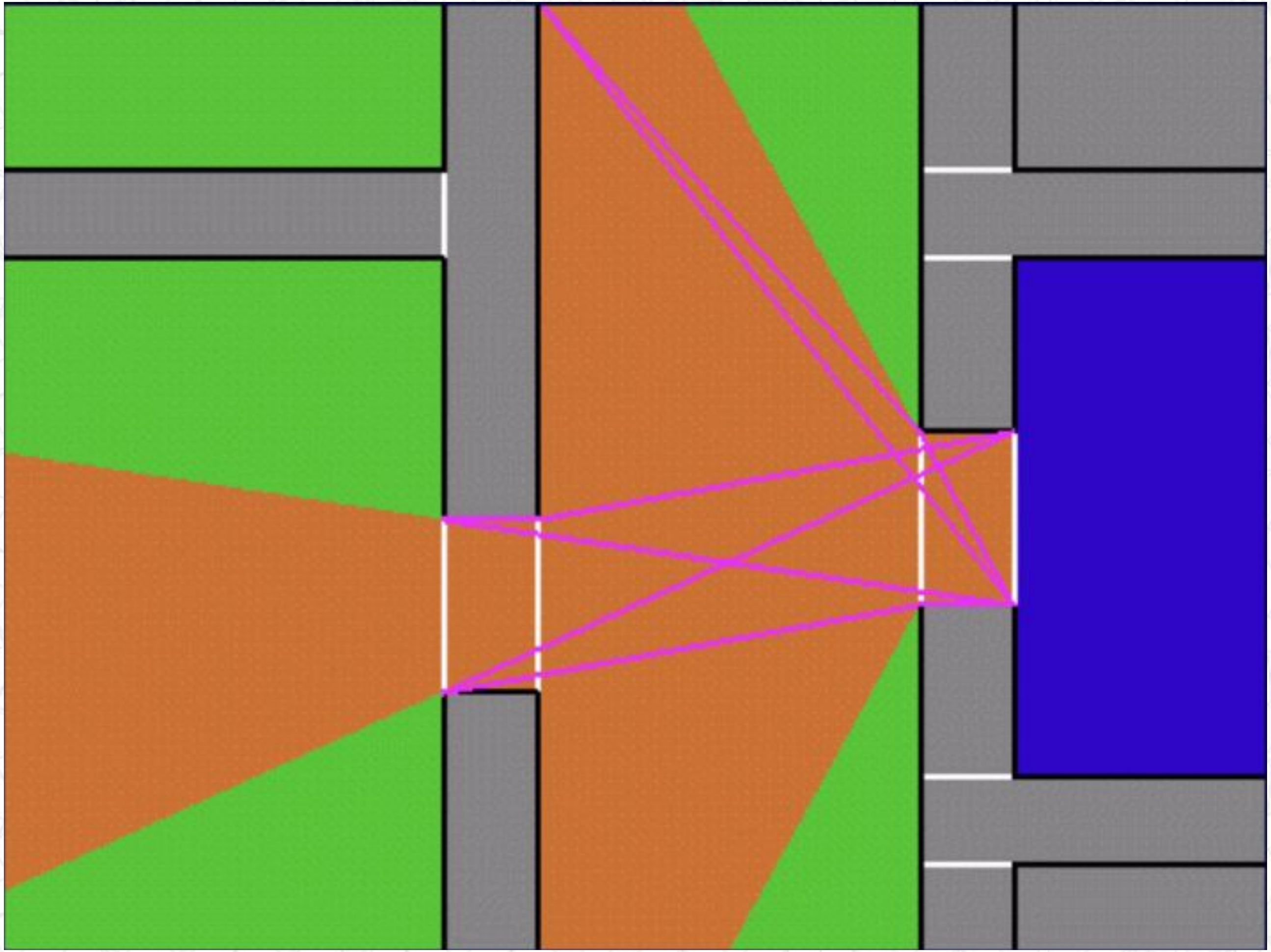


Seth Teller, PhD thesis, 1992, Berkeley Soda Hall walkthrough



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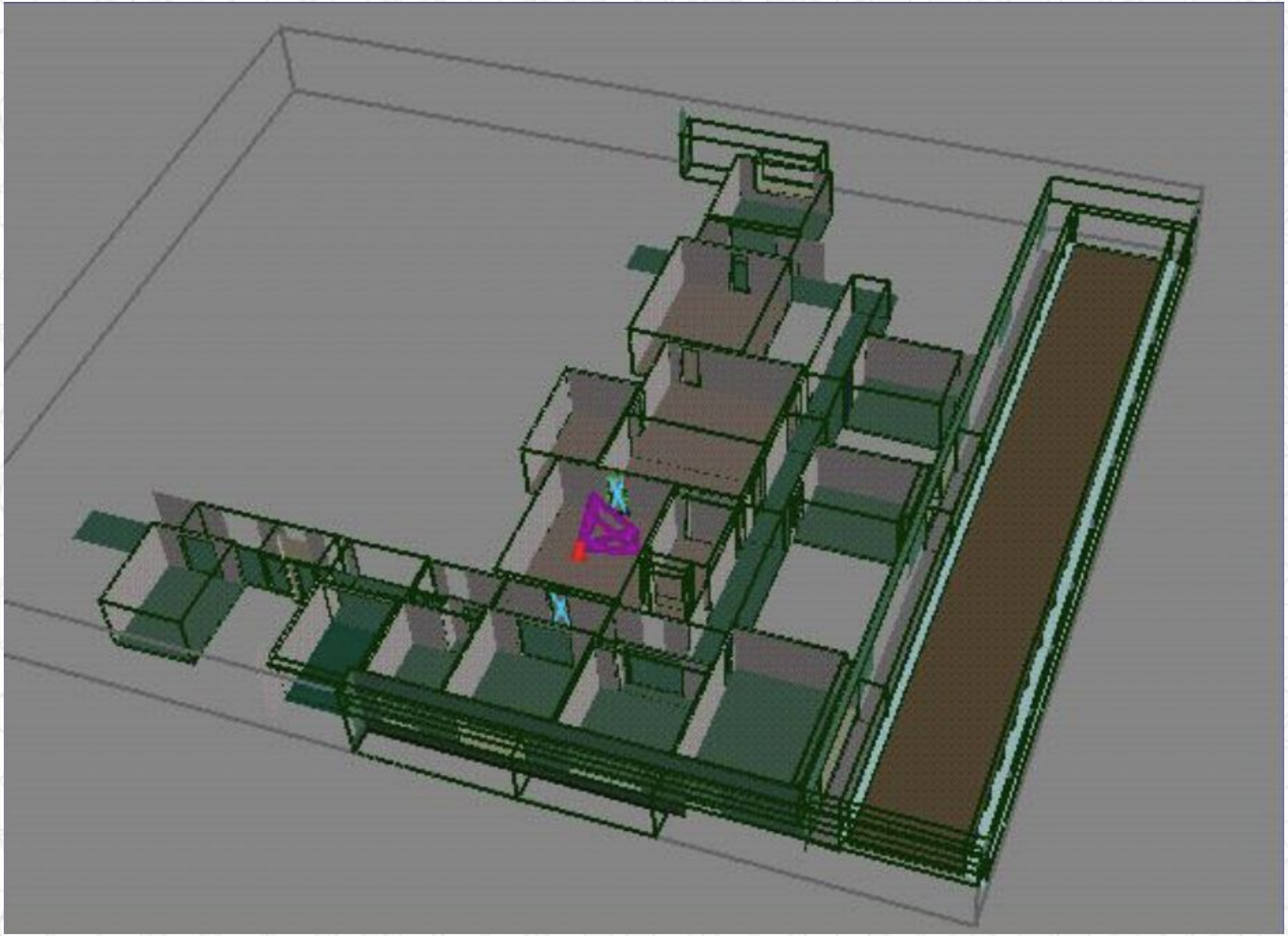
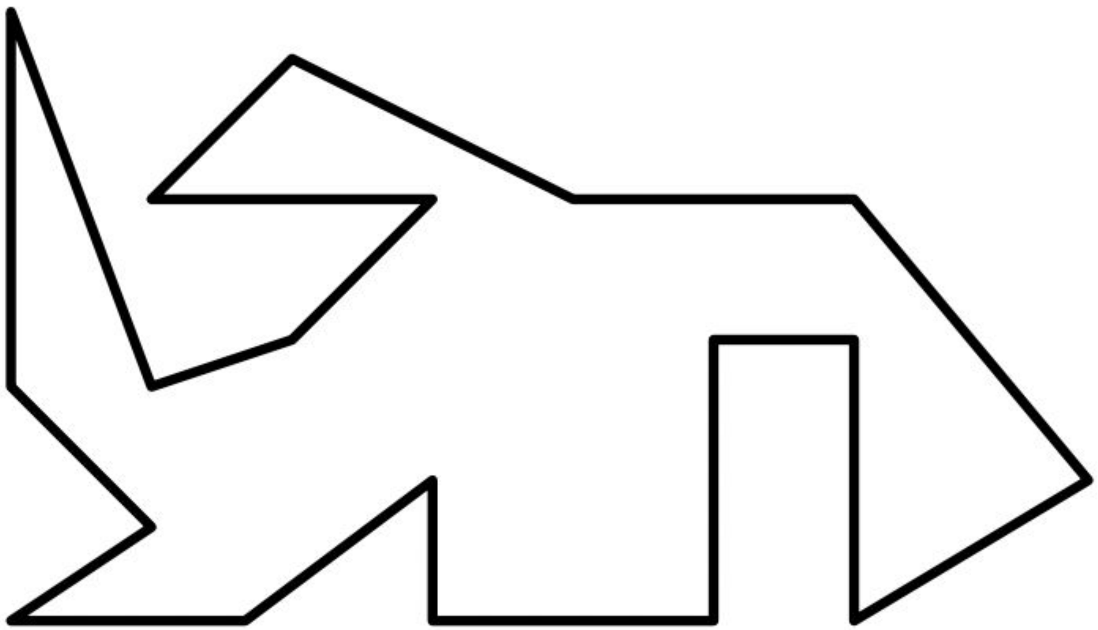


Seth Teller, PhD thesis, 1992, Berkeley Soda Hall walkthrough

Application: Art Gallery Problem

- How many cameras are necessary for 100% coverage?
- Where should we place these cameras?



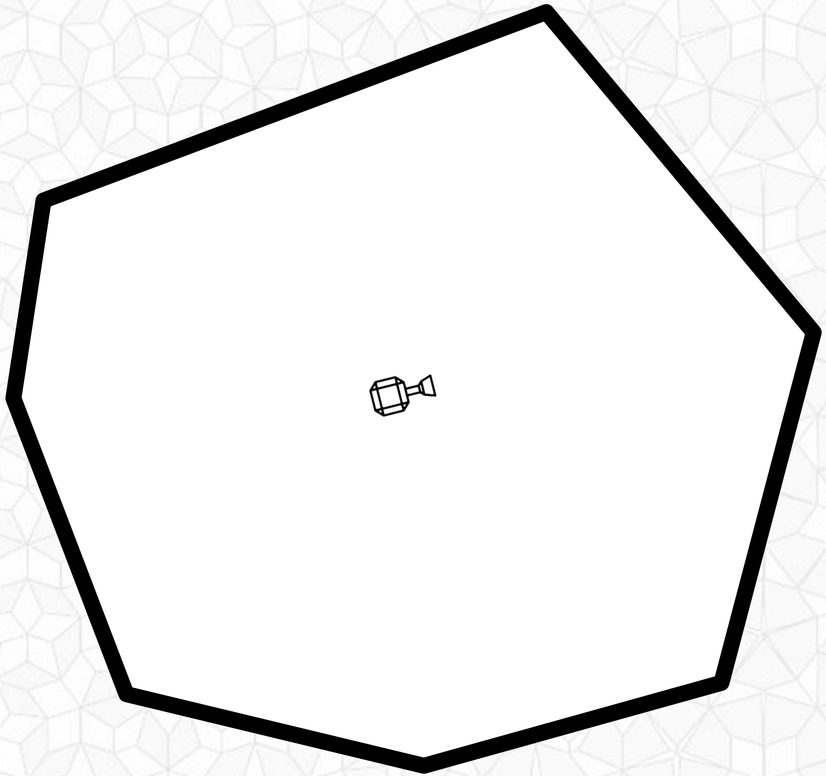
- *Note: The optimal solution is NP hard!*

Application: Art Gallery Problem

- If the gallery is convex, we can just use a single camera placed anywhere inside that polygon.

- If we chop up a non-convex polygon into convex polygons, we can place 1 camera per polygon and get 100% coverage.

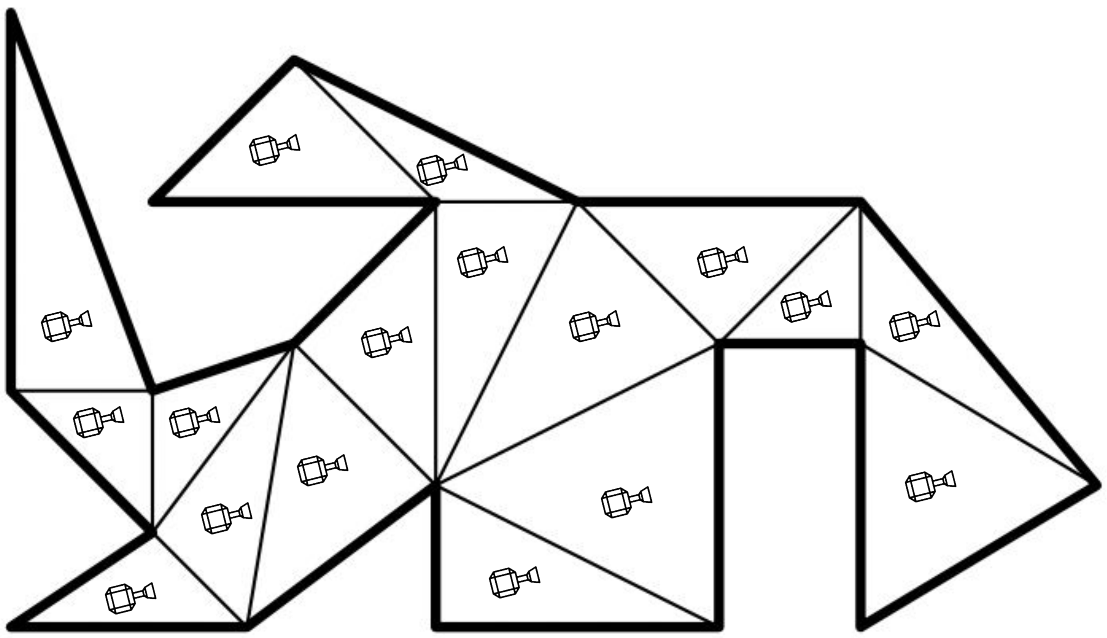
  This isn't easy to do optimally…



Application: Art Gallery Problem

- Let's chop up a non-convex polygon into triangles (which are convex).

- Place 1 camera per triangle and get 100% coverage.



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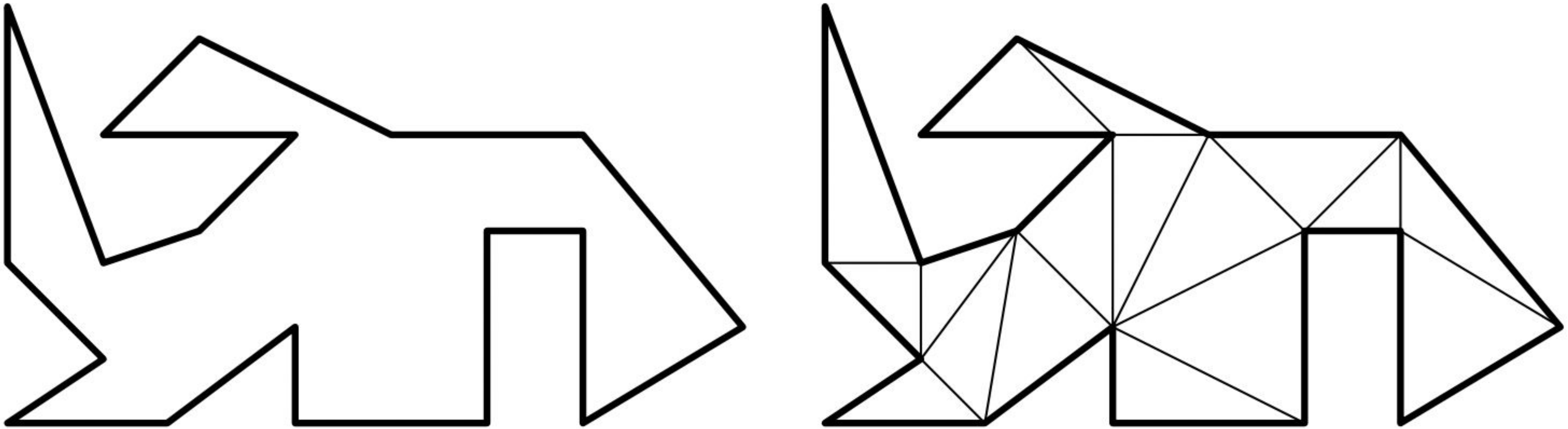
Can every shape be Triangulated?

- Theorem 3.1 (CCAA book): Every simple polygon admits a triangulation, and any triangulation of a simple polygon with n vertices consists of exactly n−2 triangles.



Can every shape be Triangulated?

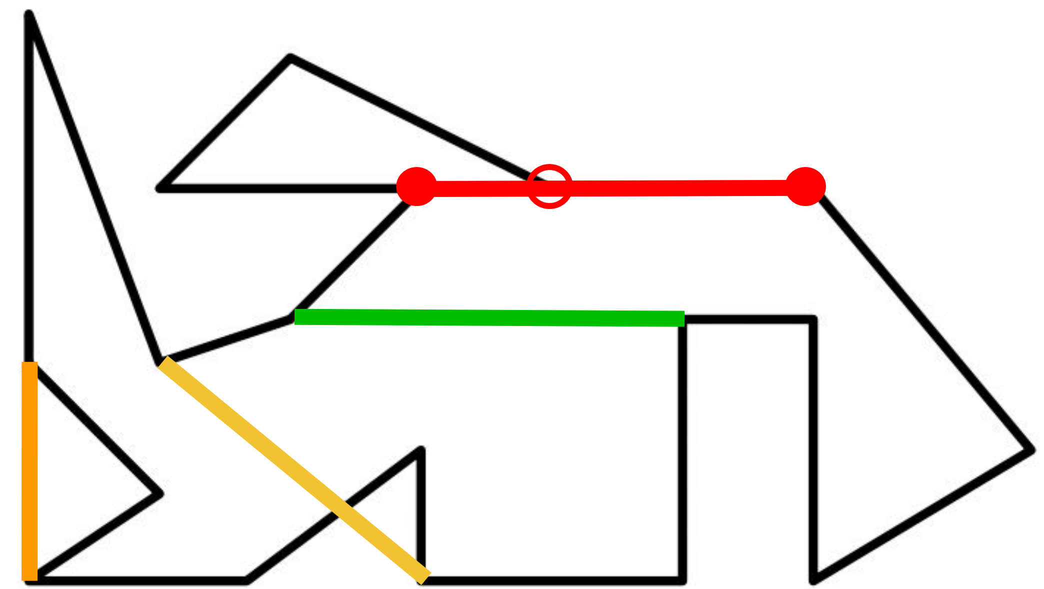
- Theorem 3.1 (CCAA book): Every simple polygon admits a triangulation, and any triangulation of a simple polygon with n vertices consists of exactly $n-2$ triangles.

- ***Proof by Induction***
- A polygon with 3 vertices is a triangle.
- Assume every polygon with $n-1$ or fewer vertices can be triangulated.
- Given a polygon with n vertices, we will draw a *diagonal line* between two vertices that cuts this shape into two smaller polygons which can be triangulated.

Which Diagonals are Allowed?

- Diagonal should connect two non-adjacent vertices on the polygonal boundary.

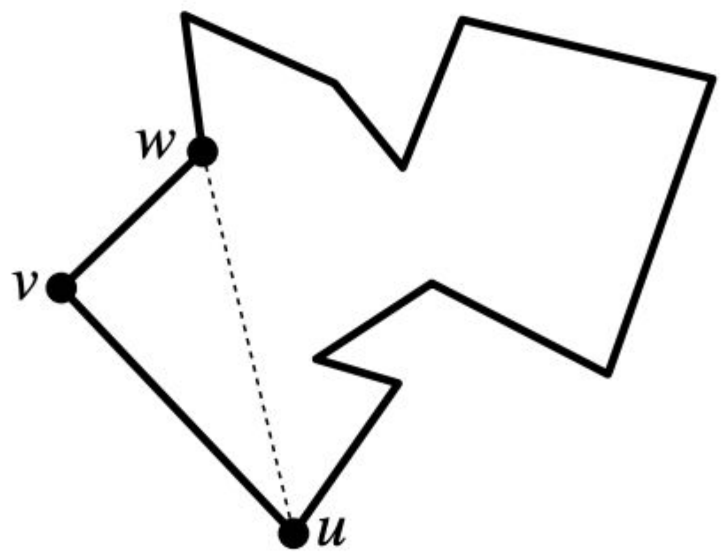
- Diagonal must not be outside the polygon.
- Diagonal may not cross any edge.
- Diagonal should not pass through any other vertex.



How do we find a Valid Diagonal?

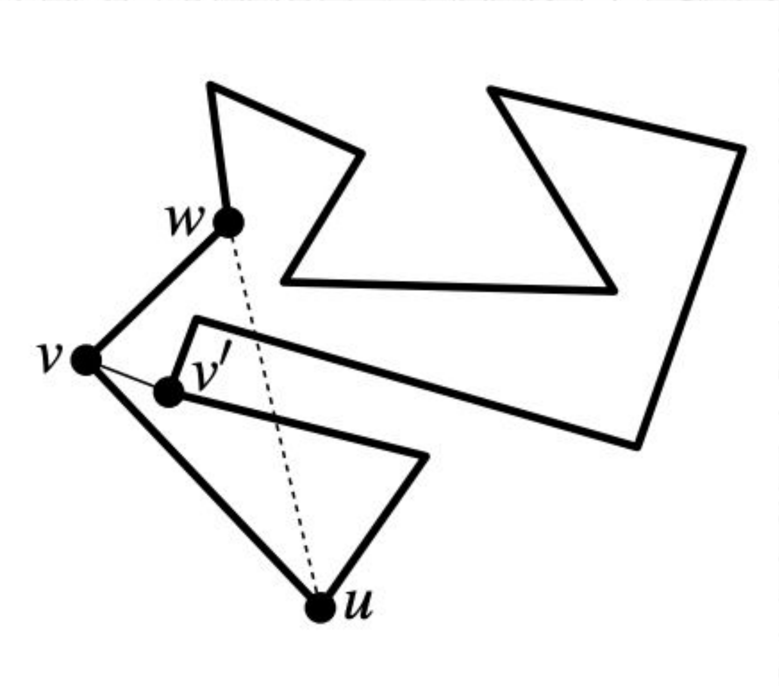
- Start at the leftmost vertex, *v*
  - *NOTE: If two or more vertices have the same x, chose the one with smaller y.*
- Find vertices *u* and *w*, adjacent to *v*

- Check if the line *uw* is a valid diagonal.
  - This line does not pass through *v.*
  - Does it intersect other line segments?
  - Does it pass through any other vertices?
  - Does it lie completely outside of the polygon? (possible if one of the vertices is the rightmost vertex)



How do we find a Valid Diagonal?

- If it does cross another line segment, there must one or more vertices inside the triangle *uvw.*
- Starting at the intersection, walk along the boundary to find those vertices.
- Choose the vertex *v'*,
  furthest from the line segment *uw*
- Draw the diagonal from *v* to *v'*



How many Triangles are Necessary?

- Theorem 3.1 (CCAA book): Every simple polygon admits a triangulation, and any triangulation of a simple polygon with n vertices consists of exactly $n-2$ triangles.

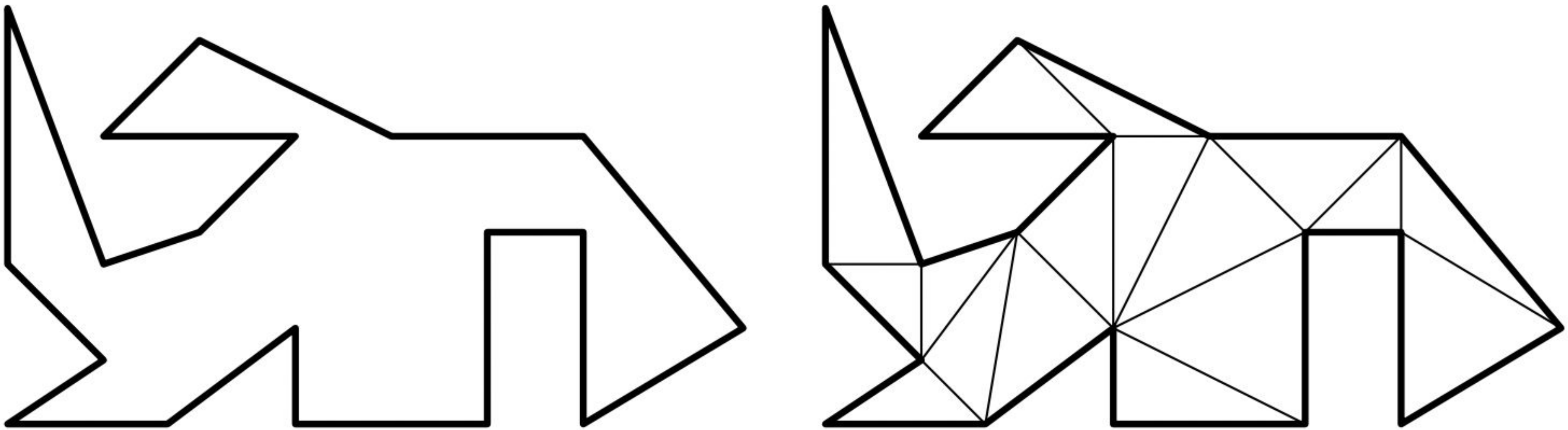
How many Triangles are Necessary?

- Theorem 3.1 (CCAA book): Every simple polygon admits a triangulation, and any triangulation of a simple polygon with n vertices consists of exactly $n-2$ triangles.

- When we draw a diagonal and split the polygon with n vertices into two smaller polygons with m_1 and m_2 vertices.
- Every vertex will be used in exactly one of the two smaller polygons, except two vertices will appear in both polygons.
- $m_1 + m_2 = n + 2$
- By induction, triangulations of these smaller polygons will have $m_1 - 2$ and $m_2 - 2$ triangles.
- Overall: $m_1 - 2 + m_2 - 2 = (m_1+m_2) - 4 = n+2-4 = n-2$ triangles

Non-Uniqueness of Triangulation

- Observation: There's more than one way to chop up this non-convex polygon into triangles *… more on this later in the term*



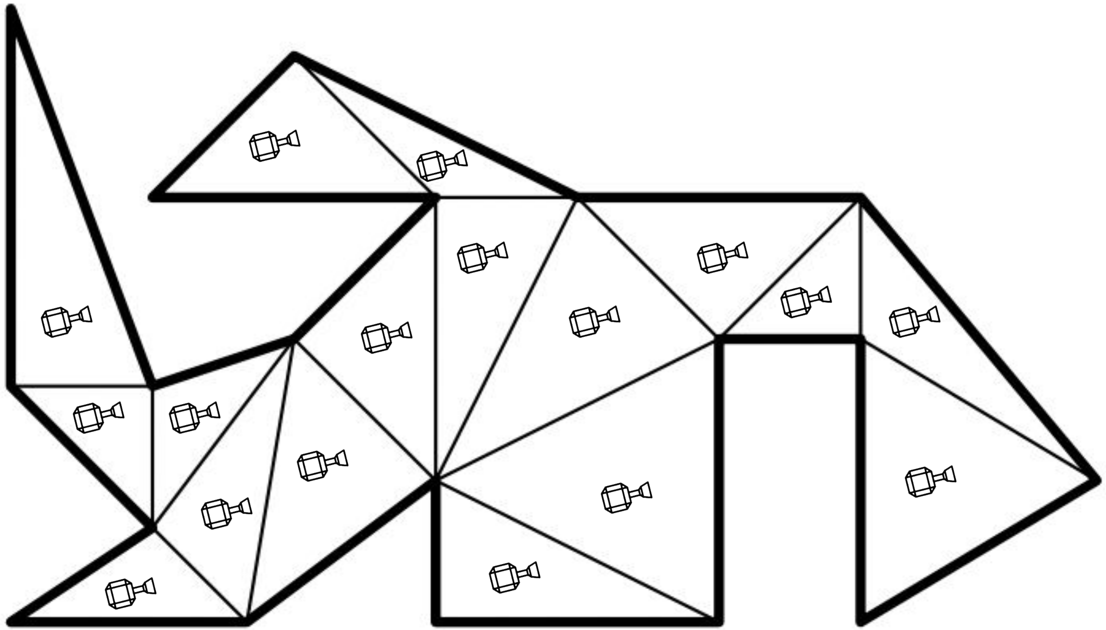
Application: Art Gallery Problem

- Non convex gallery with *n* edges, *n*-2 triangles

- Place 1 camera per triangle

- Requires *n*-2 cameras

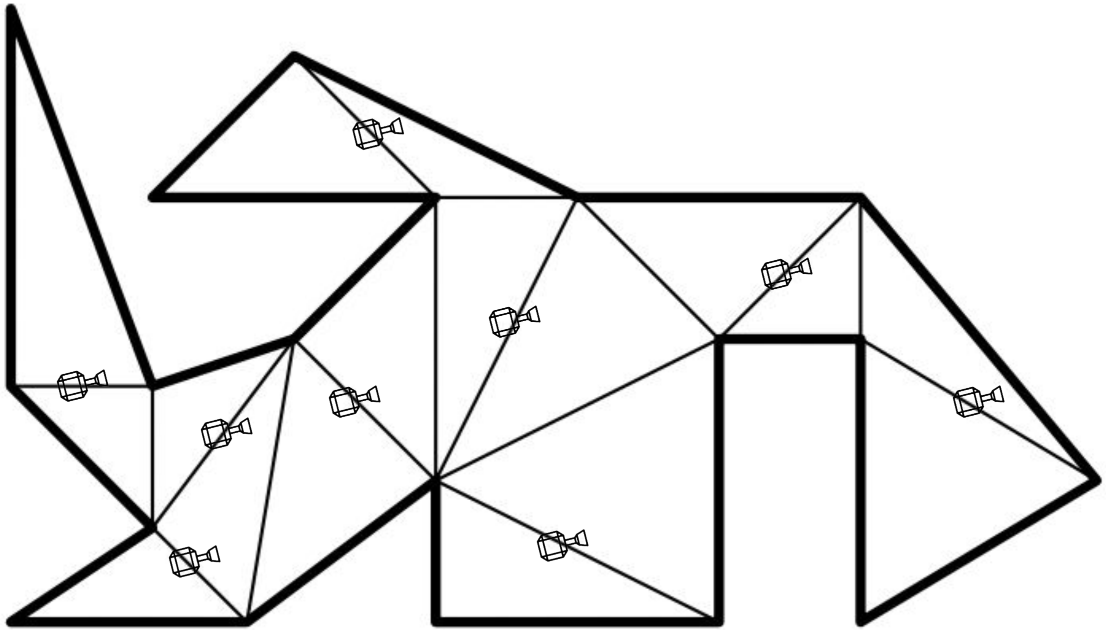
- Can we do better?



Application: Art Gallery Problem

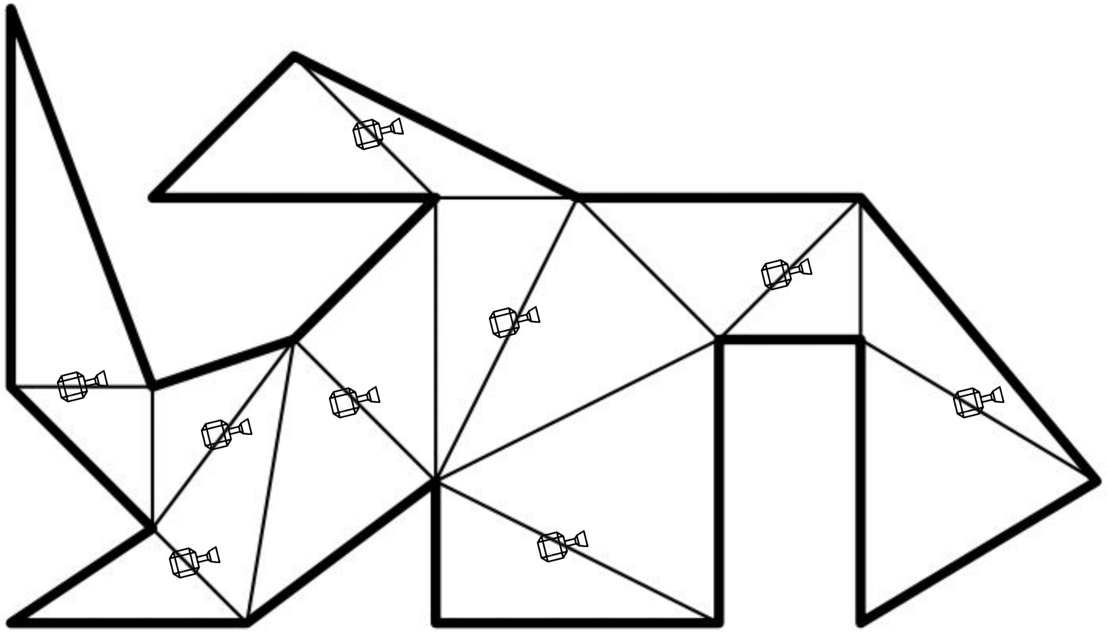
- Place cameras on edge between 2 triangles

- Covers both triangles



Application: Art Gallery Problem

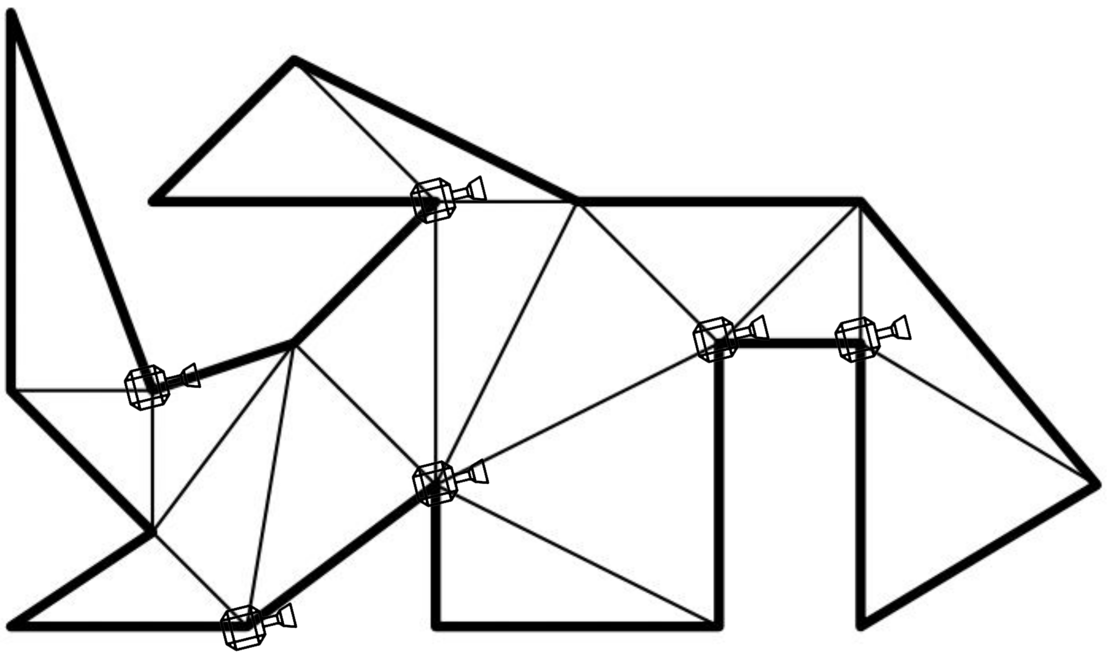
- Place cameras on edge between 2 triangles
- Covers both triangles
- Requires ≈ $n/2$ cameras
- Can we do better?



Application: Art Gallery Problem

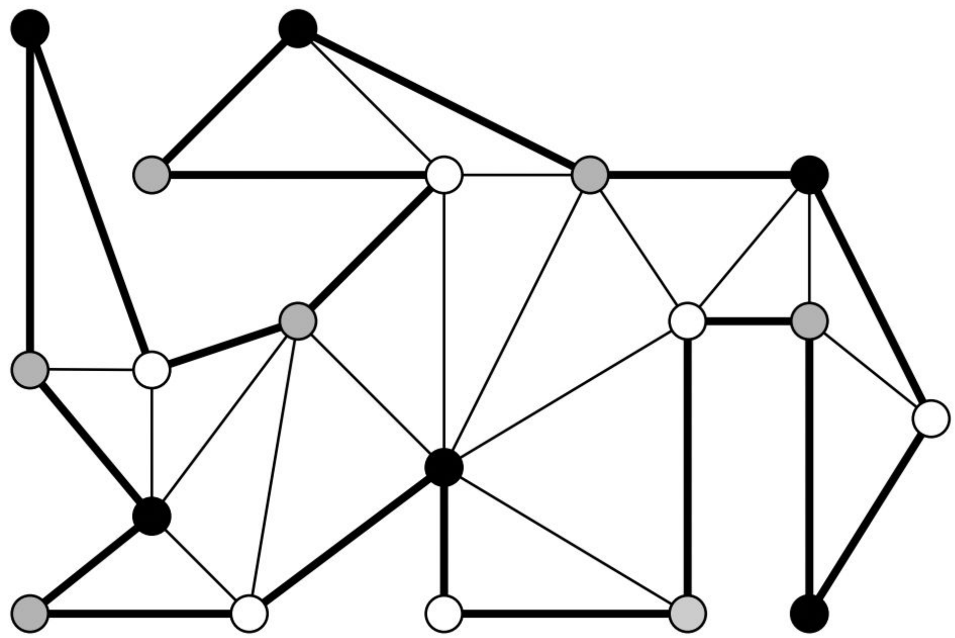
- Place cameras on vertices
- Can view all triangles that touch that vertex

- On which vertices should we place the cameras?



3 Coloring of a Triangulated Simple Polygon

- The vertices of a triangulated simple polygon can be colored with 3 colors (white, grey, black) such that each triangle has one vertex of each color (no duplicates).



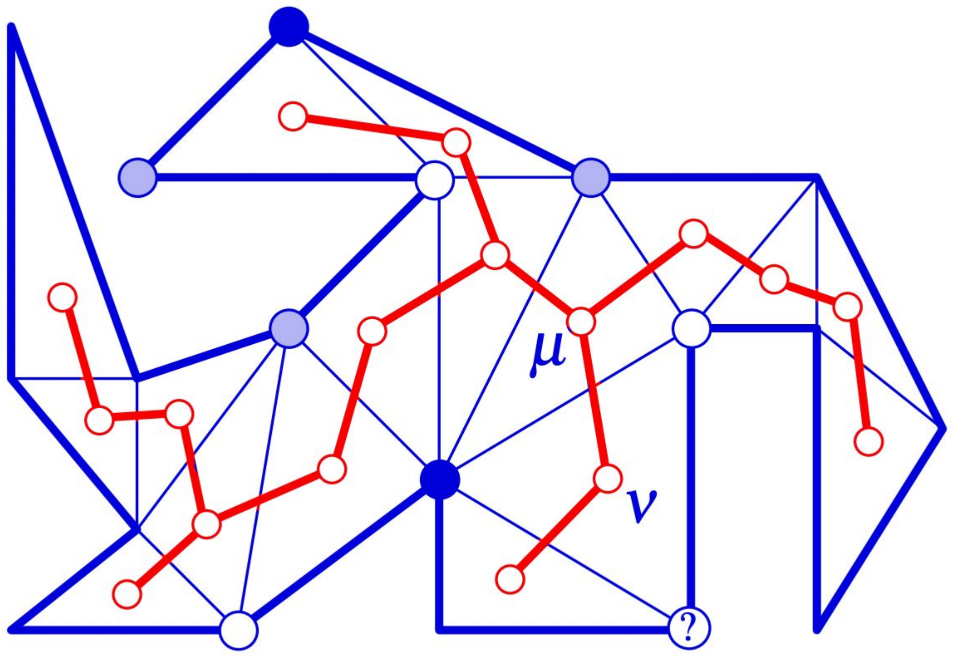
- Place cameras on color is used the least
- ≤ *n/3* cameras

Definition: Dual Graph

A common and very important tool in our Computational Geometry toolbox!

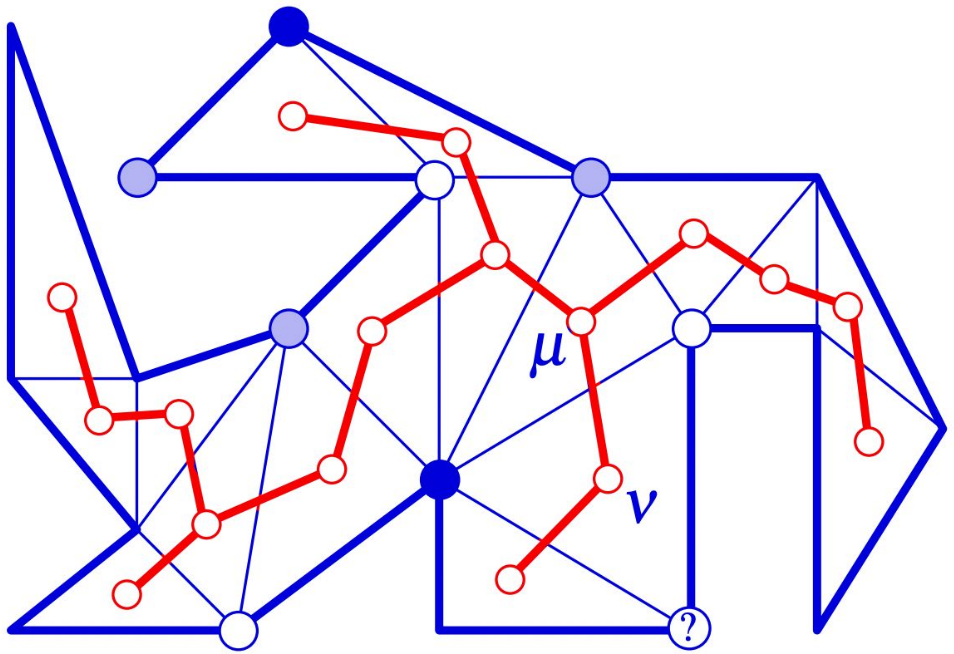
- We place a **vertex** in the dual graph at the center of every triangle in the primary graph

- We draw an **edge** in the dual graph connecting two vertices if the corresponding triangles in the primary graph share an edge.



Dual Graph - Is a 3 Coloring Always Possible?

- The dual graph for our triangulated simple polygon is a tree (no cycles!)
    - Connected
    - No interior holes in the polygon
    - No interior vertices in the triangulation
- We can perform a depth-first tree walk and color the vertices without duplicates

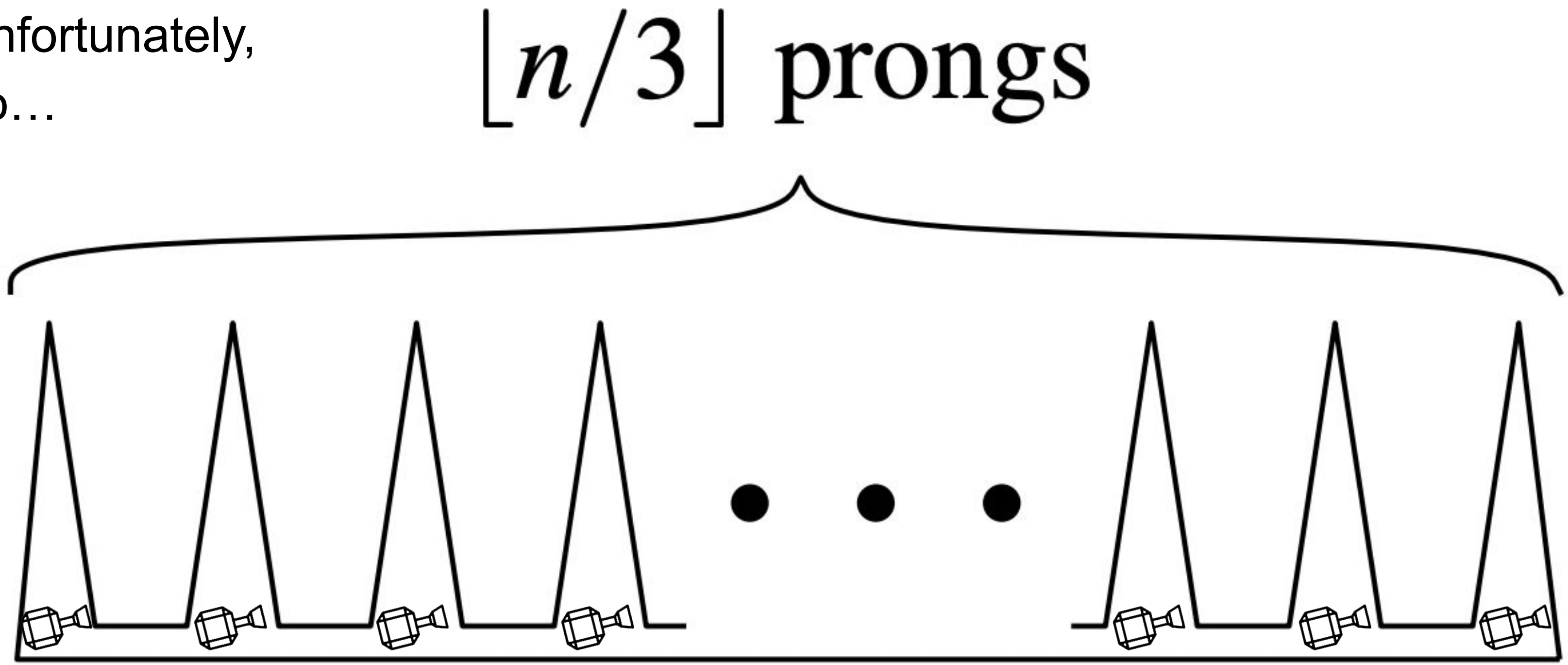


Application: Art Gallery Problem

- Can we do better than *n/3* cameras?

Application: Art Gallery Problem

- Can we do better than *n/3* cameras?
- Unfortunately, no…



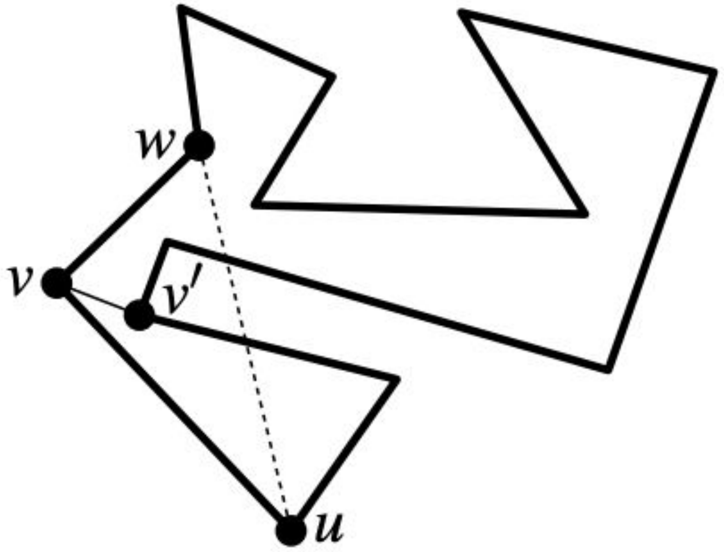
Preliminary Analysis?

- What is the worst case running time to triangulate a non-convex, simple polygon with *n* vertices?

- Identify a legal diagonal

- Split into two smaller polygons

- Overall:

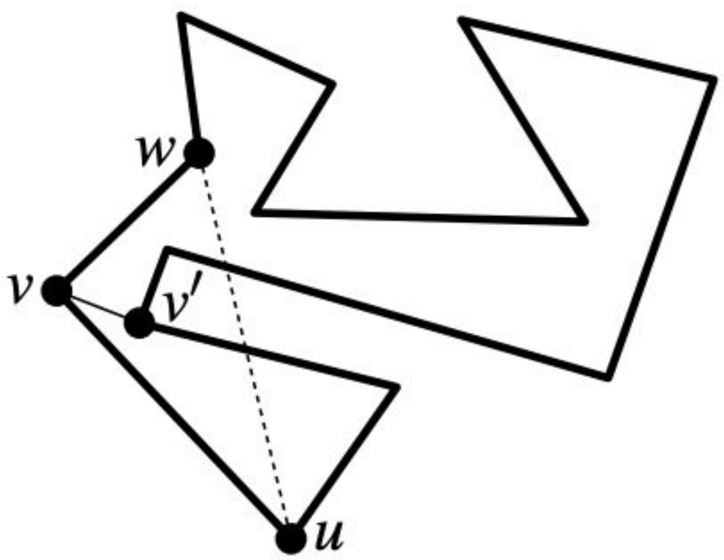


Preliminary Analysis?

- What is the worst case running time to triangulate a non-convex, simple polygon with *n* vertices?

- Identify a legal diagonal
    - O(n) in worst case
- Split into two smaller polygons
    - Worst case:
      m_1 = 3 vertices and
      m_2 = n-1 vertices

- Overall: O(n^2) running time



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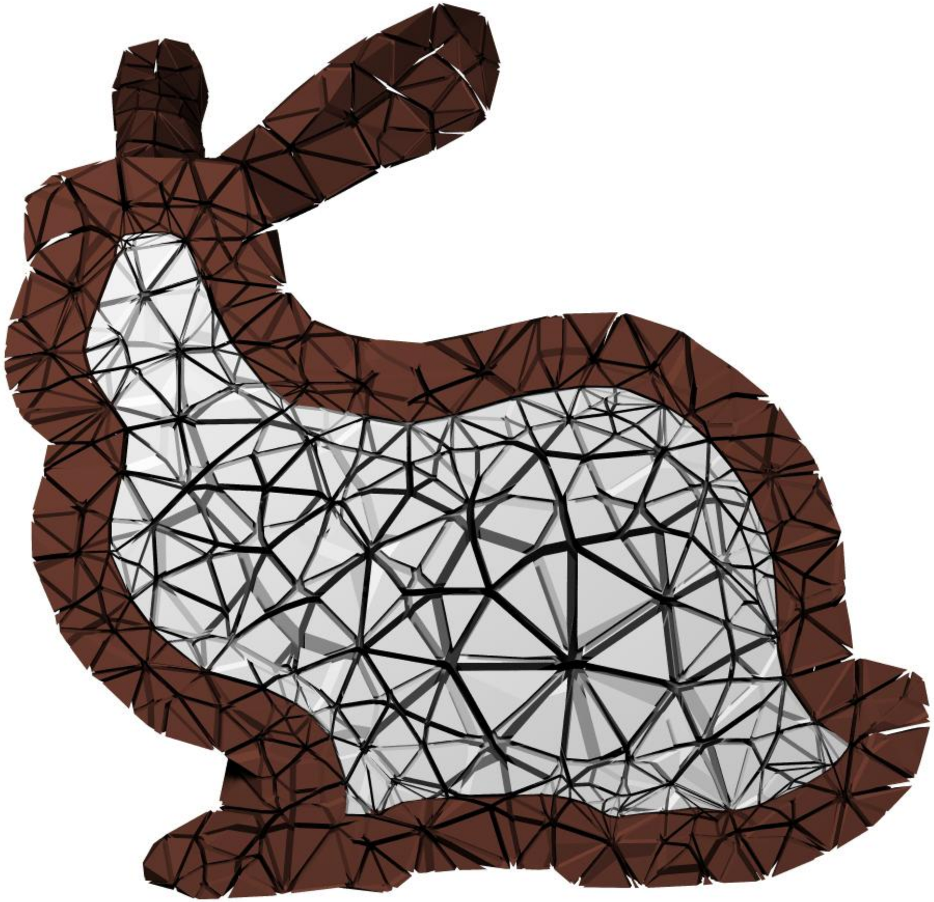
Next Time…

- Analysis: Can we do better than O(n^2)?

  YES!

- Does this work in 3D too?
- Can we triangulate or tetrahedralize the interior of a polytope?



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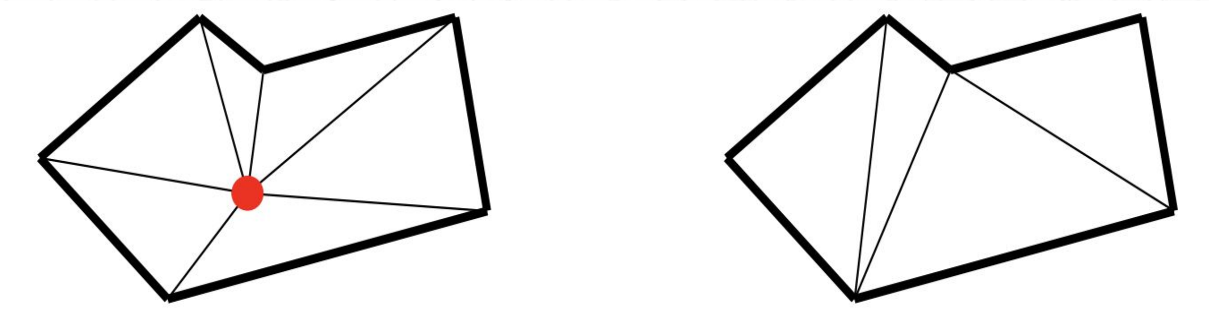
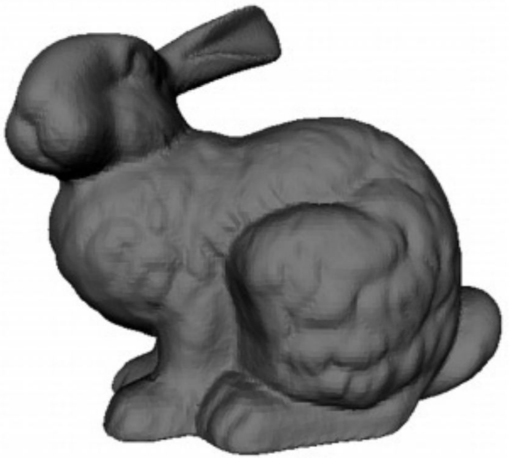
Application: Mesh Simplification

- Identify a *relatively unimportant* vertex to remove
- Remove the connected triangles
- Re-triangulate the hole



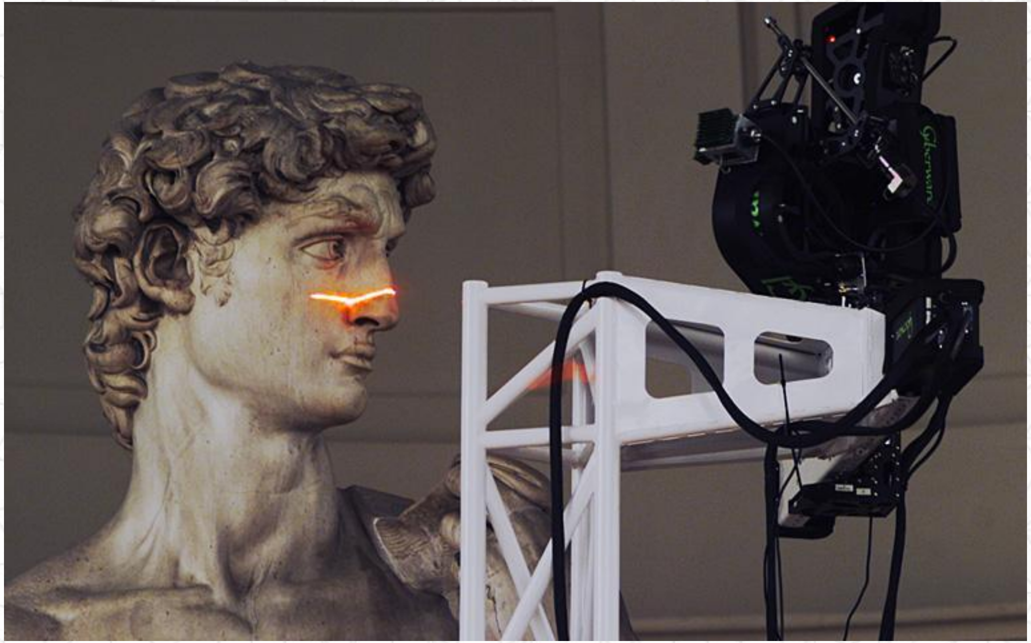
“Surface Simplification Using Quadric Error Metrics”
Garland & Heckbert, SIGGRAPH 1997



Original: 70,000 triangles

Simplified: 1,000 triangles

Application: 3D Digitizing

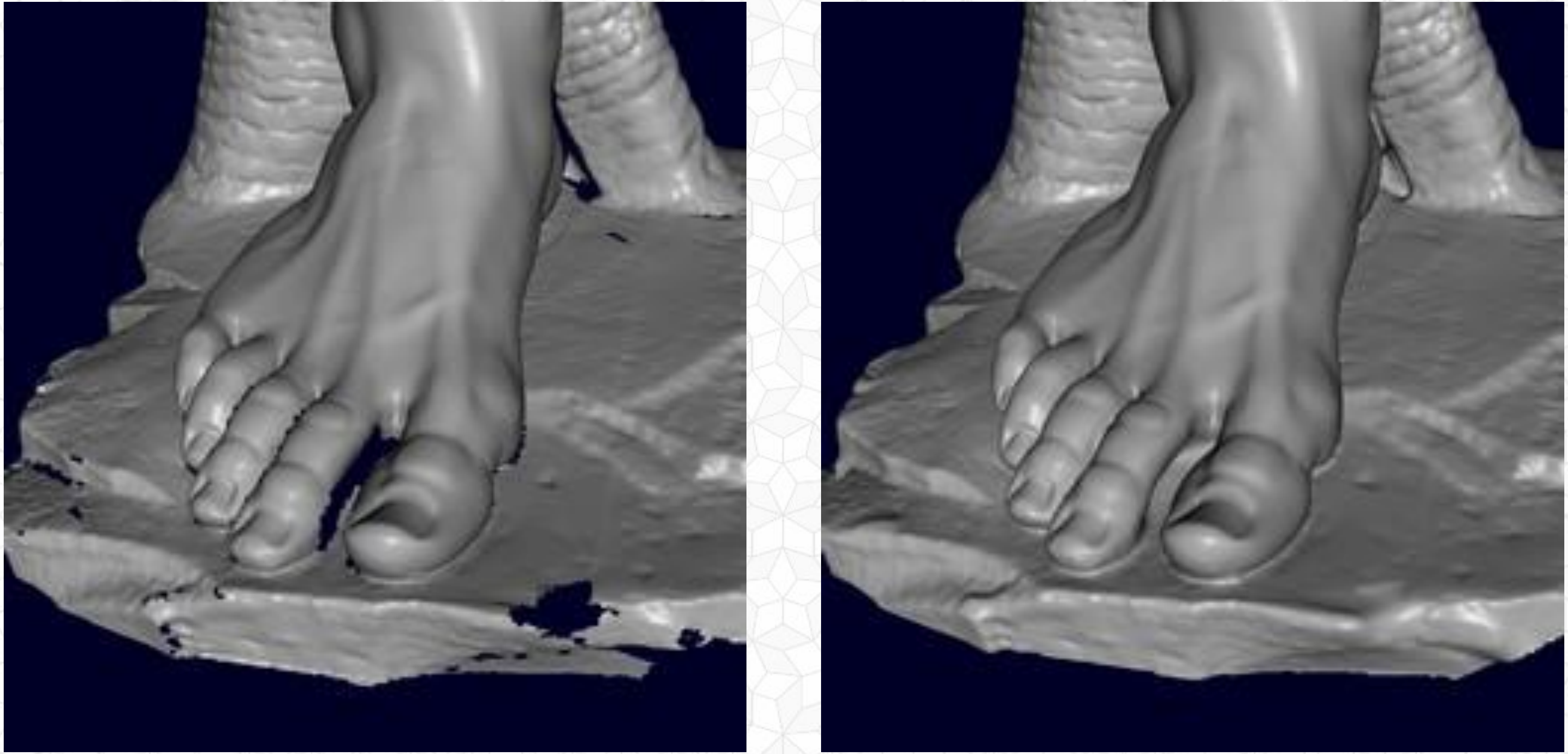


The Digital Michelangelo Project: 3D Scanning of Large Statues, Levoy et al., *SIGGRAPH 2000*

Cyberware

Application: Hole Filling

“Filling holes in complex surfaces
using volumetric diffusion”
Marschner, Davis, Garr, and Levoy



https://graphics.stanford.edu/papers/holefill-tr-2001-07/

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