CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/S22/

Lecture 12: Voronoi Diagrams, Part 3

Outline for Today

- Homework 4 Posted!
- Friday Feb 25th: Quiz 1, Logistics
- Last Time: More Voronoi Diagrams
- Higher-Order vs Higher Dimension Voronoi Diagrams
- Centroidal Voronoi Diagram
- K-Means Clustering
- Application: Architectural Geometry
- More Spatial Query & Search Problems / Applications
- Reducing Other Problems to the Voronoi Diagram
- Future Topic (after Quiz 1): Delaunay Triangulation

Quiz 1

- In class, Friday Feb 25th, 2-3:50pm
- On paper, will involve simple sketching,
  you are welcome (but not required) to bring colored pencils/markers/crayons/etc.
- If you cannot attend class in person, contact me
  at least 24 hours in advance for alternate arrangements.

- Volunteer to pick up the quizzes from Shannon or Shianne Friday morning/early afternoon &
  slide under my office door in Lally after the quiz?

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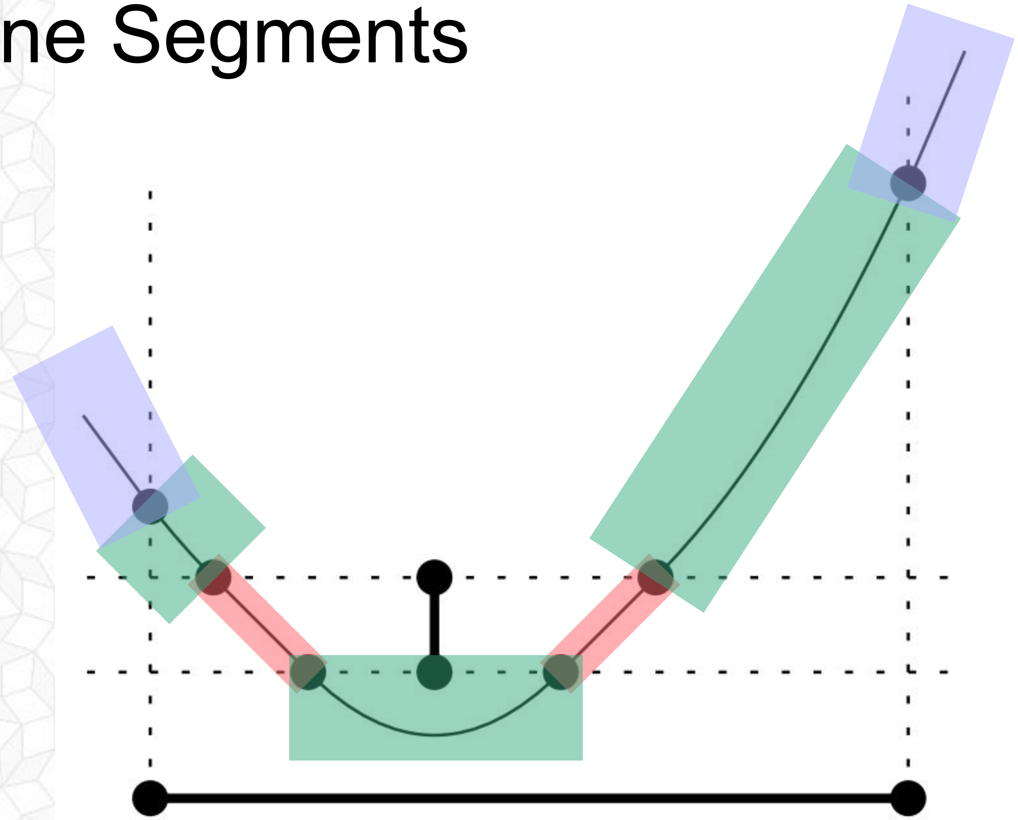
Voronoi Diagram of Line Segments

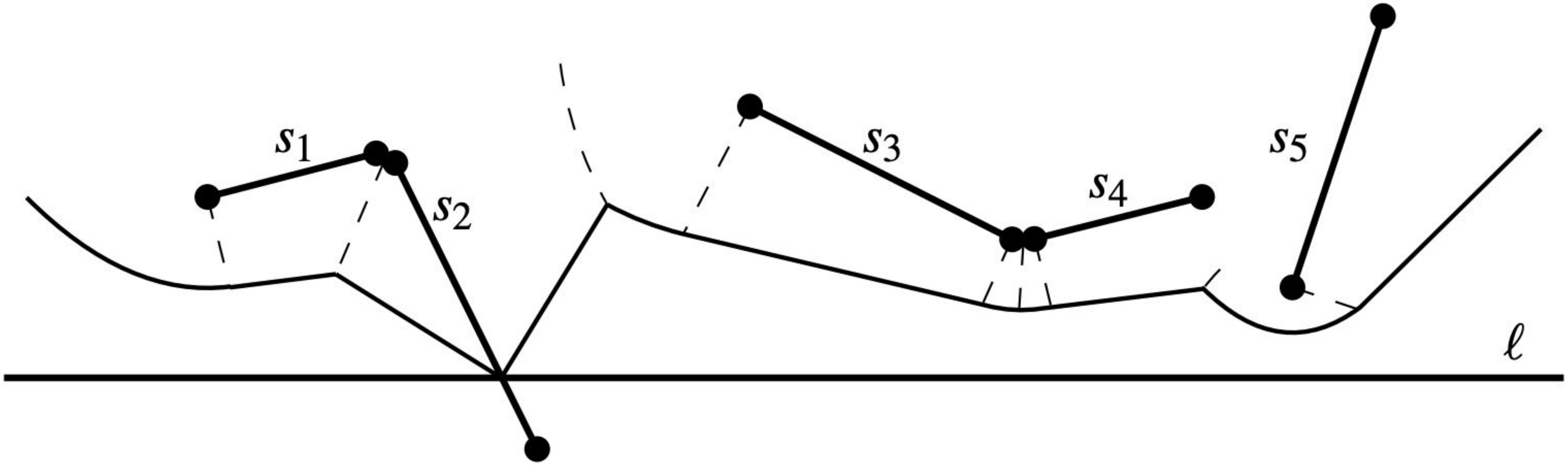
- Points equidistant between two points form a line.
- Points equidistant between a point and a line form a parabola.
- Points equidistant between two lines form a line.



Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 7

Sweep Line: More Complicated Beach Front

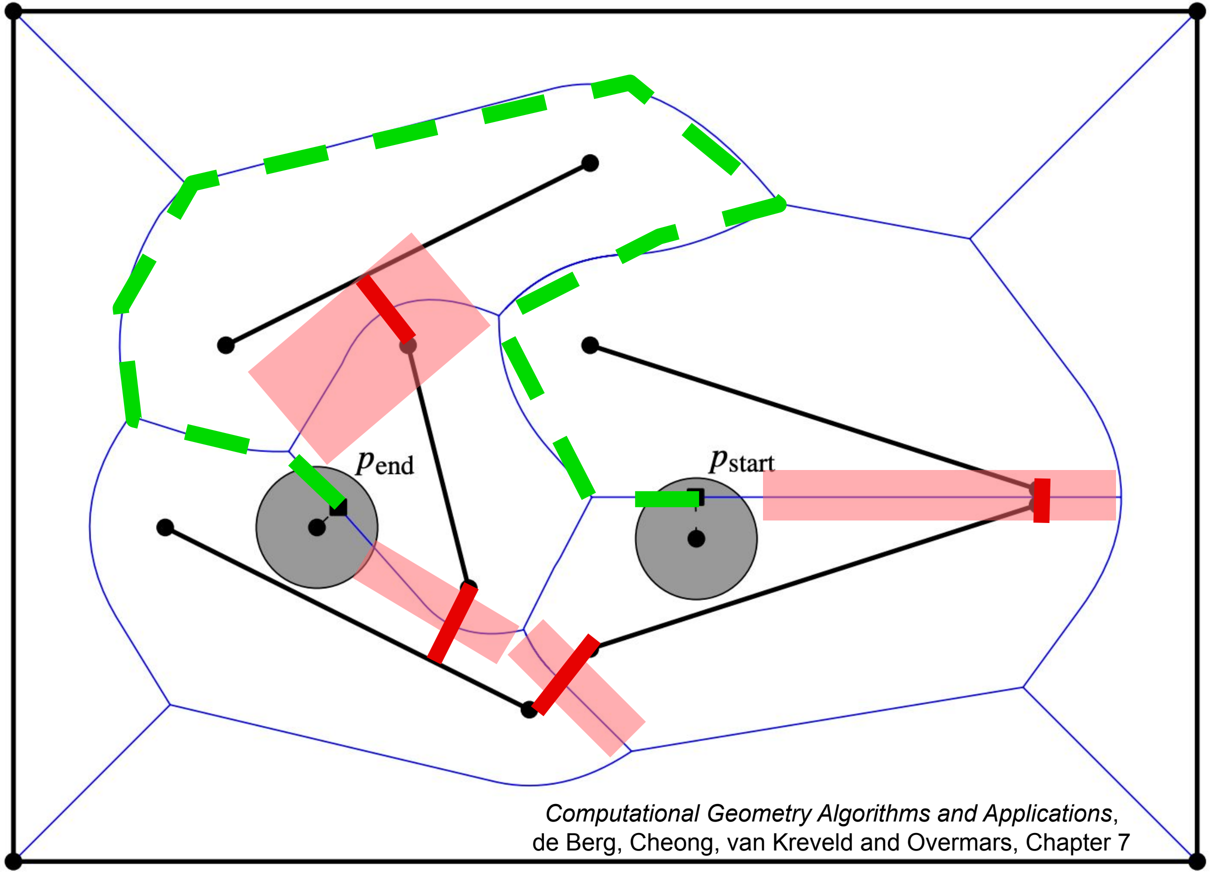
- Fortunately, the complexity (# of segments) is still *O(n)* in the size of the input – now line segments instead of just points!



Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 7

Application: Robotics & Motion Planning

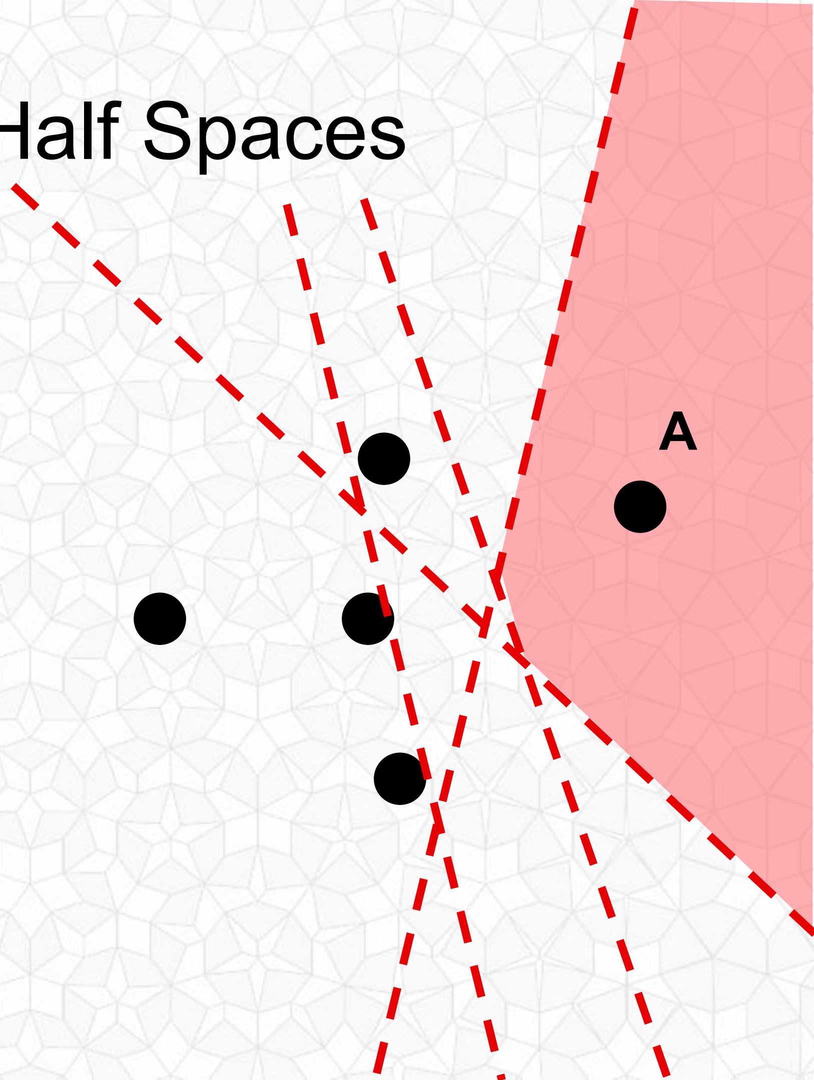
- Step 1: Project robot center to closest Voronoi edge.
- Step 2: Remove Voronoi edges from diagram graph where smallest distance to segment < radius.
- Step 3: Search the remaining graph for a connected path from start to end.



Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 7

Voronoi Cell: Intersection of Half Spaces

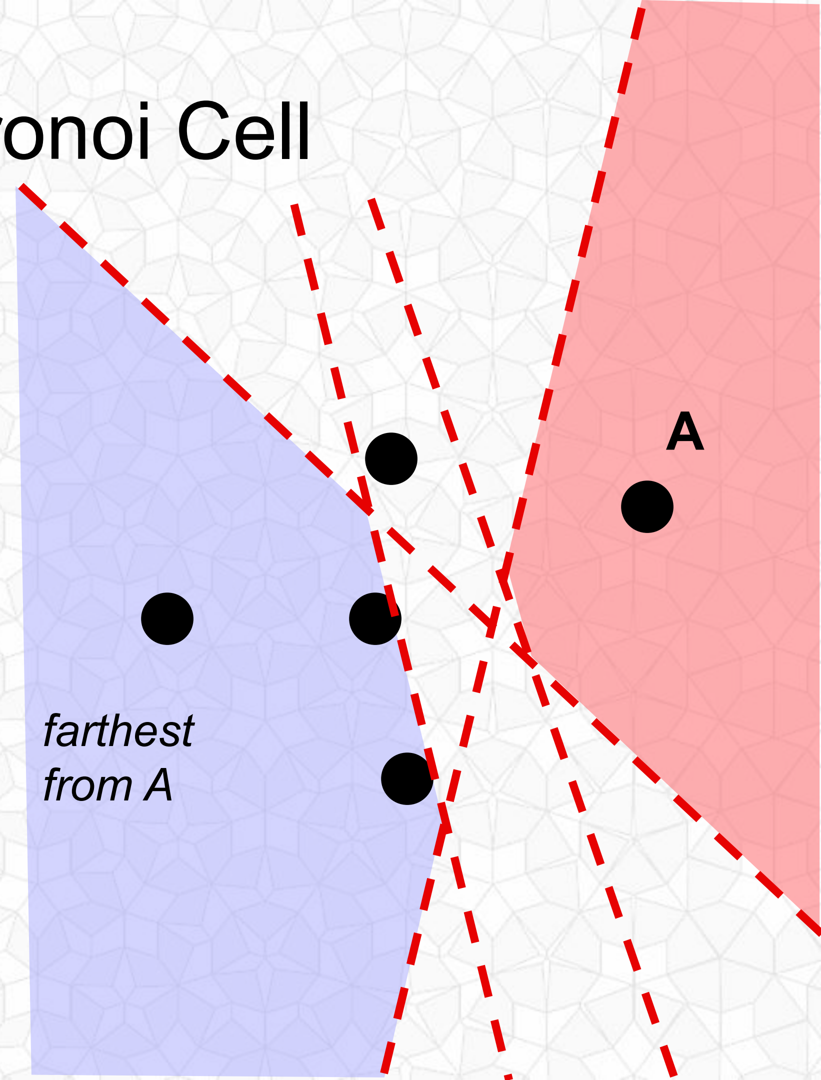
- The intersection of these half-spaces is the Voronoi Cell for A – *all points that choose A as their closest Voronoi site*.



Definition: Farthest Point Voronoi Cell

- The intersection of these half-spaces is the Voronoi Cell for A – *all points that choose A as their closest Voronoi site*.

- The intersection of the opposite half-space is the Farthest Point Voronoi Cell - all points that indicate that A is their furthest Voronoi site.

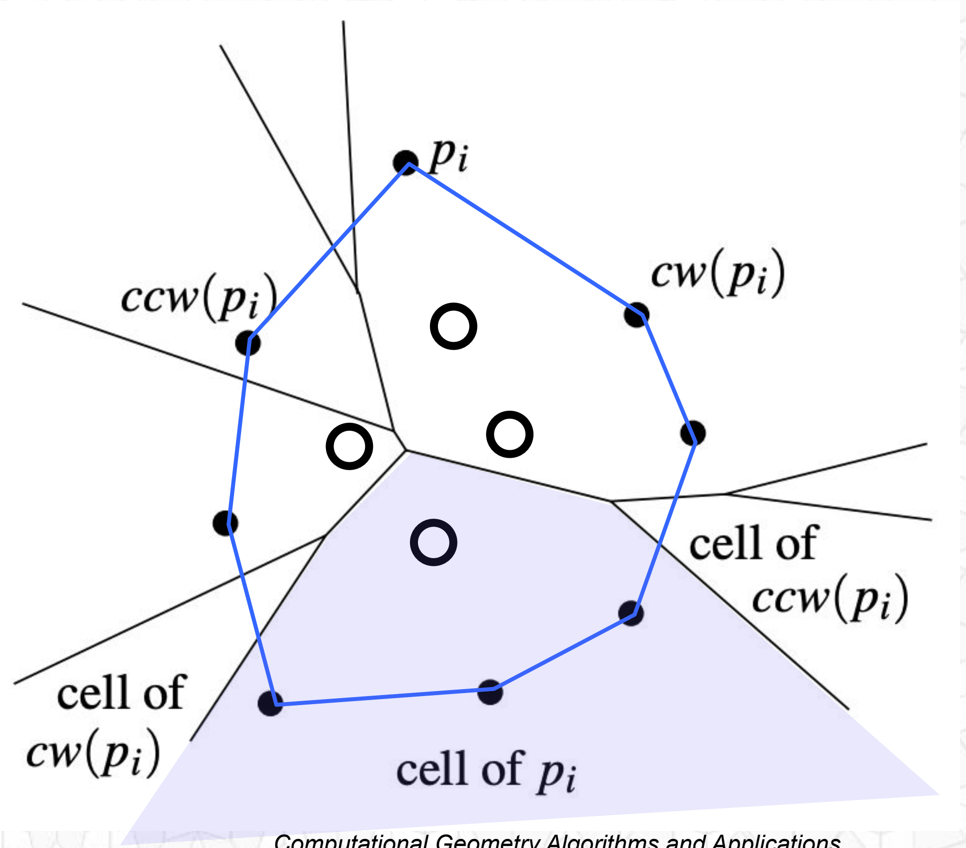


Farthest-Point Voronoi Diagram

- Observation: Only sites on the convex hull will have a cell in the farthest point diagram.

- Observation: All farthest-point cells are *unbounded.*

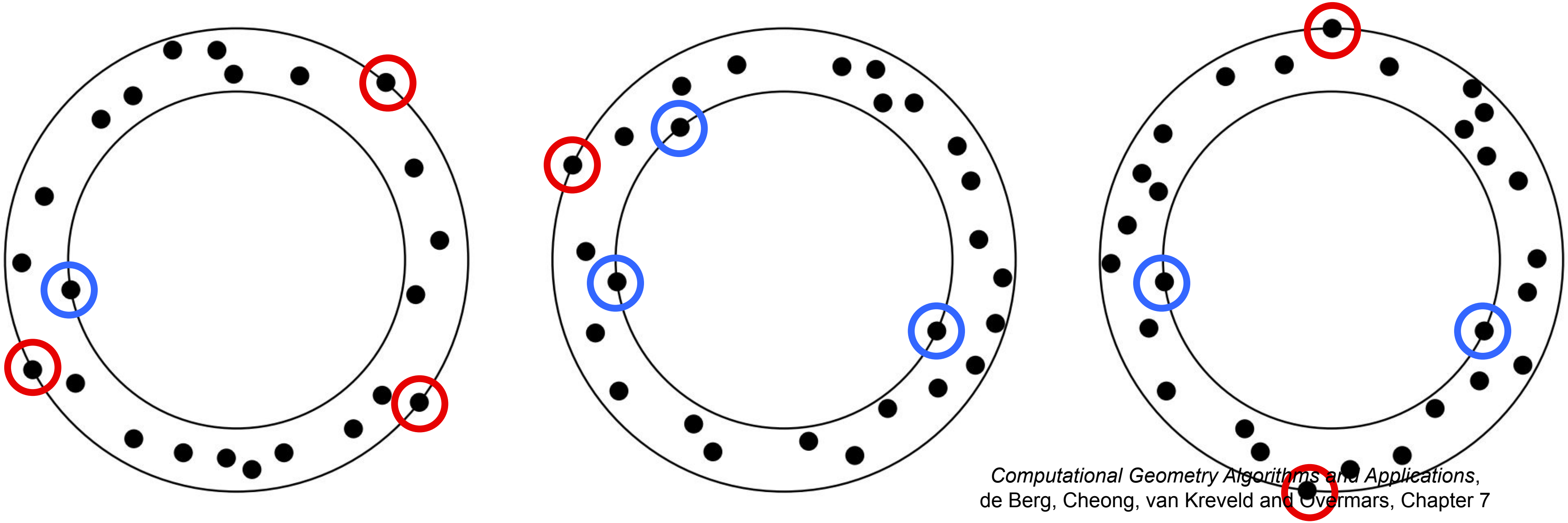
- Observation: The diagram is a tree – no cycles!
  If there were a cycle, that would mean we had a bounded cell.



Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 7

Finding the Smallest-Width Annulus

- Easy to compute once we know the center
  (it is the center of both the inner & outer circle)
- *What points might be the center? Any point on the plane?*



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de Berg, Cheong, van Kreveld and Overmars, Chapter 7

Finding the Smallest-Width Annulus

- Easy to compute once we know the center
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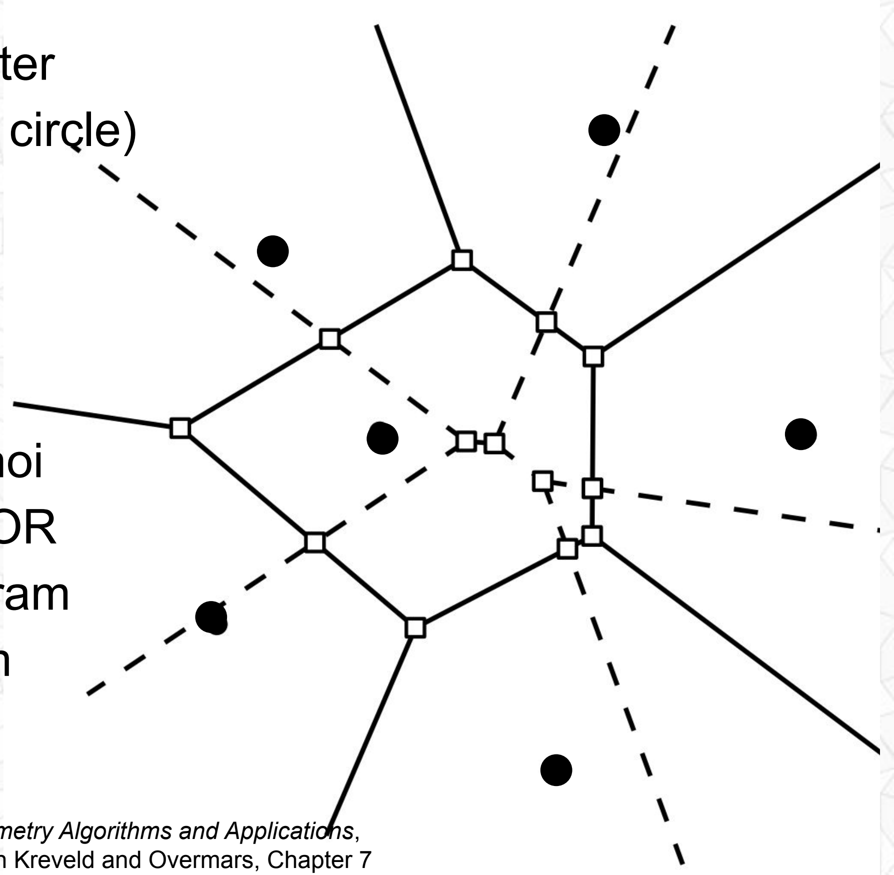
- *What points might be the center?*
  It must be:
  - A vertex of the Voronoi Diagram
    (equally close to 3 sites) OR
  - A vertex of the Farthest Point Voronoi Diagram (equally far from 3 sites) OR
  - An intersection of the Voronoi Diagram and Farthest Point Voronoi Diagram
    (equally close to 2 sites AND
    equally far from 2 sites)



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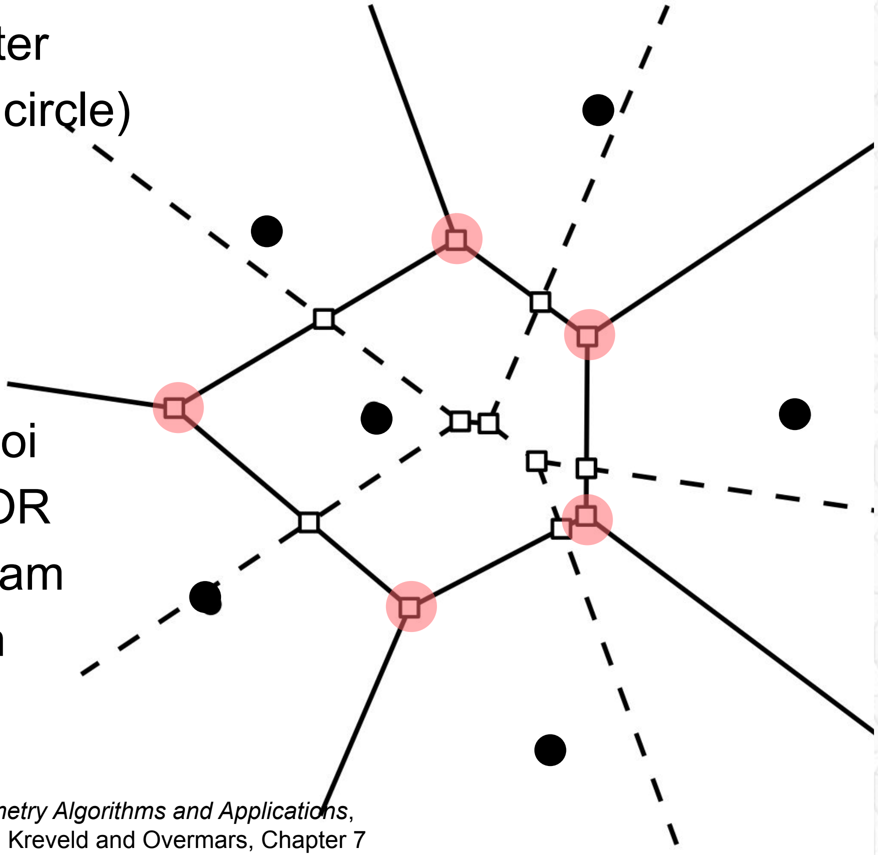

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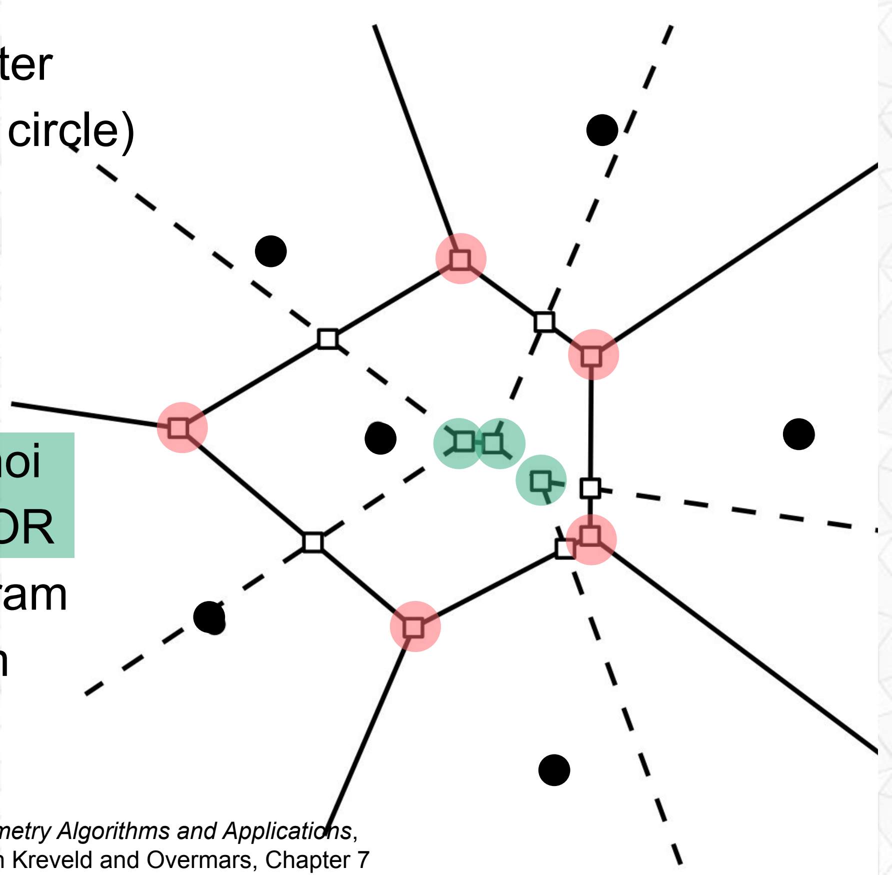

Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 7

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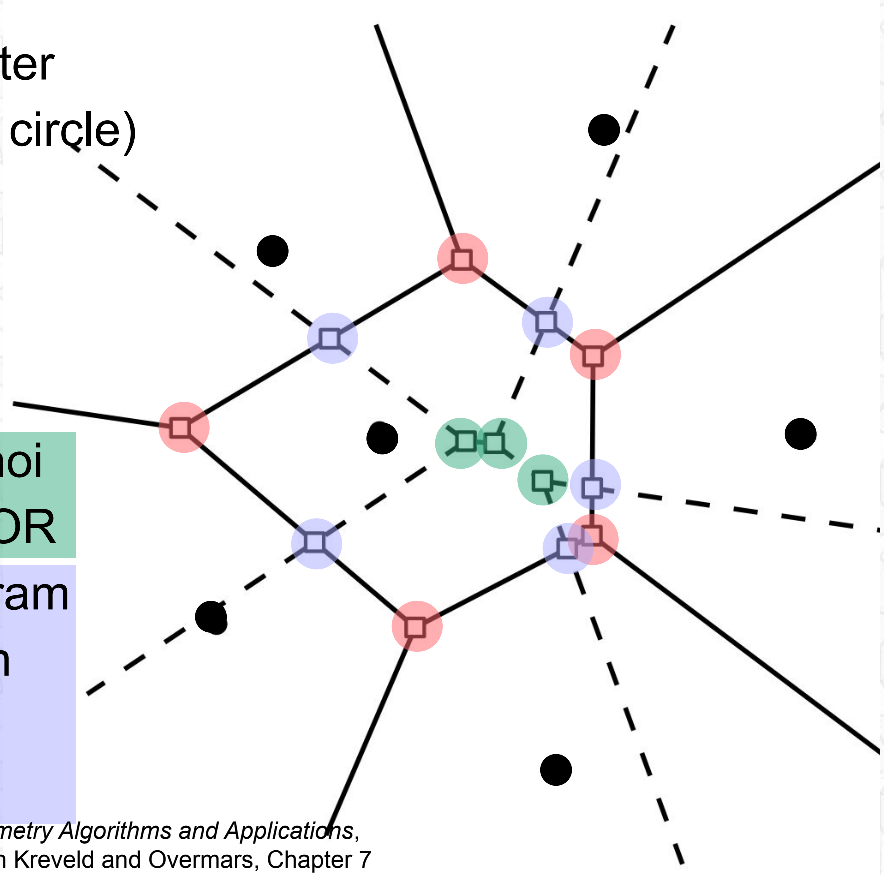

Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 7

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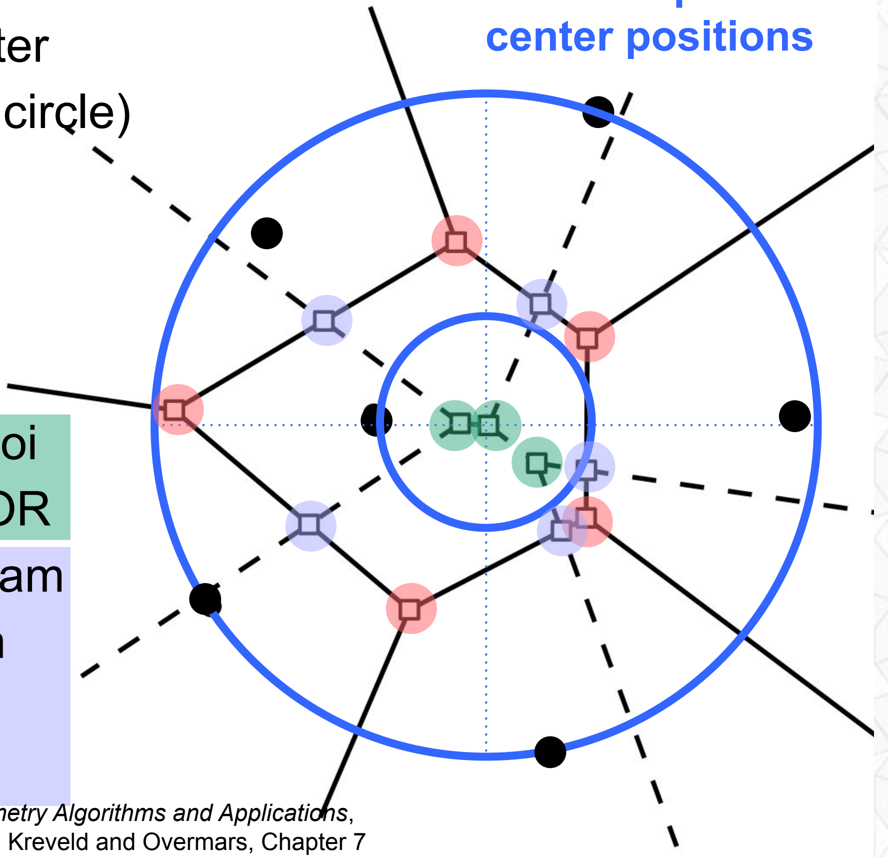
Brute force check of *O(n)* = a FINITE number of possible center positions



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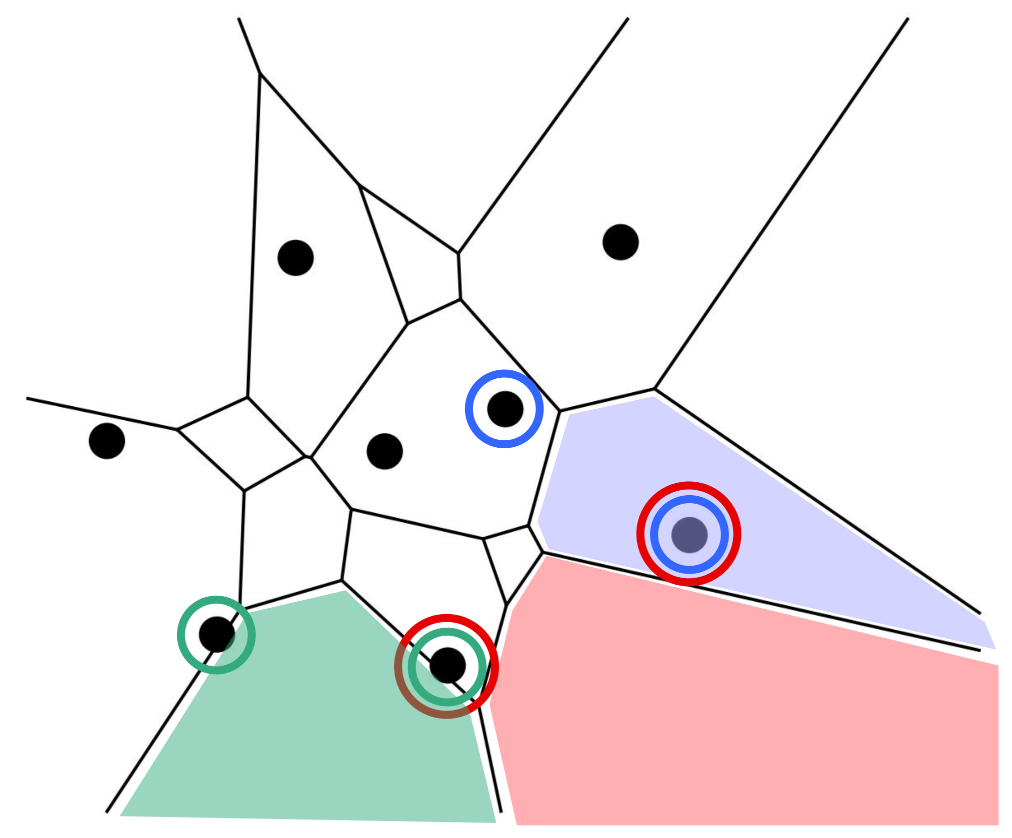
Higher-Order Voronoi / k-Closest Sites

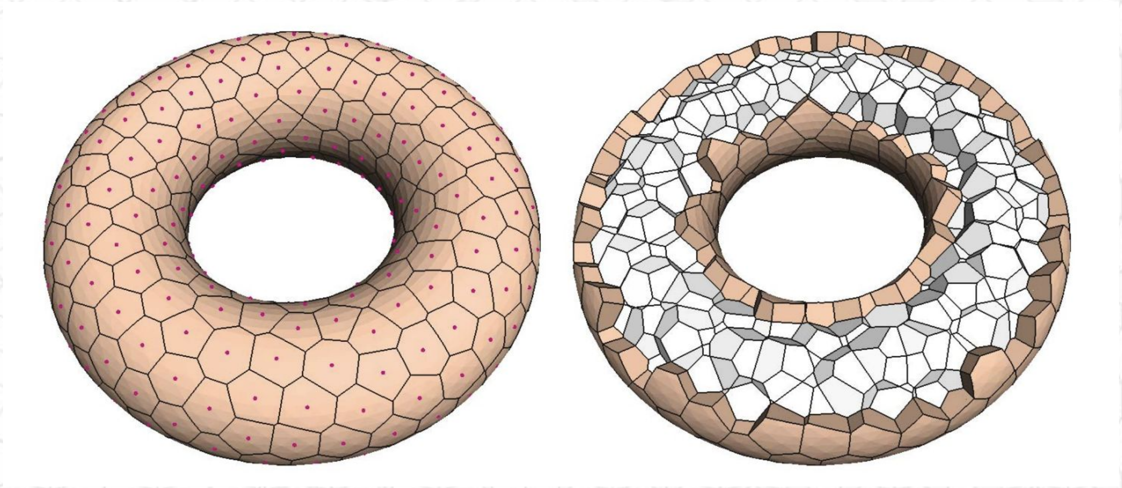
- For example, k = 2…
- Subdivide the plane into regions that have the same closest *and* second closest sites



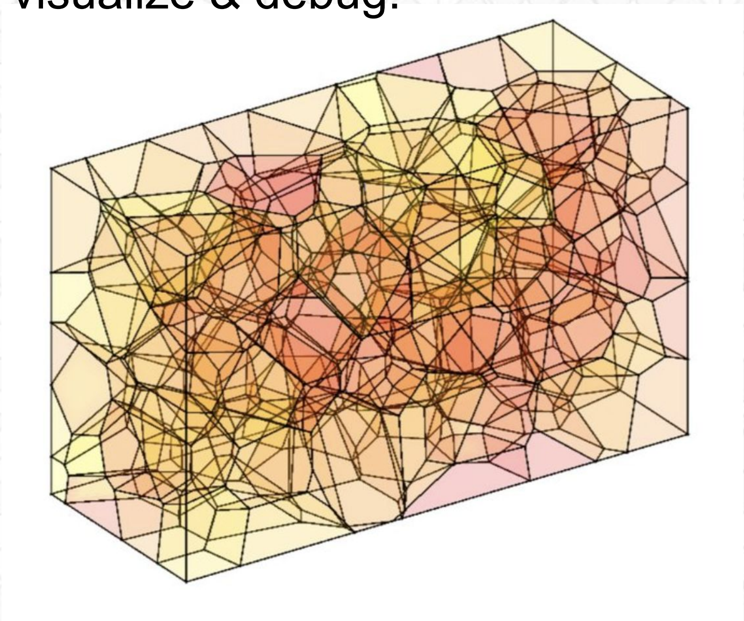
Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 7

Voronoi Diagram in 3D or Higher Dimension

- *Not the same as "Higher-Order Voronoi Diagram"*
- Well defined in higher dimensions, but hard to visualize & debug!
- Each Voronoi cell is convex!



"Efficient Computation of Clipped Voronoi Diagrams for Mesh Generation", Yan, Wang, & Liu, 2011



"Simulation and Optimization of Porous Bone-Like Microstructures with Specific Mechanical Properties", Wit, Wronski, & Tarasiuk, 2019

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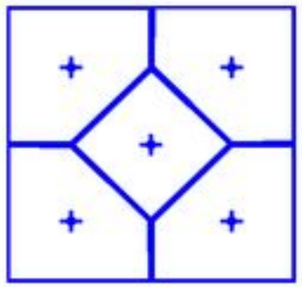
Centroidal Voronoi Diagram



- *What if we could place all of the grocery stores?*
- Where should we place the grocery stores so that they are centrally located for all of their customers?

https://en.wikipedia.org/wiki/Centroidal_Voronoi_tessellation

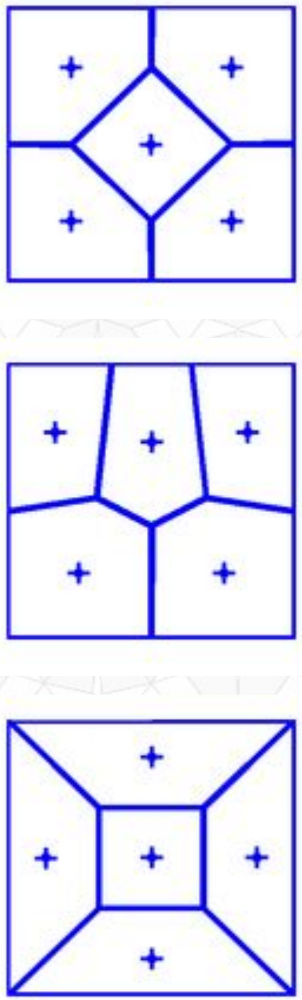
Centroidal Voronoi Diagram

- *What if we could place all of the grocery stores?*
- Where should we place the grocery stores so that they are centrally located for all of their customers?
- But if you change the position of the store, the closest store will change for some customers…

- Points are at the center of mass of their cell
- Constructed using k-means clustering / Lloyd’s algorithm - an iterative relaxation algorithm

- *Note: May be multiple solutions!*

https://en.wikipedia.org/wiki/Centroidal_Voronoi_tessellation



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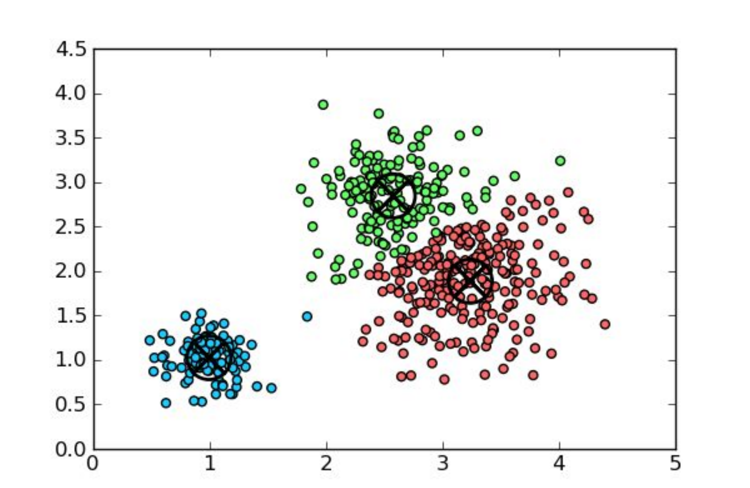
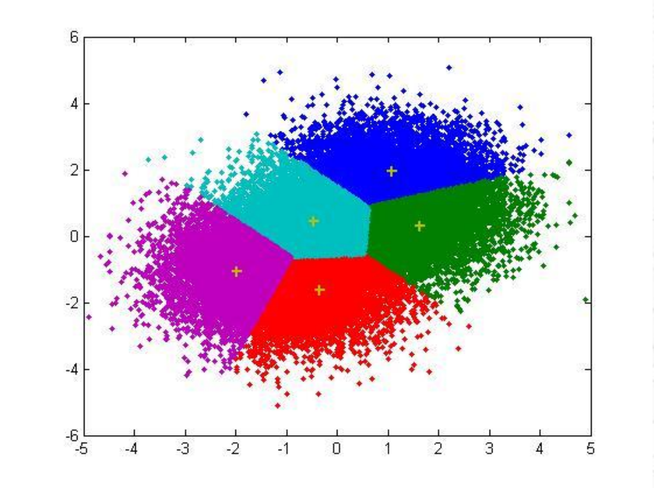
K-Means Clustering

- Works quite well, when the data can be meaningfully classified (and we know how many clusters to use).
- With dense data, output is visually similar to Voronoi diagram (k-Means chooses the data points that define the cells)



http://blog.mpacula.com/2011/04/27/k-means-clustering-example-python/



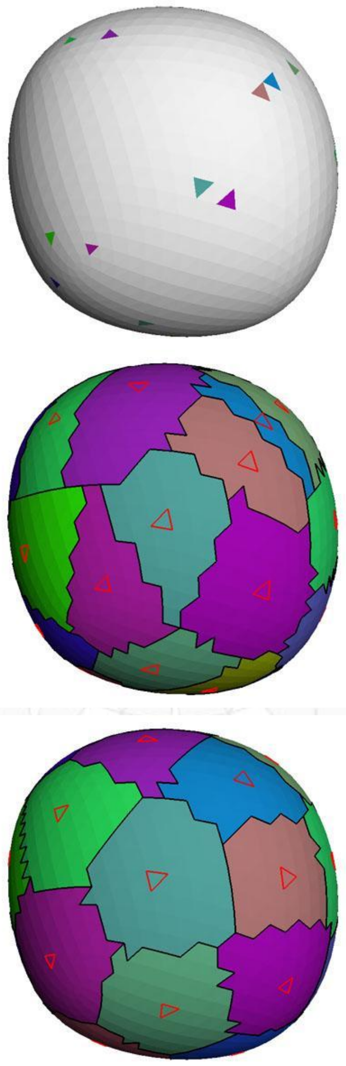
"Efficient K-Means Clustering using JIT" Yi Cao

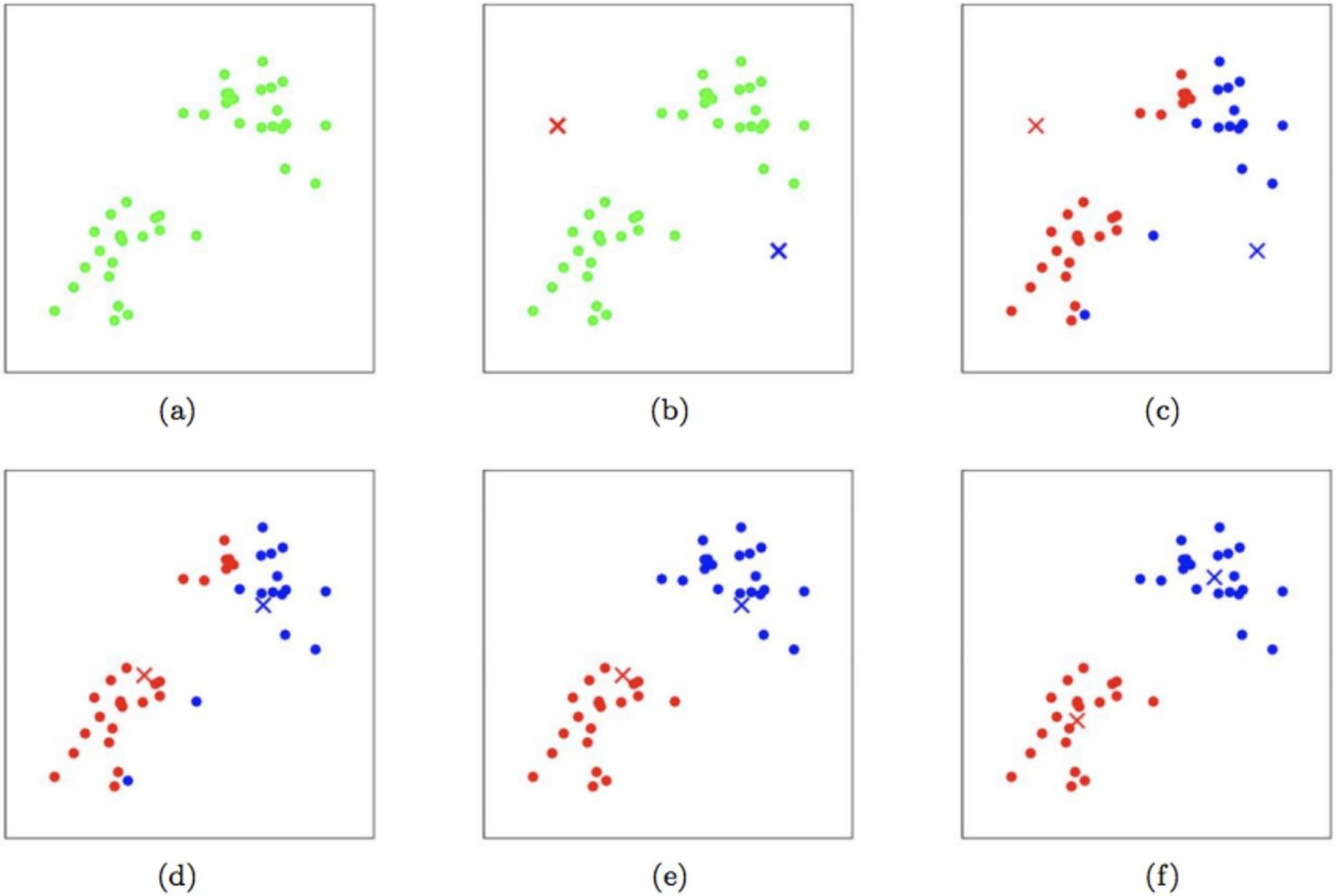
K-Means Clustering

For a set of 2D/3D/*n*D points:

- Choose *k*, # of clusters (maybe an “oracle” tells us...)
- Select *k* points from your data at random as initial team representatives
- Every other point determines which team representative it is closest to and joins that team
- The team averages the positions of all members, this is the team’s new representative
- Repeat *x* times or until change < threshold

Same/Similar to: Lloyd’s Algorithm





Wei Zhang
https://wei2624.github.io/MachineLearning/usv_kmeans/

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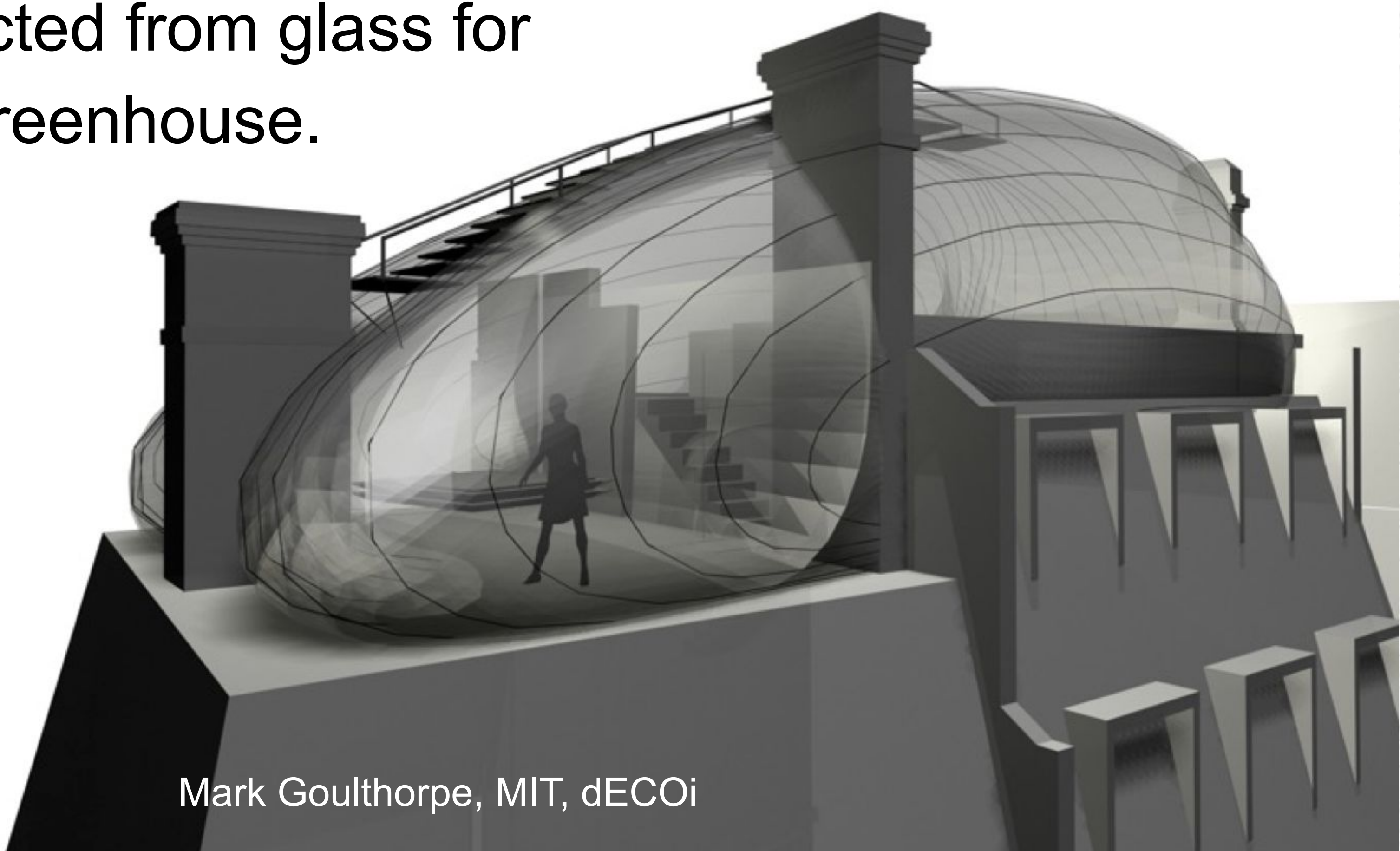
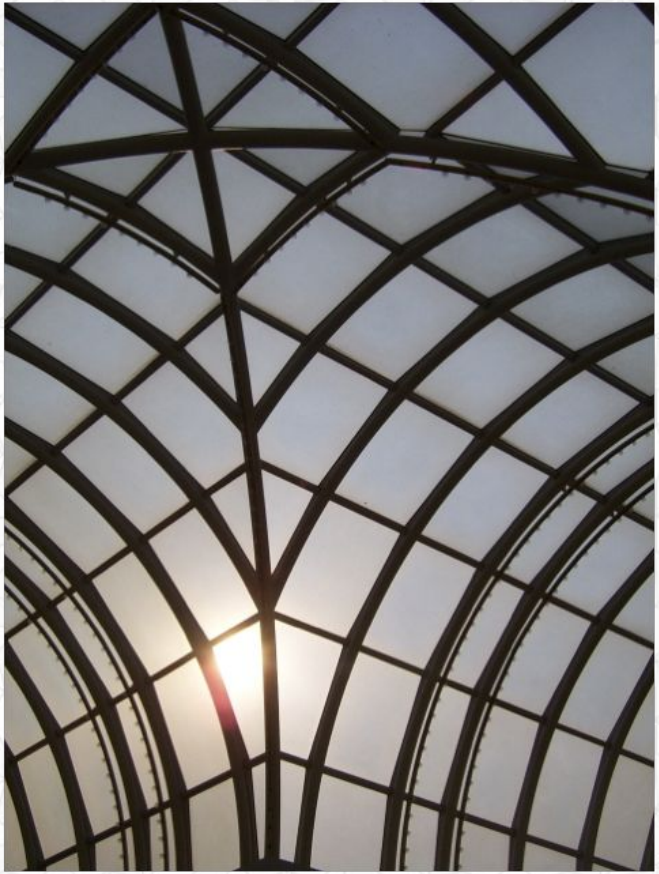
The Problem: Mesh this curved surface so that it can be constructed from glass for a rooftop greenhouse.



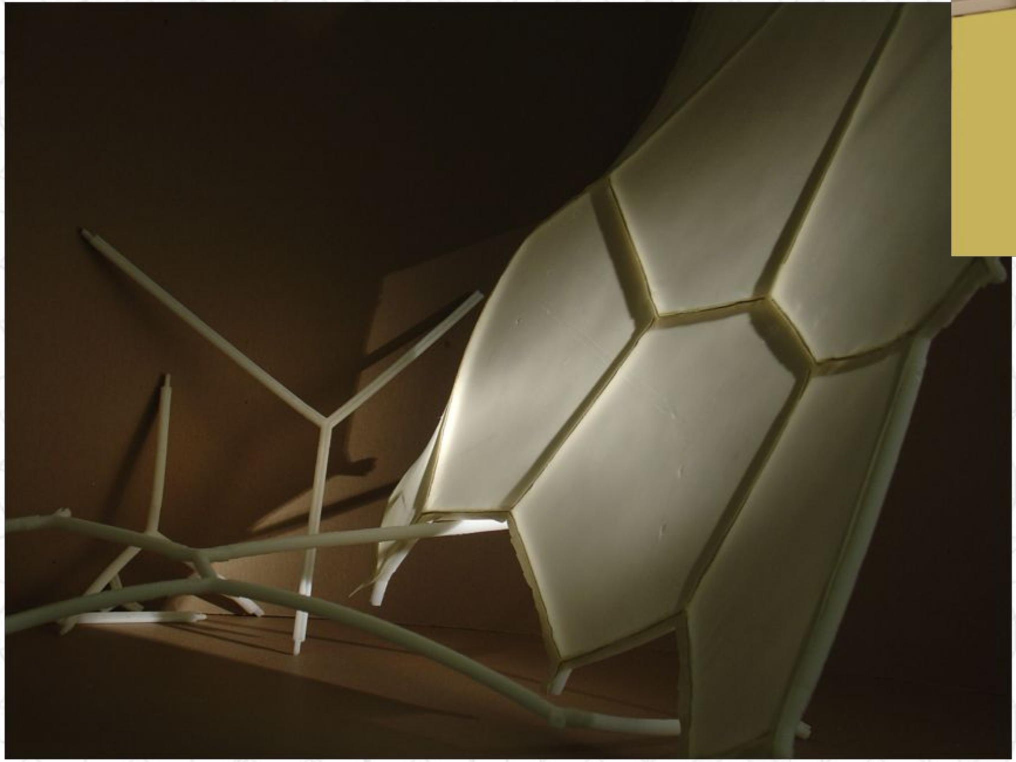
Mark Goulthorpe, MIT, dECOi

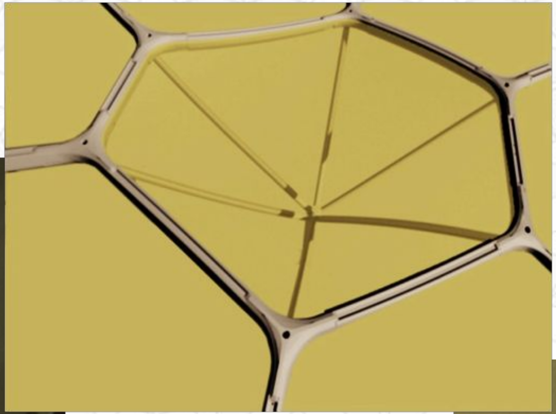
Great Court at the British Museum
London, England
Norman Foster and Partners, 2000.

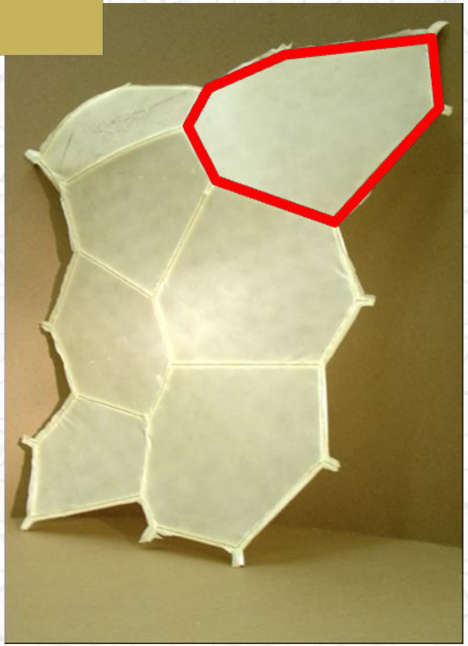


Chadstone Shopping Center
Melbourne, Australia
RTKL Associates Inc, 1999.

“Voronoi Surfaces”





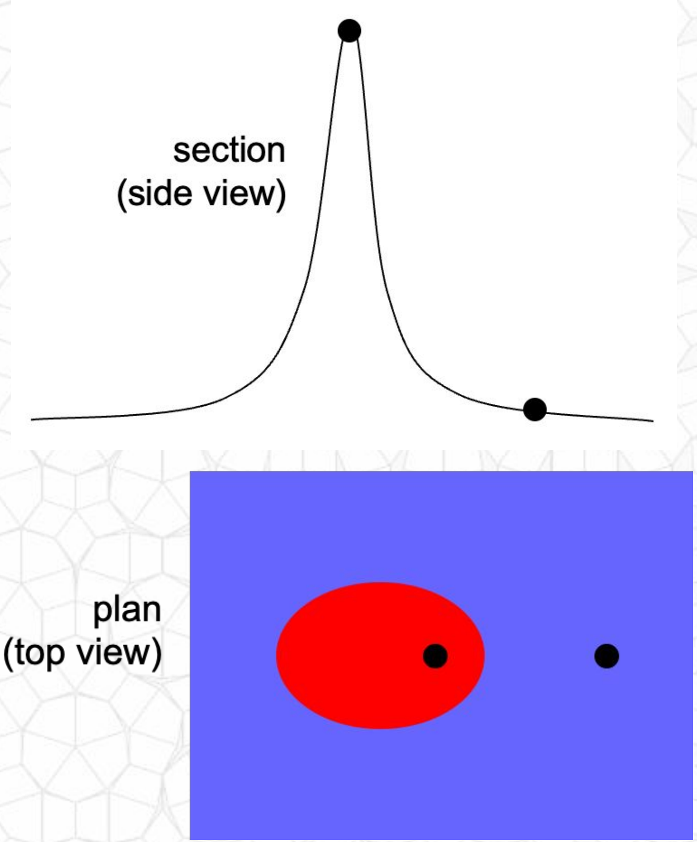


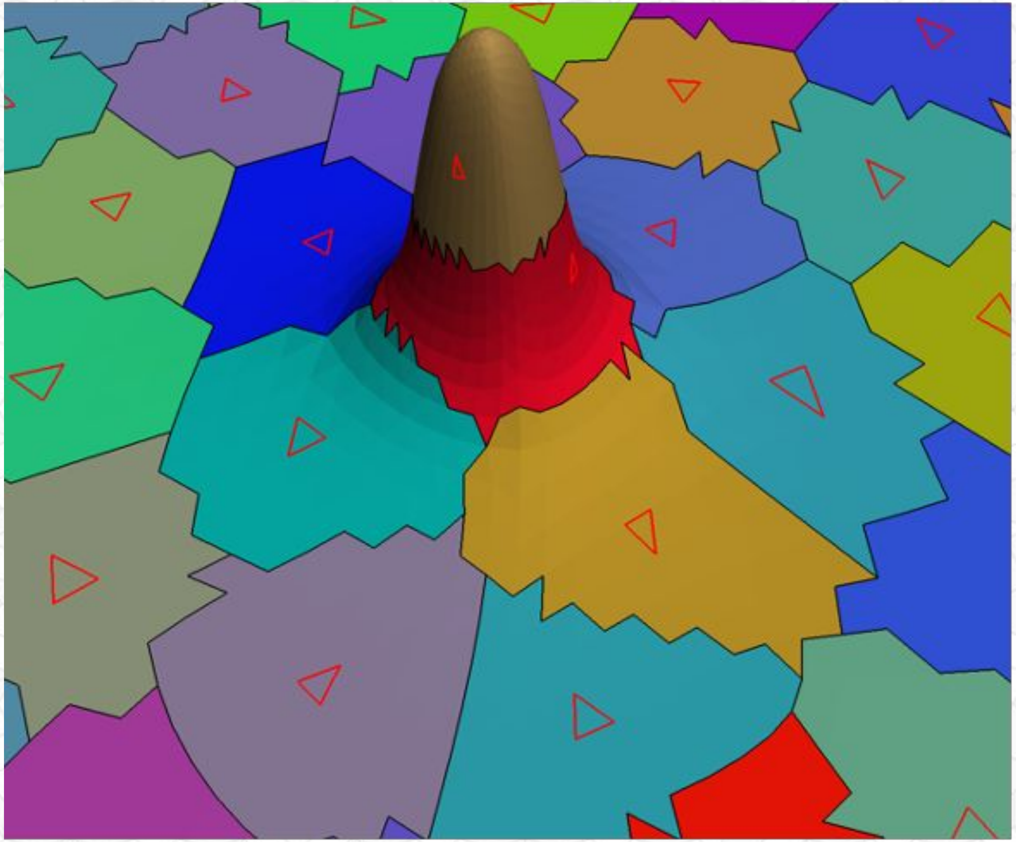
Mike Powell, MIT
Studio Project Fall 2004

Unfortunately,
the cell vertices
are rarely planar!

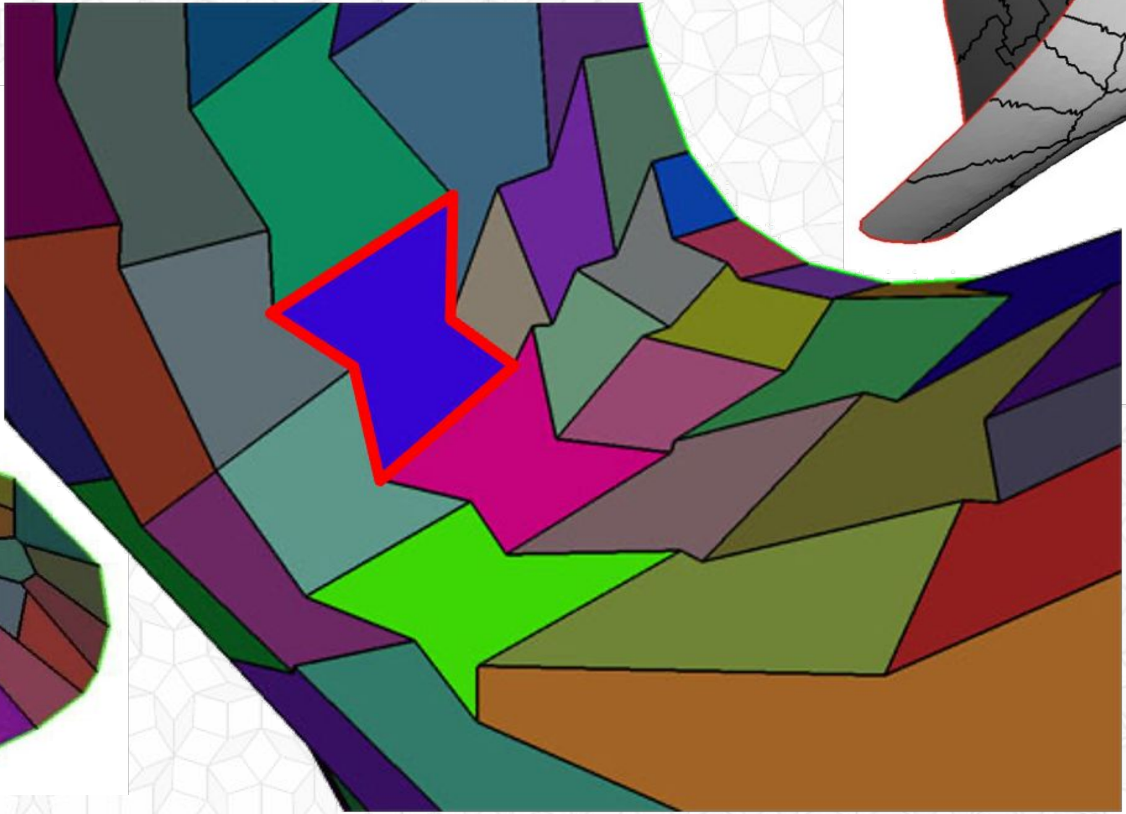
Voronoi Diagram on a 3D Surface

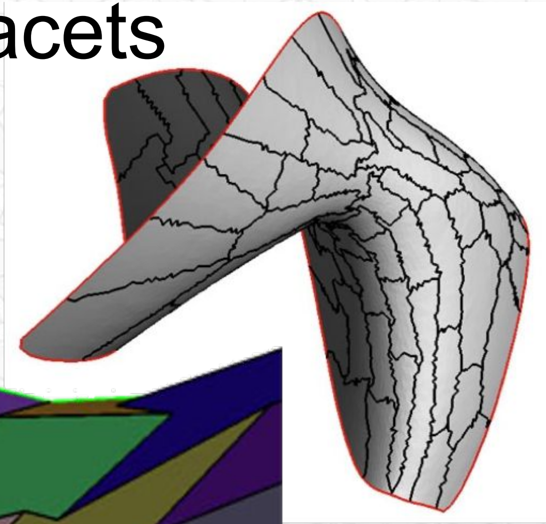
We're no longer using *Euclidean Distance!*





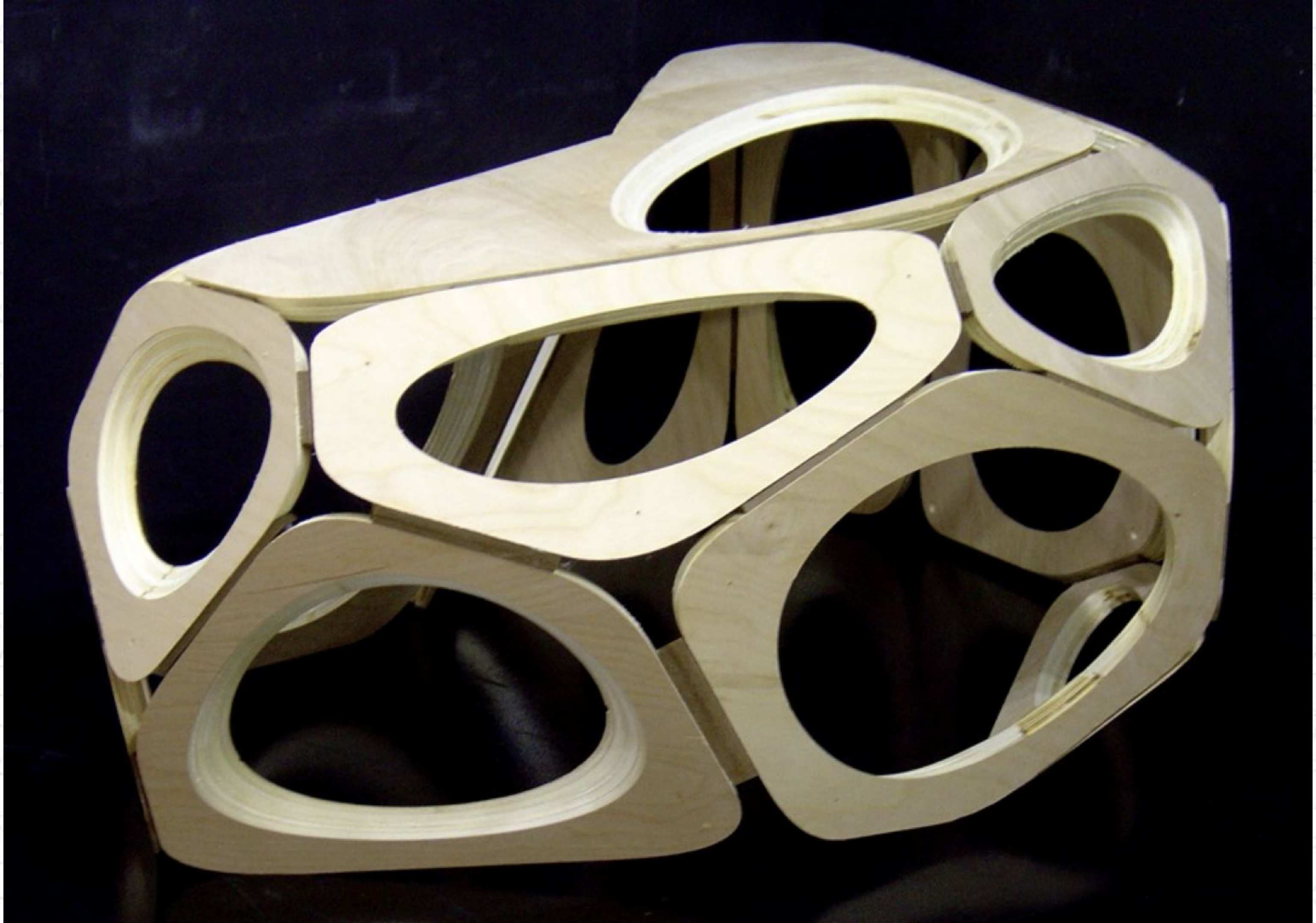
“Saddle Surface” → Non-Convex Facets



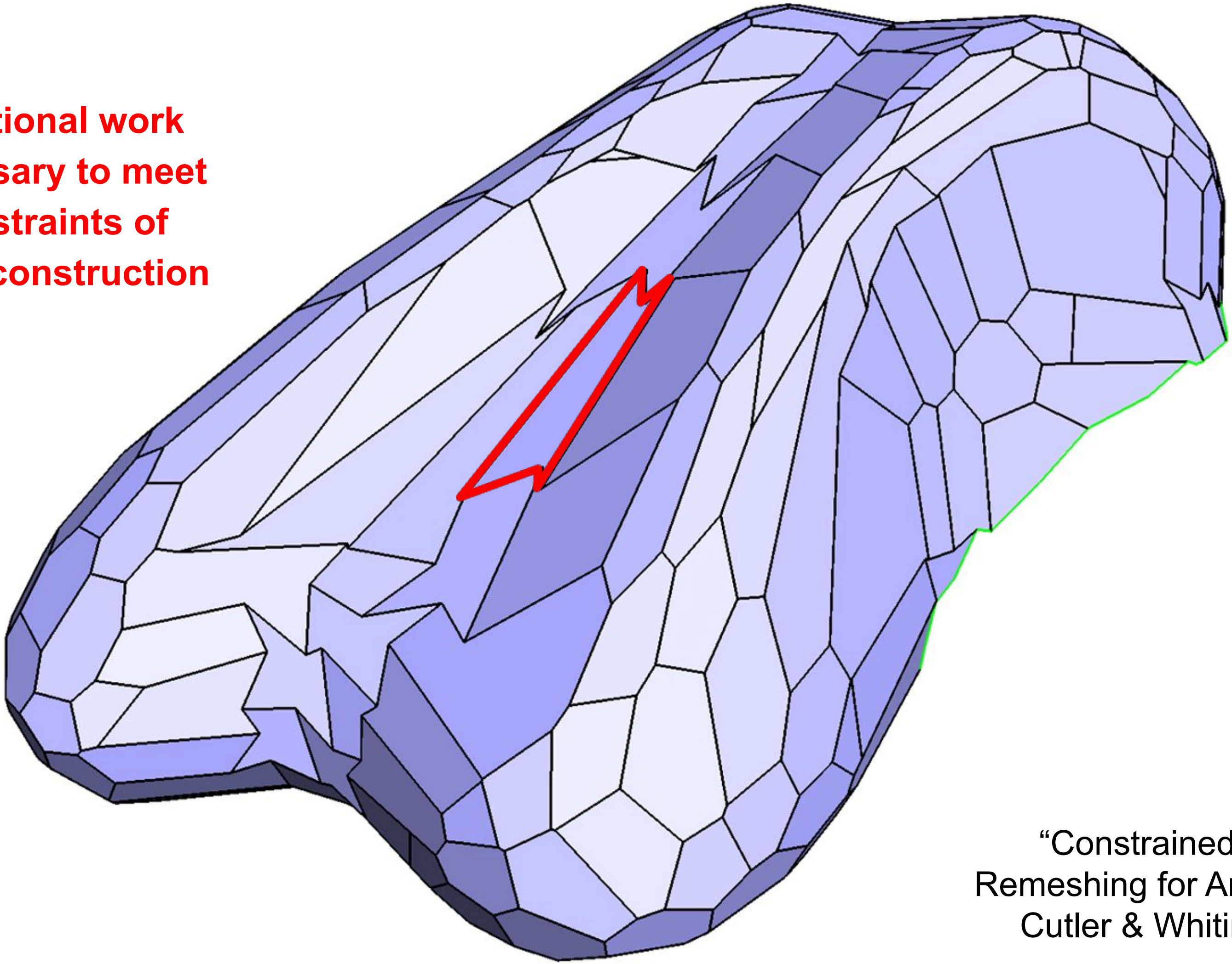


“bowtie” shapes

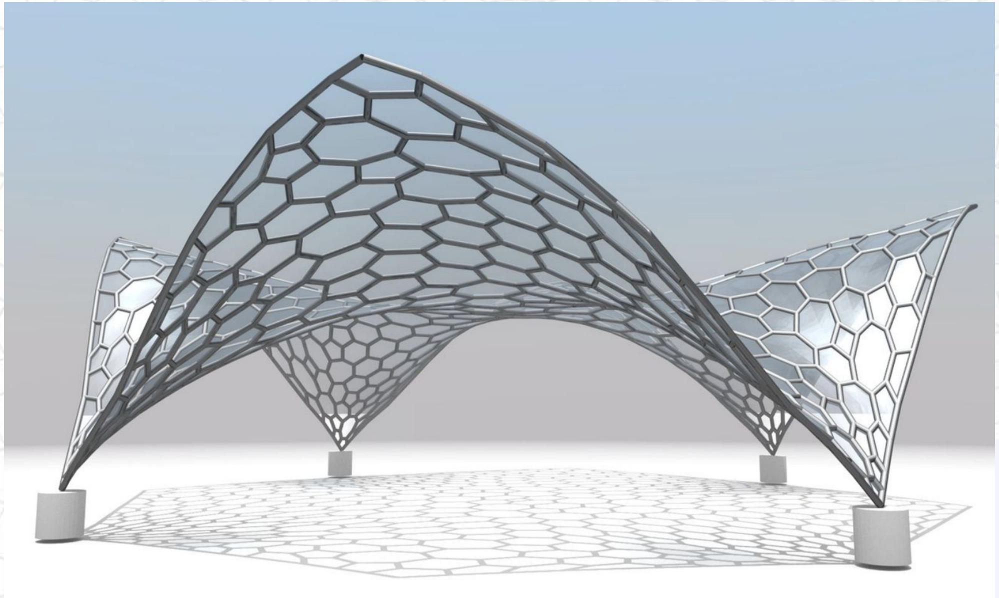
Fabrication

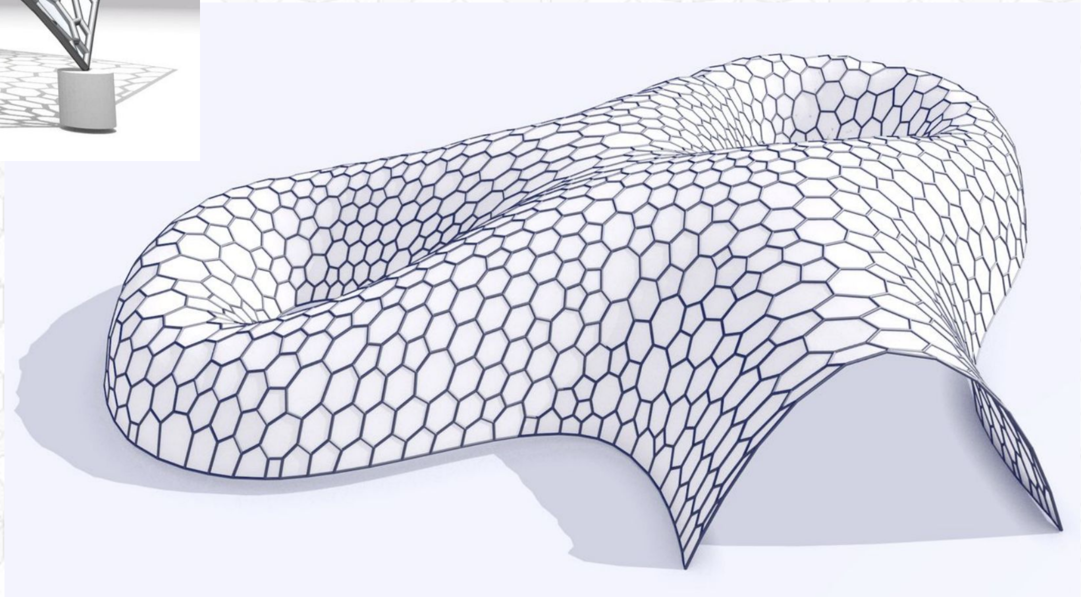


Additional work necessary to meet constraints of glass construction

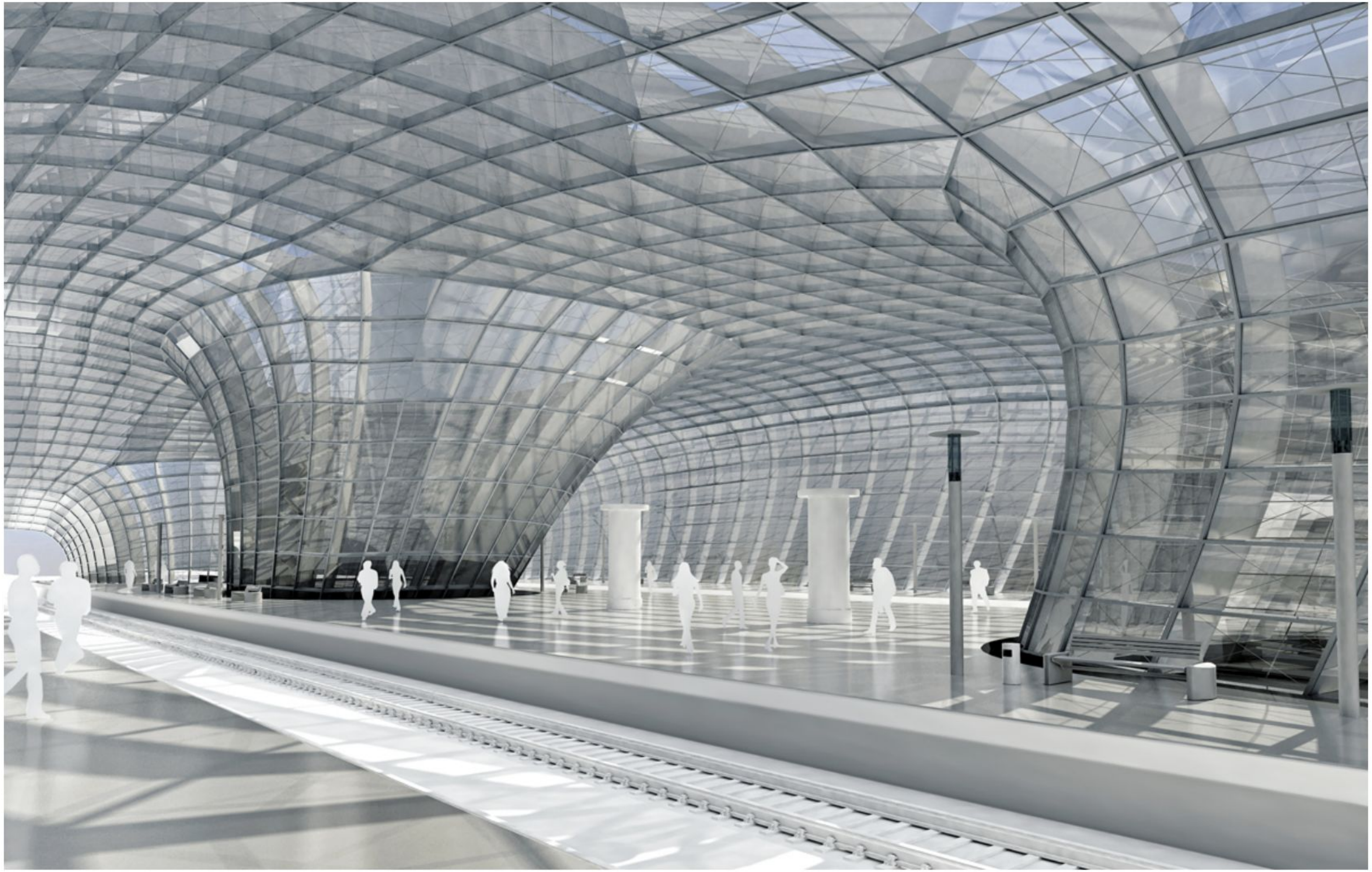


"Constrained Planar Remeshing for Architecture", Cutler & Whiting, 2007



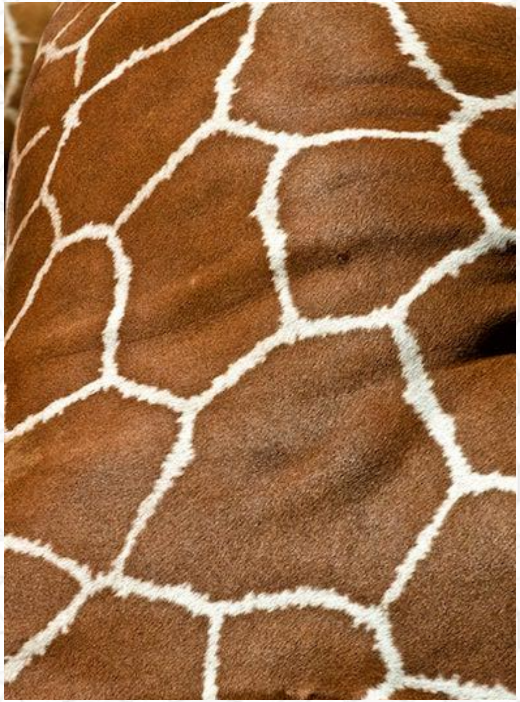


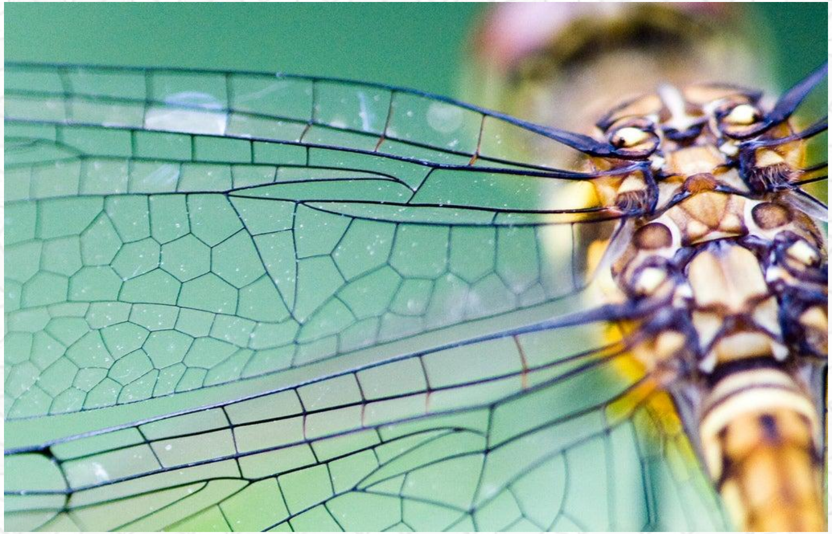
“Voronoi Grid-Shell Structures”
Pietroni, Tonelli, Puppo, Froli,
Scopigno, & Cignoni, 2014



“Geometric Modeling with Conical Meshes and Developable Surfaces”
Liu, Pottmann, Wallner, Yang &Wang, SIGGRAPH 2006

Voronoi Diagram in Nature



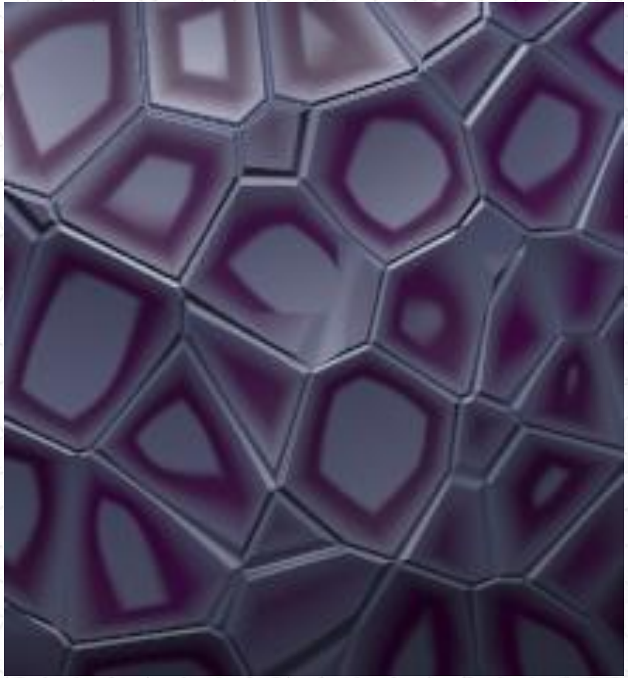


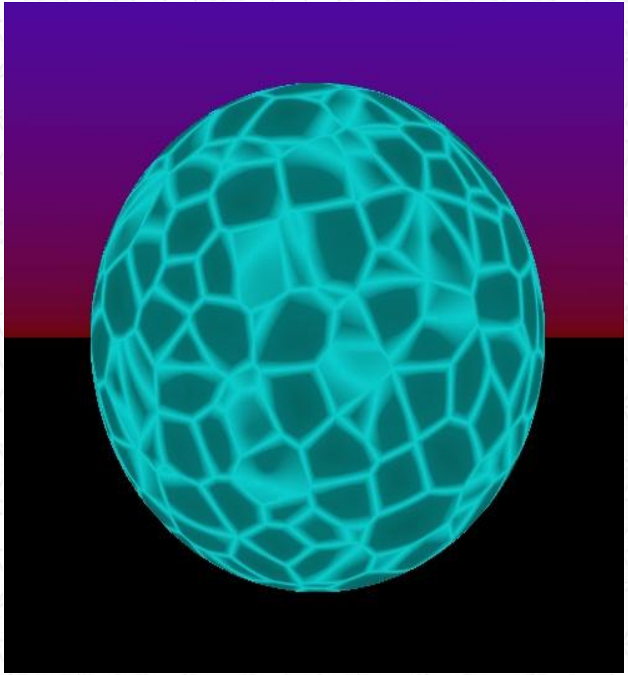
https://spring-of-mathematics.tumblr.com/post/85519358219/the-beauty-of-voronoi-diagram-in-nature-how

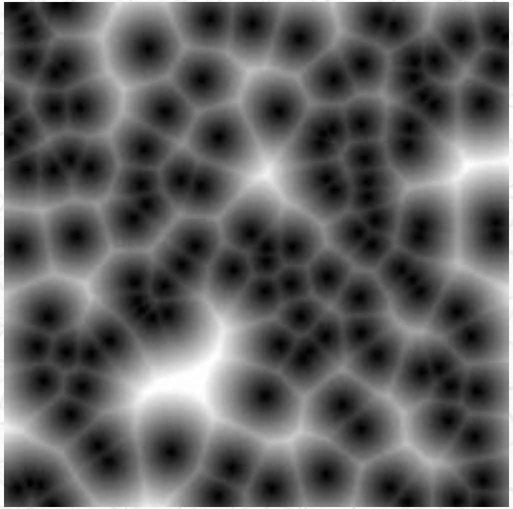
https://blogs.scientificamerican.com/observations/voronoi-tessellations-and-scutoids-are-everywhere/

Cellular Textures

https://en.wikipedia.org/wiki/Worley_noise#/media/File:Worley.jpg







“A Cellular Texture Basis Function”,
Worley, SIGGRAPH 1996

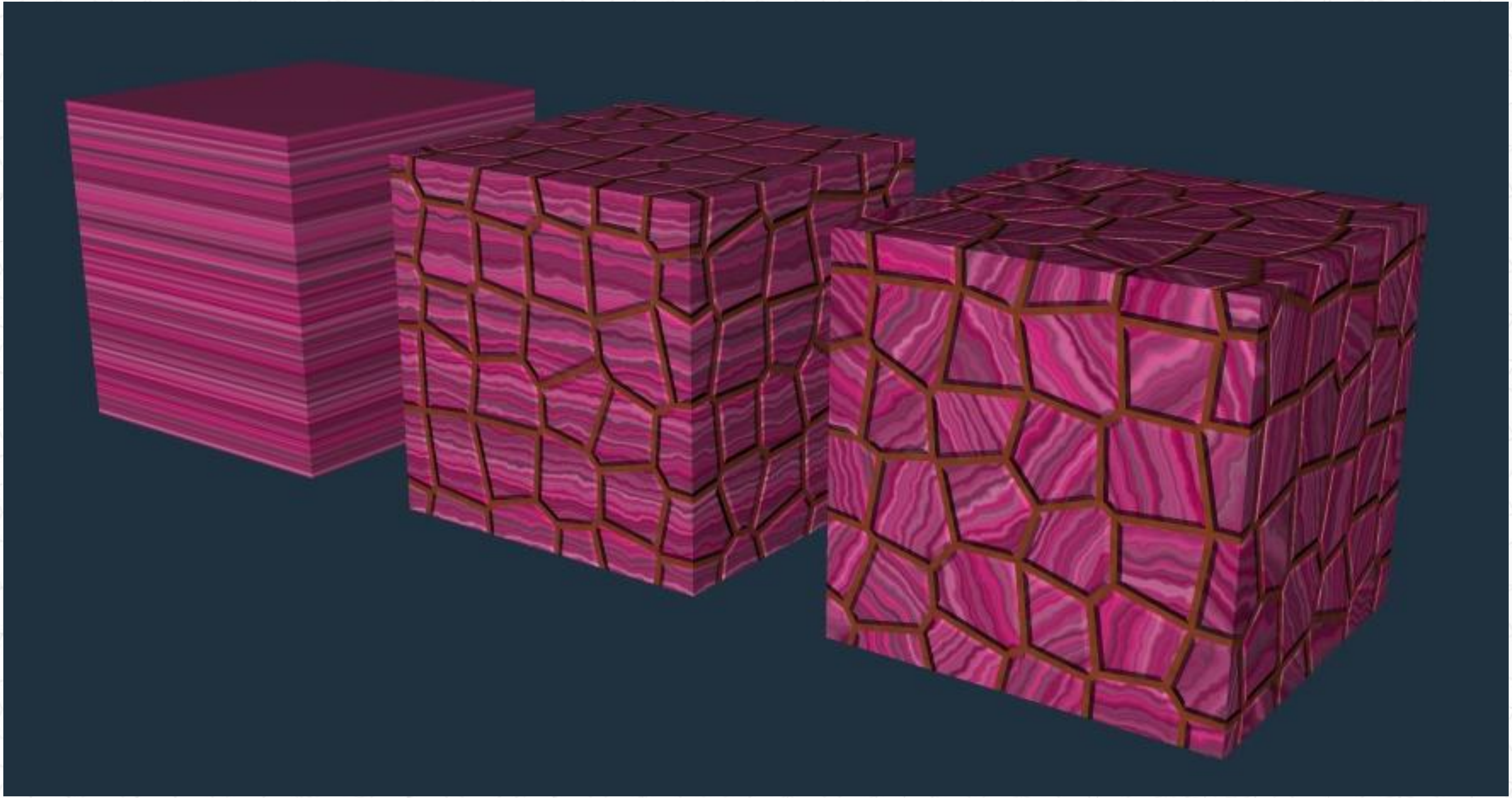


Image by Justin Legakis

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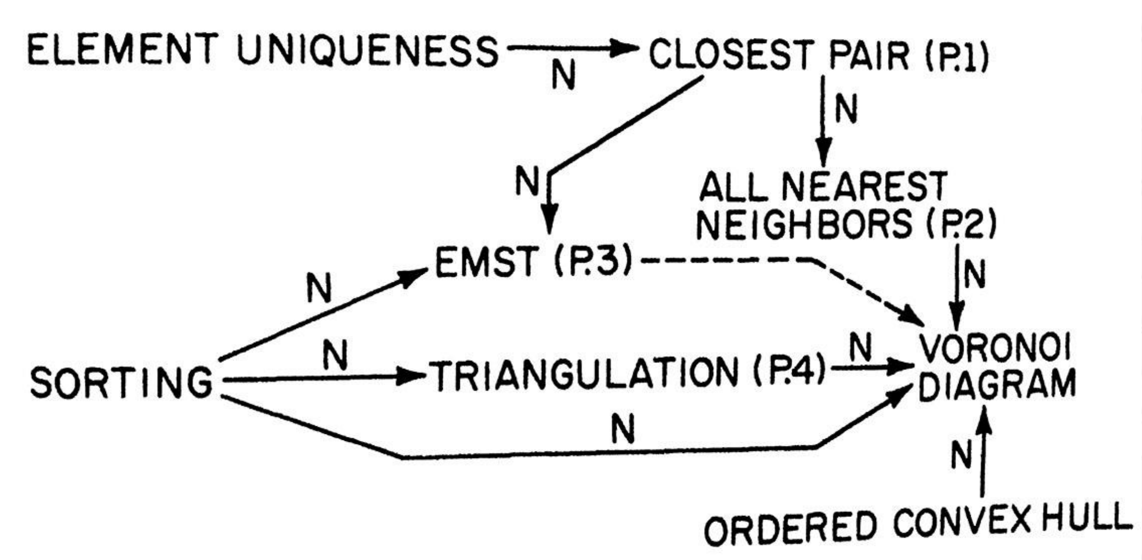
Problems that Reduce to Voronoi Diagram

- We can compute the Voronoi Diagram of *n* points in *O(n log n)* time and *O(n)* space.
- These other problems can be computed in *O(n)* additional time if given the Voronoi Diagram.

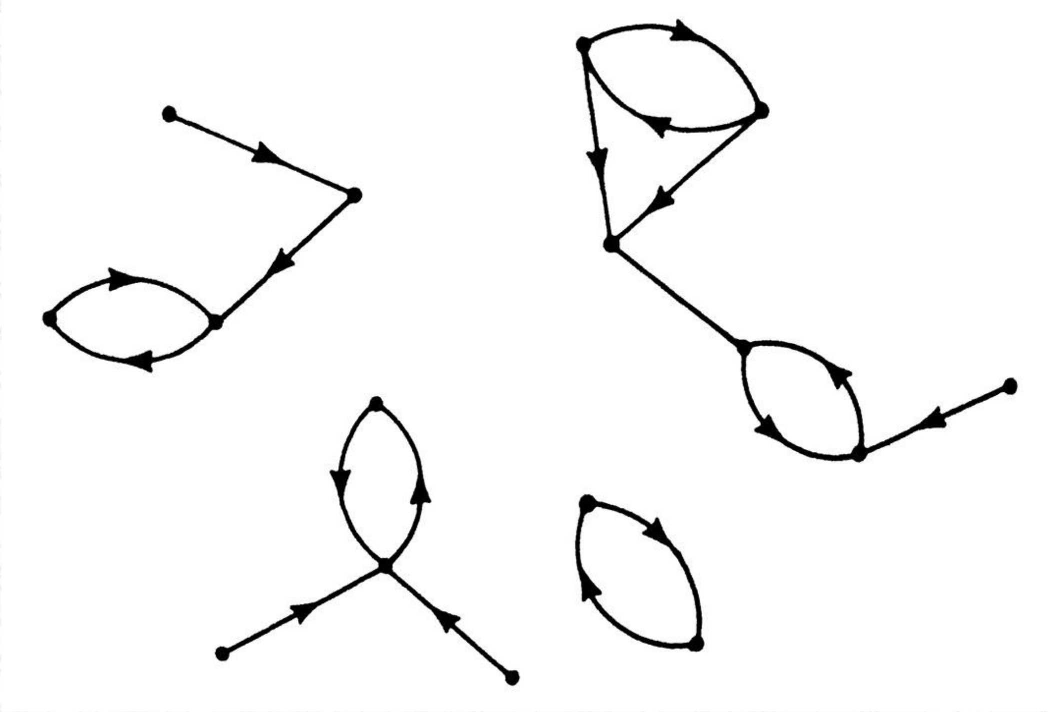


Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.30

- *Therefore they are also O(n log n) time and O(n) space.*

Problem: All Nearest Neighbors

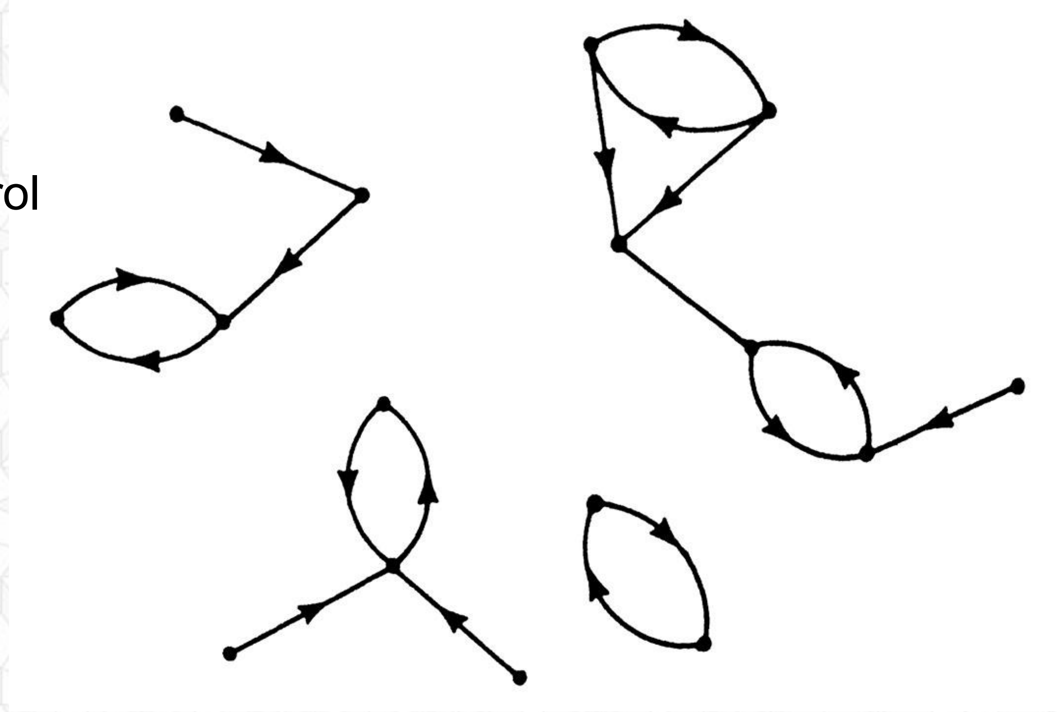
- Connect every point to its nearest point with a directed edge.
- Some points form a *reciprocal pair.*



Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.1

Problem: Closest Pair

- Which two points are the closest?
- Applications - Collision Detection & Air Traffic Control
- Which two objects have soonest potential for collision?

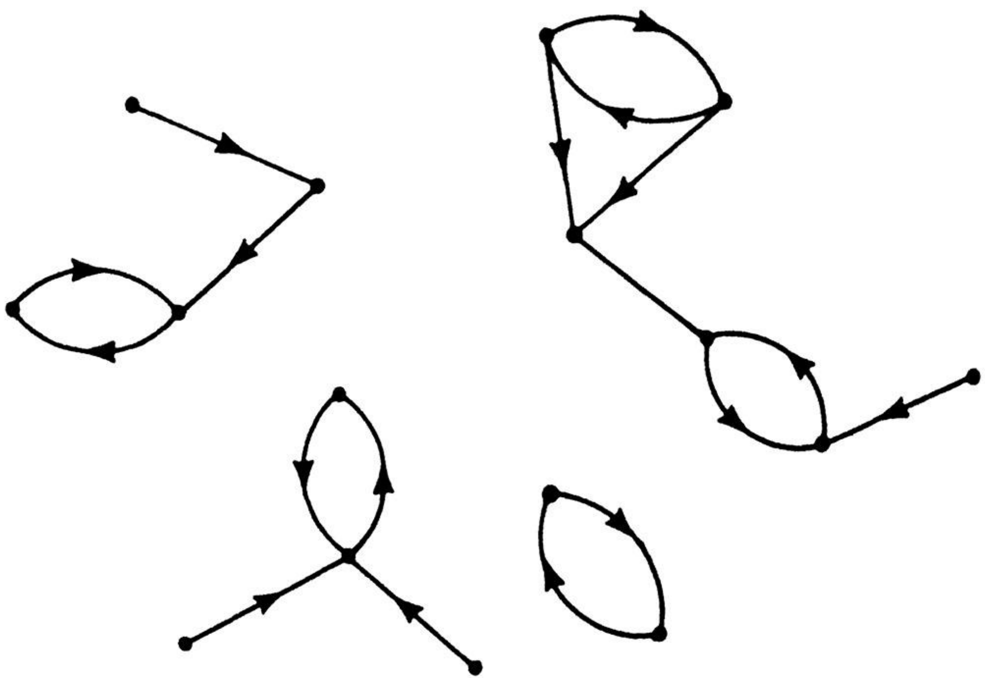


Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.1

Problem: Closest Pair

- Which two points are the closest?
- Applications - Collision Detection & Air Traffic Control
- Which two objects have soonest potential for collision?

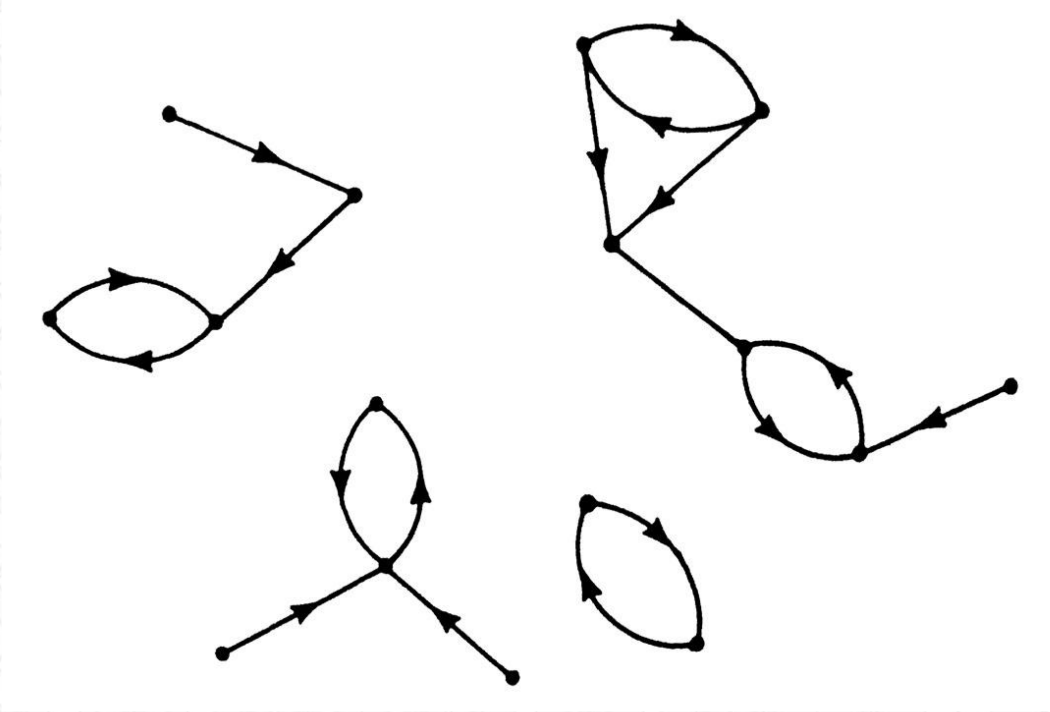
- Linear loop over all edges in the All Nearest Neighbors solution to find the shortest edge
- Will be a reciprocal pair



Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.1

Problem: Uniqueness

- Given *n* numbers, decide if any two are unique.

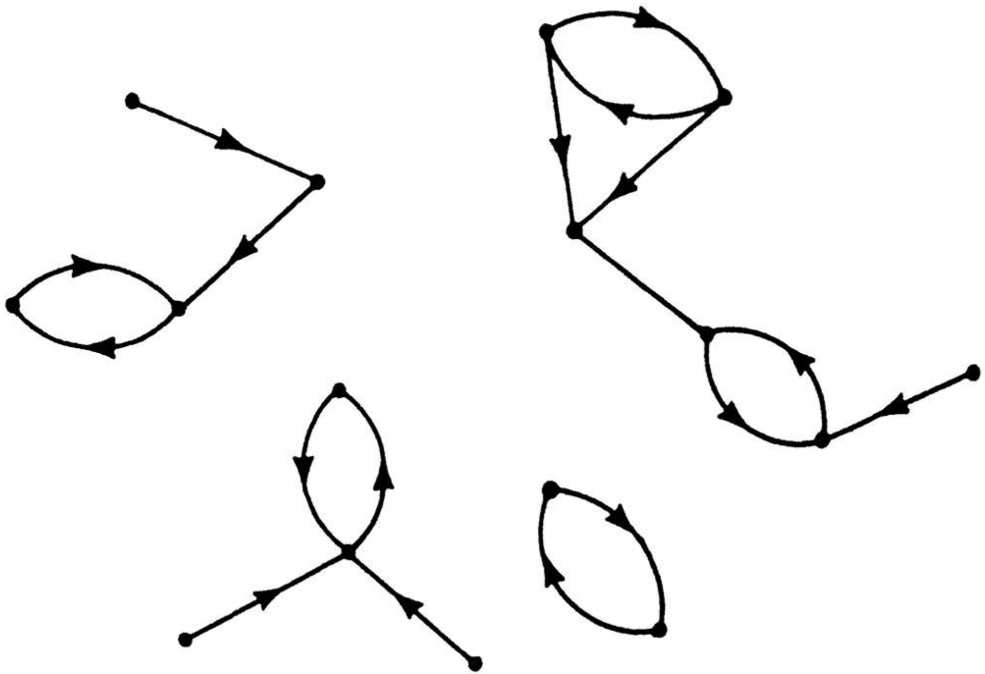


Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.1

Problem: Uniqueness

- Given *n* numbers, decide if any two are unique.

- Linear loop over all edges in the All Nearest Neighbors solution to if any edges are length zero.



Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.1

Problem: Euclidean Minimum Spanning Tree

- Given *n* points
- Draw *n-1* edges to create a tree, connecting all points without creating any cycles.
- Pick edges to minimize the sum of their lengths.

- *Application: Minimize cost of physical telephone lines*

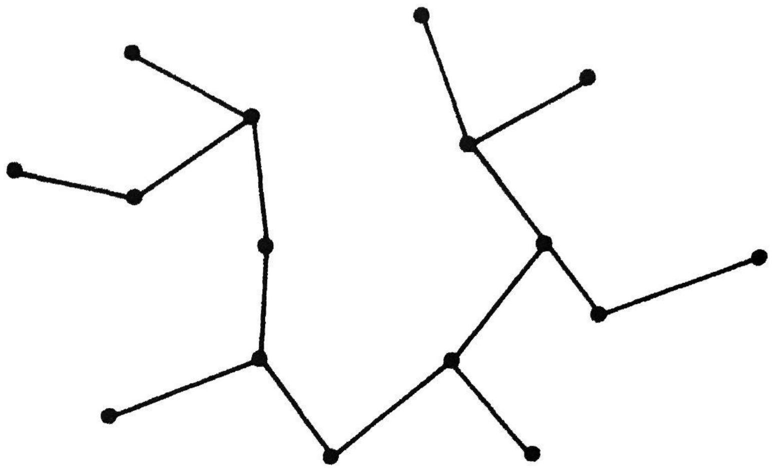


Figure 5.2 A minimum spanning tree on a planar point set.

Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.2

Problem: Minimum Spanning Tree

- *General (non-Euclidean) MST from Graph Theory*
- Each edge has a weight, *not necessarily the Euclidean distance between two points*
- Worst case, may have $m = n^2$ edges to consider
- Runtime $O(m \log n)$ $O(n^2 \log n)$ worst case

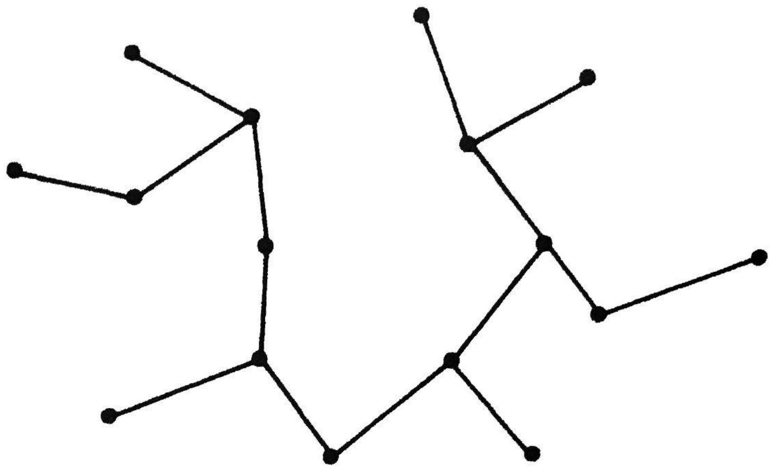


Figure 5.2 A minimum spanning tree on a planar point set.

Computational Geometry: An Introduction, Preparata & Shamos, Figure 5.2

Problem: (Euclidean) Steiner Tree

- If allowed to add additional points – so-called Steiner Points
- Minimize sum of Euclidean distance edge lengths
- *Computing the Steiner Tree is NP Complete / NP hard!*

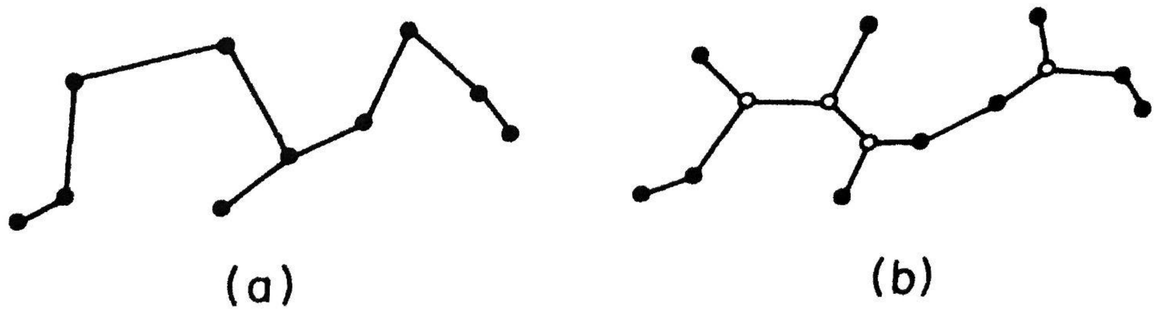
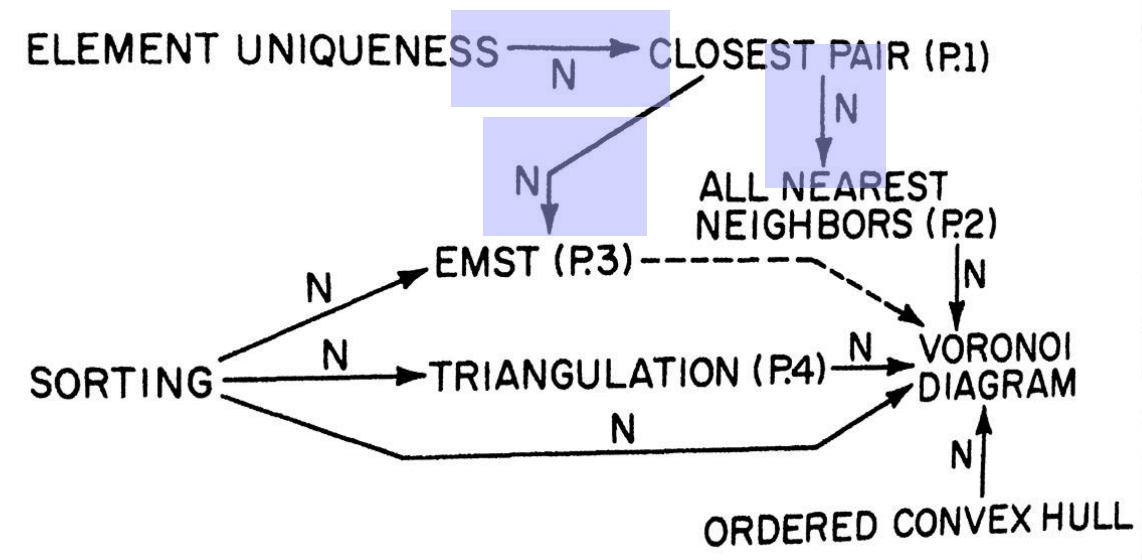


Figure 5.3 A Steiner Tree (b) may have smaller total length than the MST (a).

Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.3

Problems that Reduce to Voronoi Diagram

- We can compute the Voronoi Diagram of *n* points in *O(n log n)* time and *O(n)* space.
- These other problems can be computed in *O(n)* additional time if given the Voronoi Diagram.



Computational Geometry: An Introduction, Preparata & Shamos, Figure 5.30

- *Therefore they are also O(n log n) time and O(n) space.*

Reduce Convex Hull to Voronoi Diagram

- Theorem: Voronoi polygon V_i is unbounded if and only if Voronoi site i is on the convex hull of all sites. (proved in Preparata & Shamos)

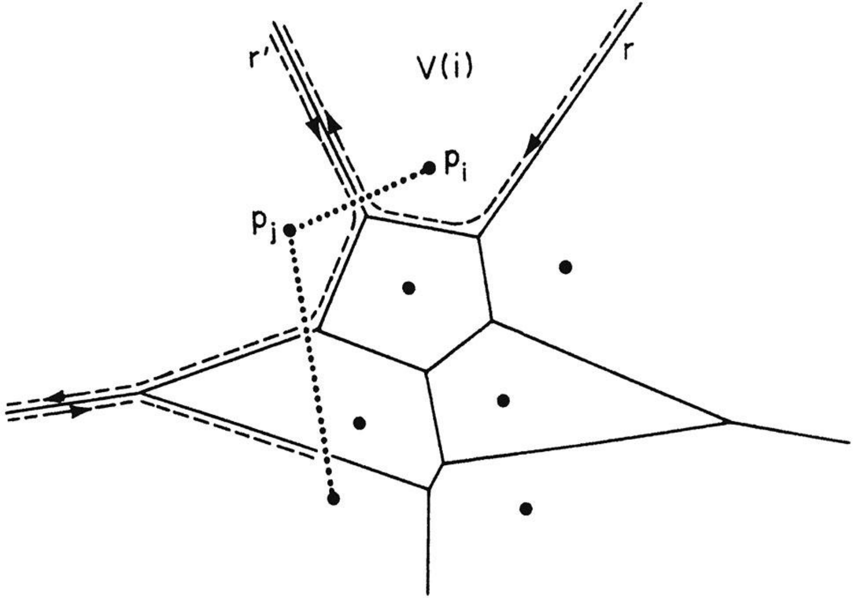


Figure 5.31 Construction of the convex hull from the Voronoi diagram.

Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.31

Reduce Convex Hull to Voronoi Diagram

- Theorem: Voronoi polygon V_i is unbounded if and only if Voronoi site *i* is on the convex hull of all sites. (proved in Preparata & Shamos)

- O(n) to convert Voronoi Diagram to Convex Hull:
  - Start with any unbounded cell
  - Walk edges clockwise to find adjacent unbounded cell
  - Voronoi sites will trace convex hull in counter-clockwise order

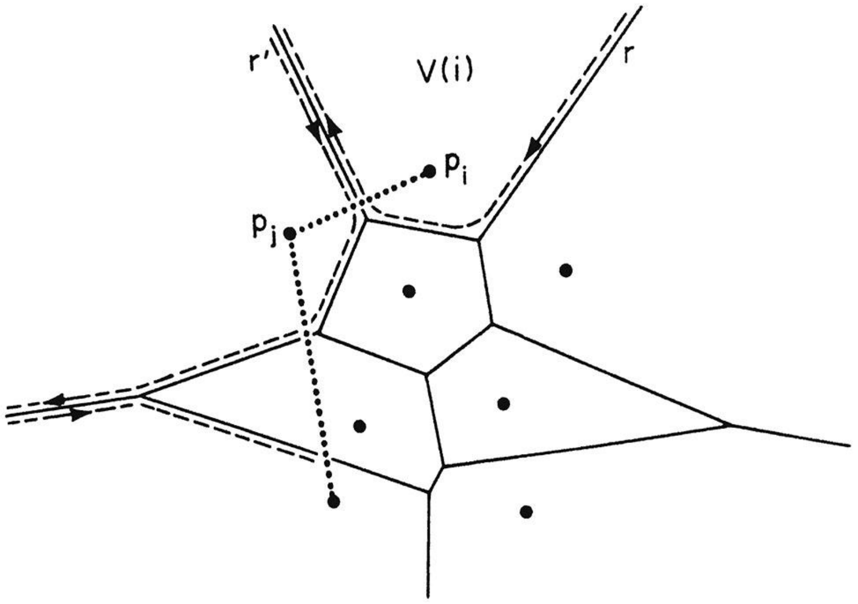


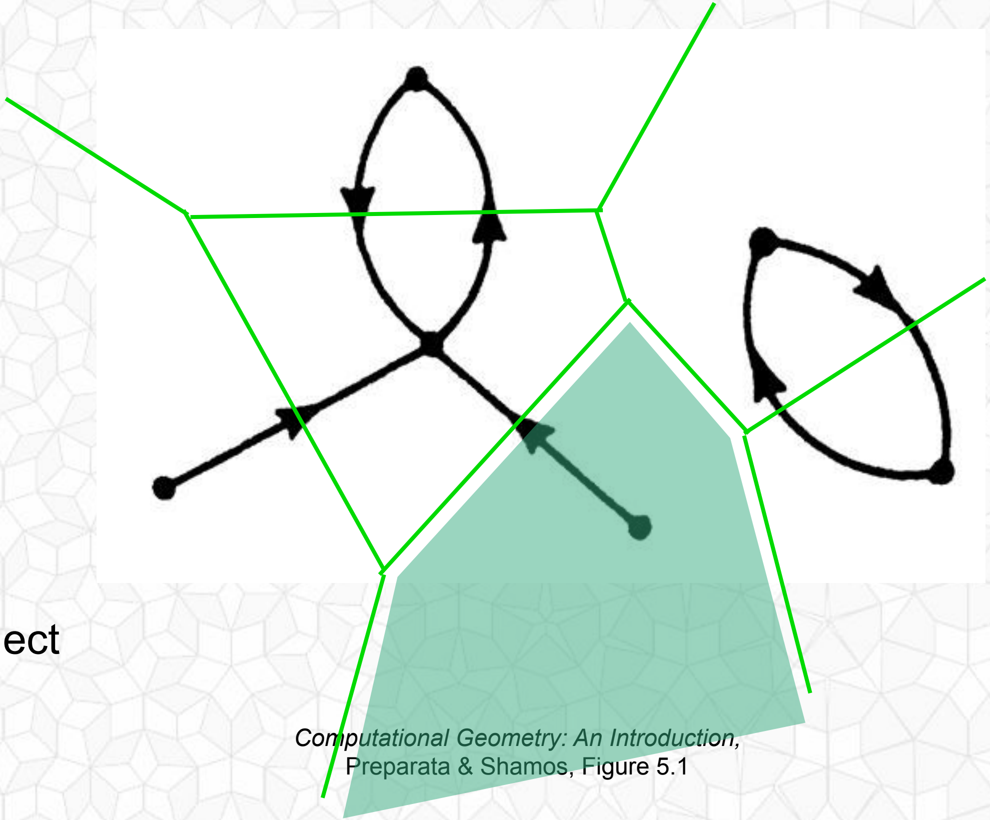
Figure 5.31 Construction of the convex hull from the Voronoi diagram.

Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.31

Reduce All Nearest Neighbors to Voronoi Diagram

- For n Voronoi Sites
- By Euler's formula: $F + V = E + 2$
  - # of Voronoi edges $\leq 3n-6$
- Theorem: Every nearest neighbor in the set of Voronoi sites defines an edge of a Voronoi polygon.
  (proved in Preparata & Shamos)

- O(n) to convert Voronoi Diagram to All Nearest Neighbors
  - For every Voronoi polygon, loop over all Voronoi edges, & select adjacent site that is closest
  - Every Voronoi edge will be considered twice



Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.1

Reduce All Nearest Neighbors to Voronoi Diagram

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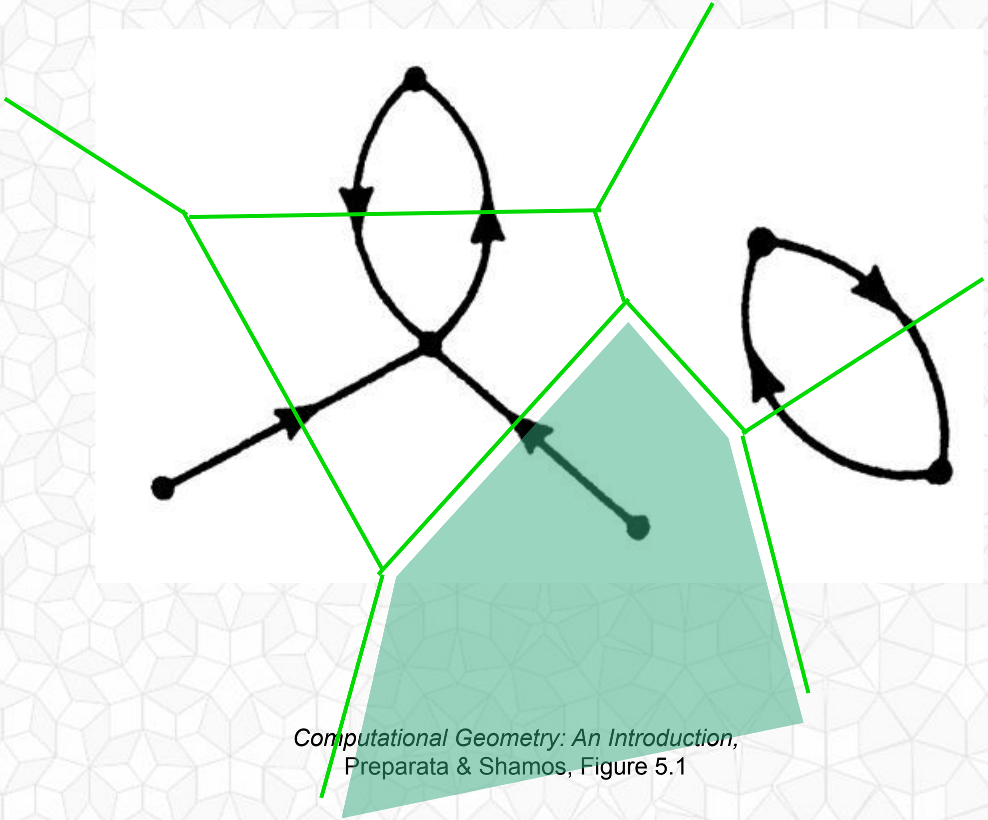

Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.1

Problems that Reduce to Voronoi Diagram

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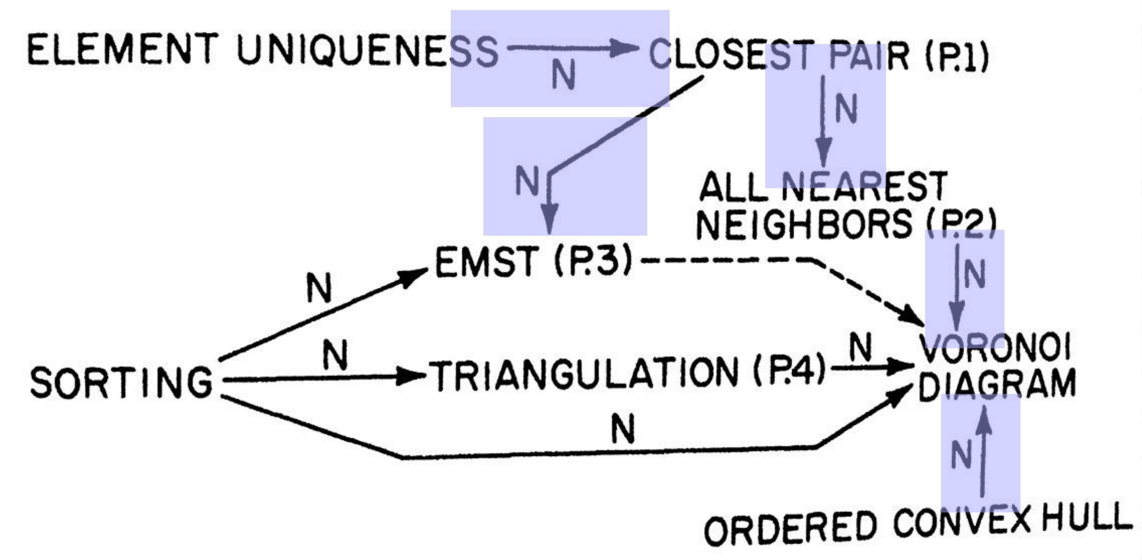


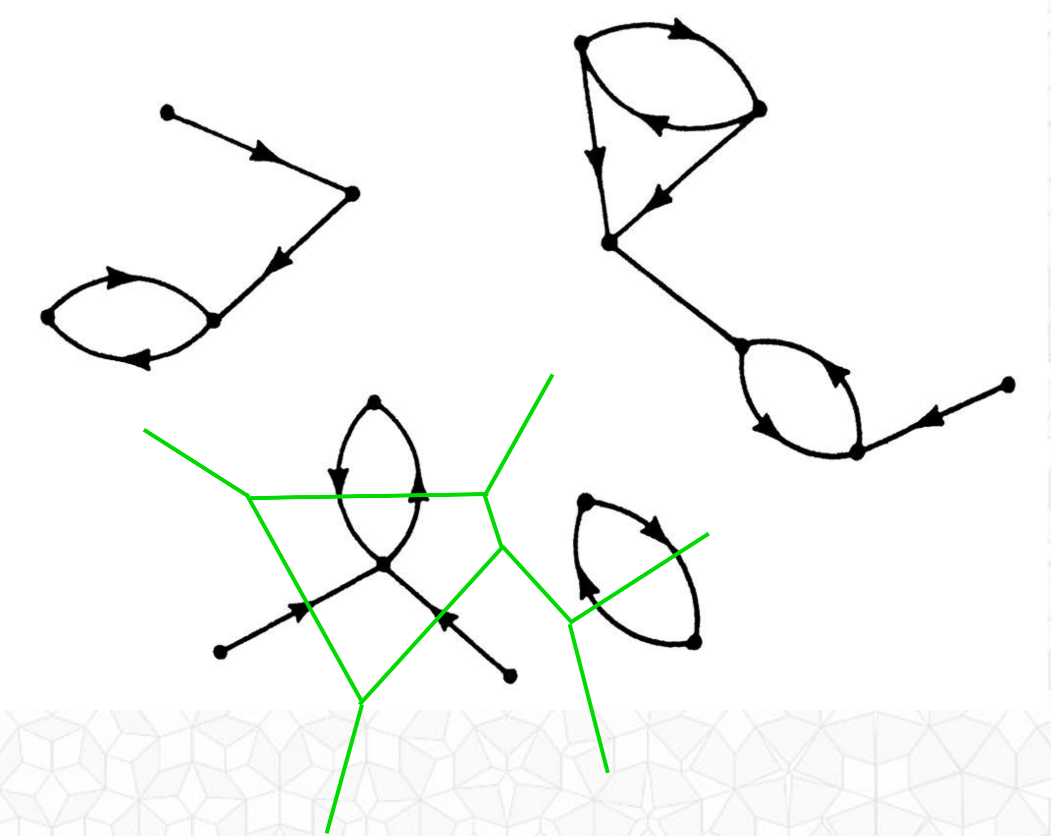
Computational Geometry: An Introduction, Preparata & Shamos, Figure 5.30

- *Therefore they are also O(n log n) time and O(n) space.*

Reduce EMST to Voronoi Diagram

- All Nearest Neighbors will have one or more cycles reciprocal pair(s)
  - Remove one edge from each cycle
- All Nearest Neighbors may be disconnected
  - *A forest of trees*

- *Core challenge: Find and add shortest edge between disconnected trees in the forest…*



Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.1

Kruskal's and Prim's Algorithm for MST

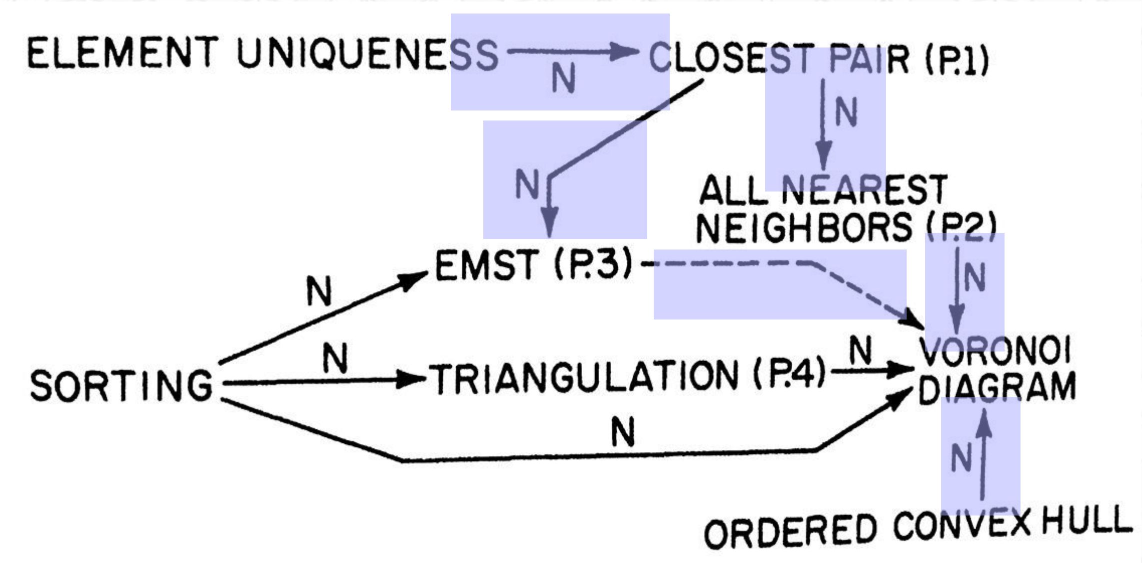
Did you cover this in Introduction to Algorithms and/or Graph Theory?

- Kruskal's - $O(E \log E)$
  - maintain a set of trees
  - find the shortest edge that merges two trees
  - repeat until there is only a single tree
- Prim's - $O(E + V \log V)$
  - maintain *one* tree, and all unconnected vertices
  - find the shortest edge from the tree to an unconnected vertex
  - repeat until there are no unconnected vertices

For a general (non-Euclidean) MST, we may have to consider $O(n^2)$ edges worst case.
For the Euclidean MST, we only need to consider the $O(n)$ Voronoi/Delaunay edges.

Problems that Reduce to Voronoi Diagram

- We can compute the Voronoi Diagram of *n* points in *O(n log n)* time and *O(n)* space.
- These other problems can be computed in *O(n)* additional time if given the Voronoi Diagram.



Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.30

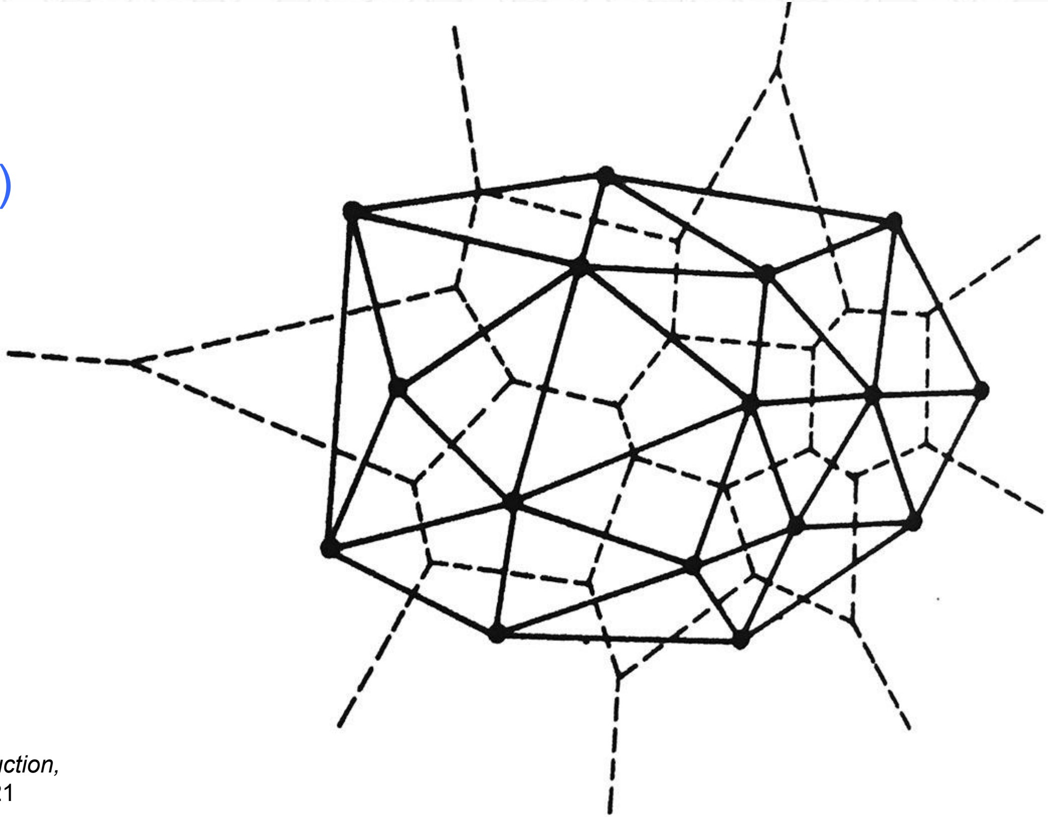
- *Therefore they are also O(n log n) time and O(n) space.*

Outline for Today

- Homework 4 Posted!
- Friday Feb 25th: Quiz 1, Logistics
- Last Time: More Voronoi Diagrams
- Higher-Order vs Higher Dimension Voronoi Diagrams
- Centroidal Voronoi Diagram
- K-Means Clustering
- Application: Architectural Geometry
- More Spatial Query & Search Problems / Applications
- Reducing Other Problems to the Voronoi Diagram
- Future Topic (after Quiz 1): Delaunay Triangulation

Next Time: Delaunay Triangulation!

- The Voronoi Diagram (VD) *is the dual of the* Delaunay Triangulation (DT)
- Every Voronoi Site is a face in Voronoi Diagram and a vertex in the DT
- Every Voronoi Edge is an edge in the DT
- Every Voronoi Vertex is a triangle in the DT



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