Lecture 15: More (Delaunay) Triangulations
Outline for Today

- Final Project: Brainstorming Ideas & Partner Matching
- Motivation Review: Terrain Height Maps
- Polygon Triangulation vs. Point Set Triangulation
- Counting the Number of Triangulations
- Incremental Triangulation by Point Insertion
- Incremental Triangulation by Line Sweep
- Flip Graph & Connectedness of all Triangulations
- Delaunay Construction by Edge Flips
- Randomized Incremental Delaunay Triangulation
- Delaunay Triangulation Construction Analysis Summary
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Motivation: Terrain Height Map

Nearest Neighbor

Bi-Linear Interpolation

*Computational Geometry Algorithms and Applications*, de Berg, Cheong, van Kreveld and Overmars, Chapter 9
Motivation: Terrain Height Map

Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 9
Not all Triangulations are the same!

this triangulation is better

this triangulation is worse

height = 985

height = 23
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Polygon Triangulation from Lecture 4

- Boundary specified
  - Typically non-convex
- No interior points
  - Vertices are 3 colorable
- Typically has many solutions
- For $n$ input points
  - Each solution has $n-2$ triangles
Point Set Triangulation

- A triangulation is a *Maximal Planar Subdivision* of a vertex set
- No edge connecting two vertices can be added without destroying planarity
- Every face will have 3 vertices
- For \( n \) input points, with \( k \) points on hull boundary
  - Each solution has \( 2n - 2 - k \) triangles

Convex hull boundary

*Computational Geometry Algorithms and Applications*, de Berg, Cheong, van Kreveld and Overmars, Chapter 9
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How many Different Triangulations?

- a point set & its convex hull
- a (non-maximal) planar subdivision

Several triangulations of this point set

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3
How many Different Triangulations?

```
<table>
<thead>
<tr>
<th>a point set &amp; its convex hull</th>
<th>a (non-maximal) planar subdivision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Several triangulations of this point set</td>
<td></td>
</tr>
</tbody>
</table>
```

- Actually, counting the number of triangulations is hard!
- For $n$ points, the number of triangulations is exponential
- Open Problem: Can we count the number of triangulations in $O(n)$ time?

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3
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Construction by Point Insertion

- Start with convex hull
  - Triangulate it
  - $k-2$ triangles
- For some ordering of the other points
  - Determine which triangle the point lies inside of
  - Replace that triangle with 3 triangles
  - $(n - k) * 2$ additional triangles
- $2*n - k - 2$ total triangles!

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3
Construction by Point Insertion

- Every solution has \(2^n - k - 2\) total triangles!

- If we enumerate every triangulation of the hull and every sequence of point intersections…

- Can we guarantee to generate every triangulation of these points?

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3
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Construction by Line Sweep

- Sort the input points by x
- Form a triangle with the 3 leftmost points
- Add every other point from left to right
  - Determine which points on the current hull are visible from the new point
  - Add a fan of triangles connecting the new point to the visible hull points

"Discrete and Computational Geometry", Devadoss & O'Rourke, Princeton University Press 2011, Chapter 3
Construction by Line Sweep

- If we enumerate every sweep orientation (rotate the coordinate system)...

- *Can we generate every triangulation?*

"Discrete and Computational Geometry", Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3
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Triangle Swap (a.k.a. Flip the Edge)

- Replace any edge between two triangles with the edge connecting the other vertices of those two triangles (only possible if the combined area of the two triangles is convex)
The Flip Graph

- If we did generate every triangulation...
- Let’s organize the triangulations as nodes in a graph
- We’ll put an edge between two nodes if flipping a single edge converts one triangulation into the other triangulation

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3
The Flip Graph

- Is this graph guaranteed to be connected?

- Are we always able to find a sequence of edge flips that converts one triangulation into another triangulation?

"Discrete and Computational Geometry", Devadoss & O'Rourke, Princeton University Press 2011, Chapter 3
The Flip Graph *is Connected*

- Let’s show that every triangulation can be converted by edge flips to the triangulation that results from the x axis sweep line construction.

- Which will prove the flip graph is connected.

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3
Proof by Induction & Construction in Reverse

- Given any target triangulation, let’s deconstruct the triangulation by removing one vertex at a time, from right to left
- Identify all triangles that touch the current rightmost vertex, $p_n$

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3
Proof by Induction & Construction in Reverse

- Identify a vertex touching $p_n$ that is not on the hull of the Line Sweep triangulation without $p_n$
- *Flip that edge (if that quadrilateral is not convex, find one that is!)*

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Definition: Angle-Optimal Triangulation

- We want to maximize the smallest angle
- Consider replacing each edge between two triangles with the edge connecting the other vertices of those two triangles (only possible if the combined area of the two triangles is convex)

- Edge $p_ip_j$ is said to be *illegal* if:
  \[
  \min_{1 \leq i \leq 6} \alpha_i < \min_{1 \leq i \leq 6} \alpha'_i
  \]
Inscribed Angle Theorem

The inscribed angle $\theta$ is half of the central angle $2\theta$ that subtends the same arc on the circle. The angle $\theta$ does not change as its vertex is moved around on the circle.

https://en.wikipedia.org/wiki/Inscribed_angle#Theorem
Constructing an Angle-Optimal Triangulation

- Brute Force

- Try all combinations of 3 vertices
- Construct the circumscribed circle
- If no other vertex is inside of that circle, keep it
- Only works if no more than 3 vertices are on the circle

- Analysis?

*Computational Geometry Algorithms and Applications*, de Berg, Cheong, van Kreveld and Overmars, Chapter 9
Walking the Flip Graph

- The Delaunay Triangulation is the Angle-Optimal Triangulation

“Discrete and Computational Geometry”, Devadoss & O’Rourke, Princeton University Press 2011, Chapter 3
Guaranteed to Terminate? Yes!

- Create a sorted vector of all of the angles of every triangle
  vector length = 3 * # of triangles
- Each edge flip replaces one of the smaller angles
- New sorted vector representation is the same up to that angle..
  (it comes lexicographically after the previous vector representation)
Walking the Flip Graph

- The Delaunay Triangulation is the Angle-Optimal Triangulation

- How many flips are necessary to reach the Delaunay Triangulation?

"Discrete and Computational Geometry", Devadoss & O'Rourke, Princeton University Press 2011, Chapter 3
Walking the Flip Graph

- The Delaunay Triangulation is the Angle-Optimal Triangulation

- How many flips are necessary to reach the Delaunay Triangulation?

- What is the “diameter” of the flip graph?
  - At most \((n-2) \times (n-3)\)

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See book for proof…
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Randomized Incremental Construction of Delaunay Triangulation

- Randomize order of points and insert one at a time
- Identify which triangle contains \( p_r \)
- Split into 3 smaller triangles
- Flip neighboring edges as necessary

Hopefully the footprint of impact is small!
Lecture 9: Point Location by Directed Acyclic Graph

- Intermediate notes are vertices (vertical lines) and line segments
- The leaves are the trapezoidal regions (map back to original polygons)
Lecture 9: Point Location by Directed Acyclic Graph

- Intermediate notes are vertices (vertical lines) and line segments
- The leaves are the trapezoidal regions (map back to original polygons)

Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 6
• Similarly… we’ll construct a directed acyclic graph (DAG) of triangles
• The leaves will be the final triangulation
• We can use this to identify which triangle contains \( p_r \)
• And then split this triangle into 3 smaller triangles

*Computational Geometry Algorithms and Applications*, de Berg, Cheong, van Kreveld and Overmars, Chapter 9
- Check the "legality" of the edges of the new triangles
- Flip edges if necessary
- Add new triangles to DAG
- & recurse
- Check the “legality” of the edges of the new triangles

- Flip edges if necessary

- Add new triangles to DAG

- & recurse

*Computational Geometry Algorithms and Applications*, de Berg, Cheong, van Kreveld and Overmars, Chapter 9
Randomized Incremental Construction of Delaunay Triangulation Analysis

- For \( n \) points, inserted one at a time
- Point location in DAG
- Split triangle
- Check edge legality
- Do edge flips & Recurse
- Overall
Randomized Incremental Construction of Delaunay Triangulation Analysis

- For $n$ points, inserted one at a time
- Point location in DAG
  \[\rightarrow O(\log n)\]
- Split triangle
  \[\rightarrow O(1)\]
- Check edge legality
  Do edge flips & Recurse
  \[\rightarrow O(1)\text{ expected}\]
- Overall
  \[\rightarrow O(n \log n)\]
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Delaunay Construction Analysis Summary

● Brute force (enumerate all triangles, construct circles, reject… )

● Construct any triangulation & Flip until all edges are legal

● Randomized Incremental Construction

● By duality, reduce to problem of Constructing the Voronoi Diagram
Dual: Voronoi Diagram & Delaunay Triangulation

- The Voronoi Diagram (VD) is the dual of the Delaunay Triangulation (DT)
- Every Voronoi Site is a face in Voronoi Diagram and a vertex in the DT
- Every Voronoi Edge is an edge in the DT
- Every Voronoi Vertex is a triangle in the DT
Lecture 10: Voronoi Sweep Line Algorithm

- For \( n \) Voronoi sites
- New Arc Events: Sort Voronoi sites vertically \( \rightarrow O(n \log n) \)
- Keep a horizontal sorted ordering of the parabolic arcs on the current beachline. \( 2n \) arcs maximum
- (Potential) Arc Absorption Events: For each triple of neighboring arcs \( \alpha, \alpha', \alpha'' \) on the beachline, compute the circle, and tangent sweep line \( \rightarrow O(n) \) Voronoi vertices
- Move sweep line to the next event…
- Overall: \( \rightarrow O(n \log n) \)
Lecture 12: Problems that Reduce to Voronoi Diagram

- We can compute the Voronoi Diagram of \( n \) points in \( O(n \log n) \) time and \( O(n) \) space.
- These other problems can be computed in \( O(n) \) additional time if given the Voronoi Diagram.
- Therefore they are also \( O(n \log n) \) time and \( O(n) \) space.

*Computational Geometry: An Introduction, Preparata & Shamos, Figure 5.30*
Delaunay Construction Analysis Summary

- Brute force (enumerate all triangles, construct circles, reject… )
  \[ \rightarrow O(n^3 * n) = O(n^4) \]
- Construct any triangulation & Flip until all edges are legal
  \[ \rightarrow O(n^2) \]
- Randomized Incremental Construction
  \[ \rightarrow O(n \log n) \]
- By duality, reduce to problem of Constructing the Voronoi Diagram
  \[ \rightarrow O(n \log n) \]