CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/S22/

Lecture 15: More (Delaunay) Triangulations

Outline for Today

- Final Project: Brainstorming Ideas & Partner Matching
- Motivation Review: Terrain Height Maps
- Polygon Triangulation vs. Point Set Triangulation
- Counting the Number of Triangulations
- Incremental Triangulation by Point Insertion
- Incremental Triangulation by Line Sweep
- Flip Graph & Connectedness of all Triangulations
- Delaunay Construction by Edge Flips
- Randomized Incremental Delaunay Triangulation
- Delaunay Triangulation Construction Analysis Summary

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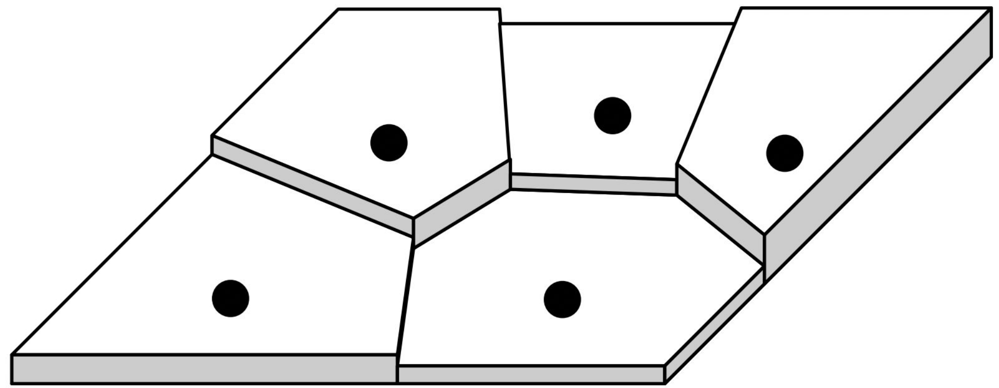
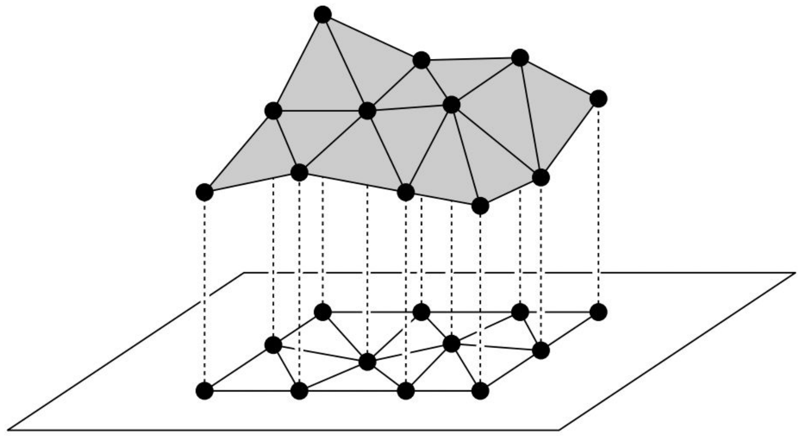
Motivation: Terrain Height Map



Nearest Neighbor



Bi-Linear Interpolation

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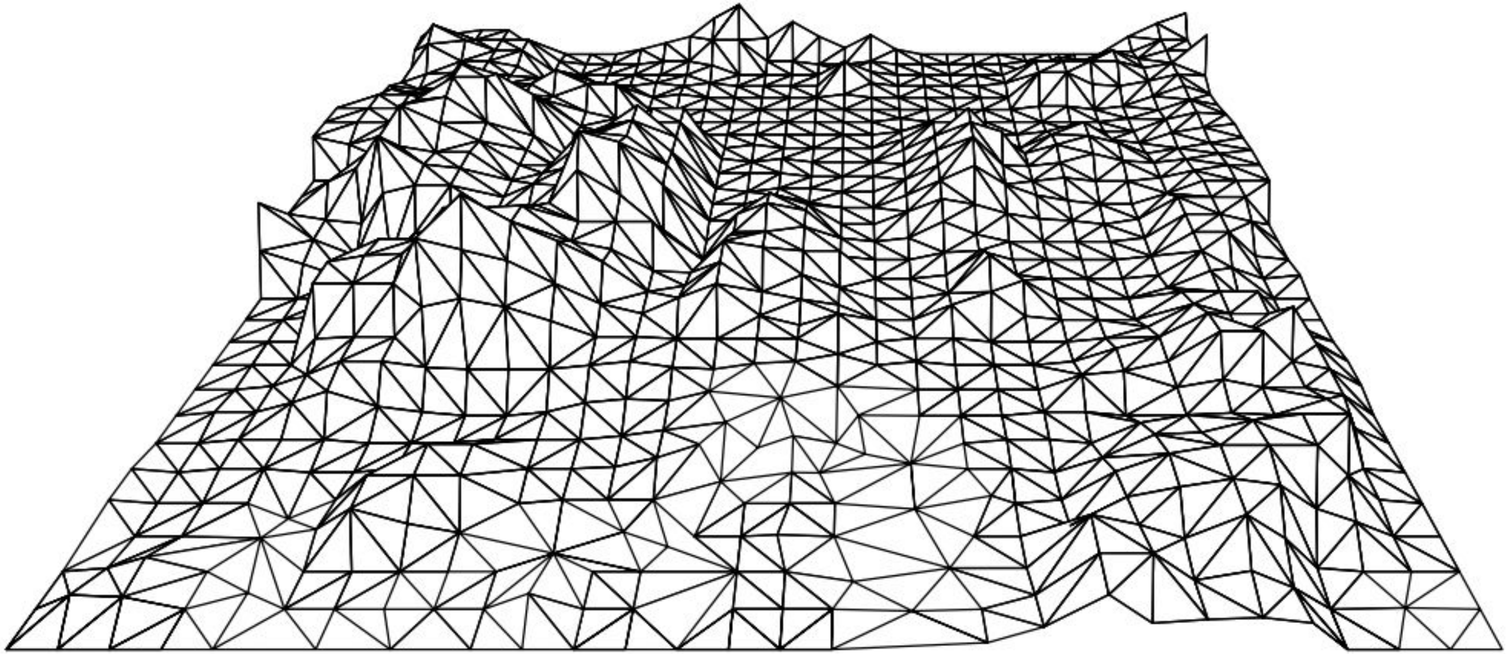
Motivation: Terrain Height Map



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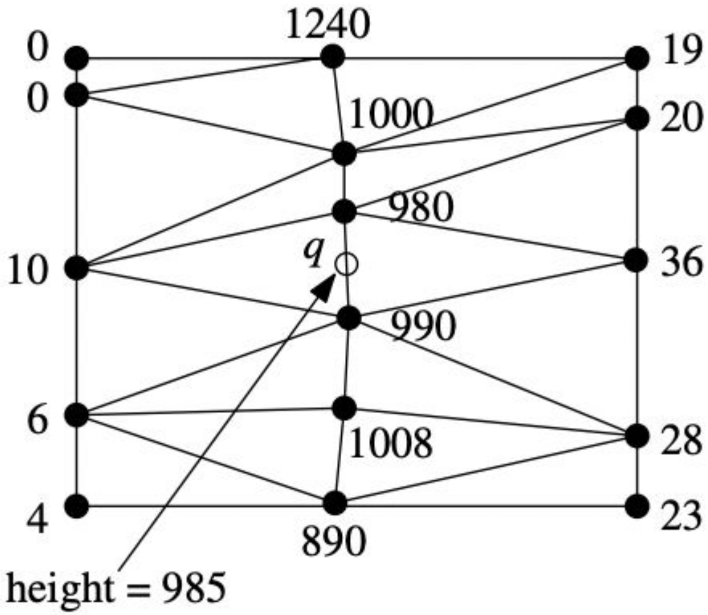
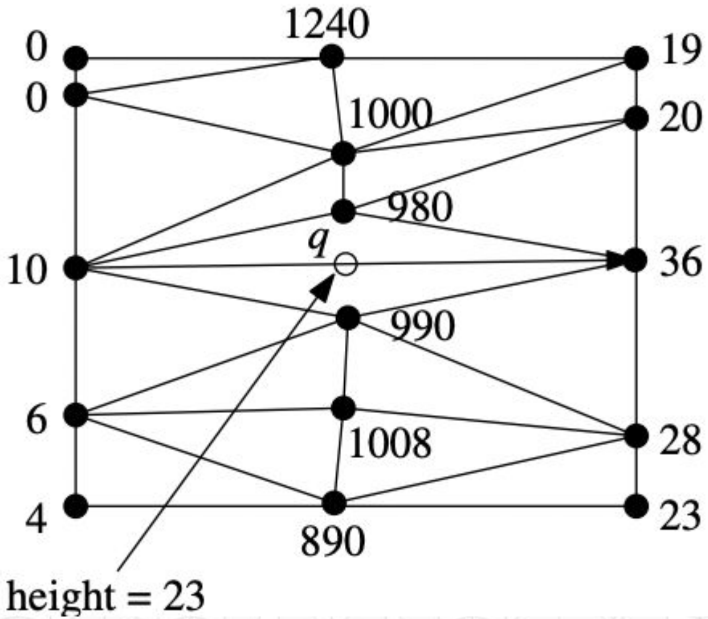
Not all Triangulations are the same!

this triangulation is better



this triangulation is worse



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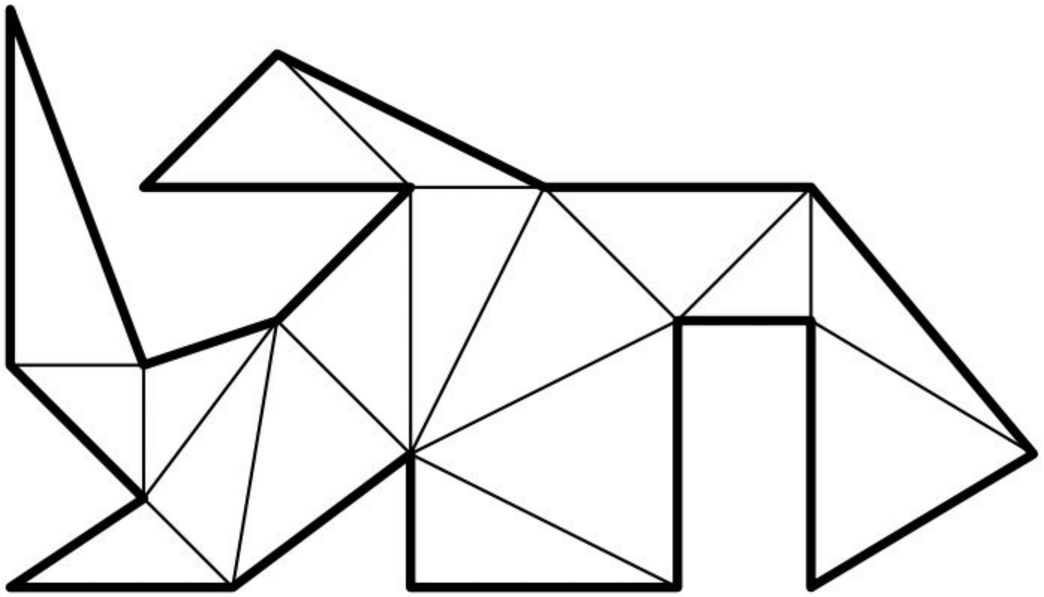
Polygon Triangulation from Lecture 4

- Boundary specified
  - Typically non-convex
- No interior points
  - vertices are 3 colorable
- Typically has many solutions
- For *n* input points
  - Each solution has n-2 triangles



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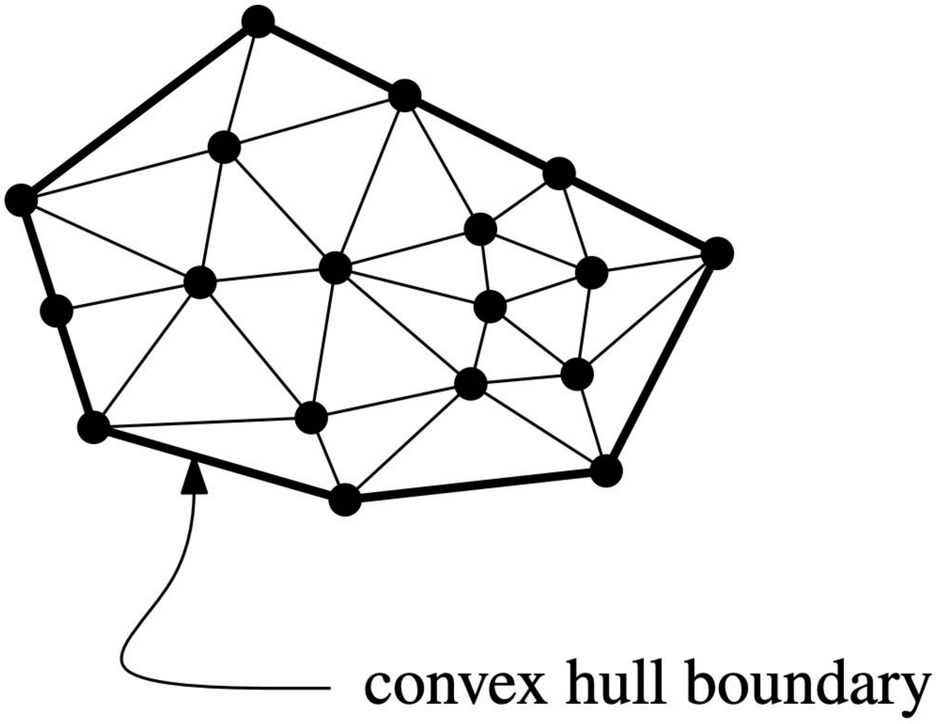
Point Set Triangulation

- A triangulation is a *Maximal Planar Subdivision* of a vertex set
- No edge connecting two vertices can be added without destroying planarity
- Every face will have 3 vertices
- For *n* input points, with *k* points on hull boundary
    - Each solution has $2n - 2 - k$ triangles



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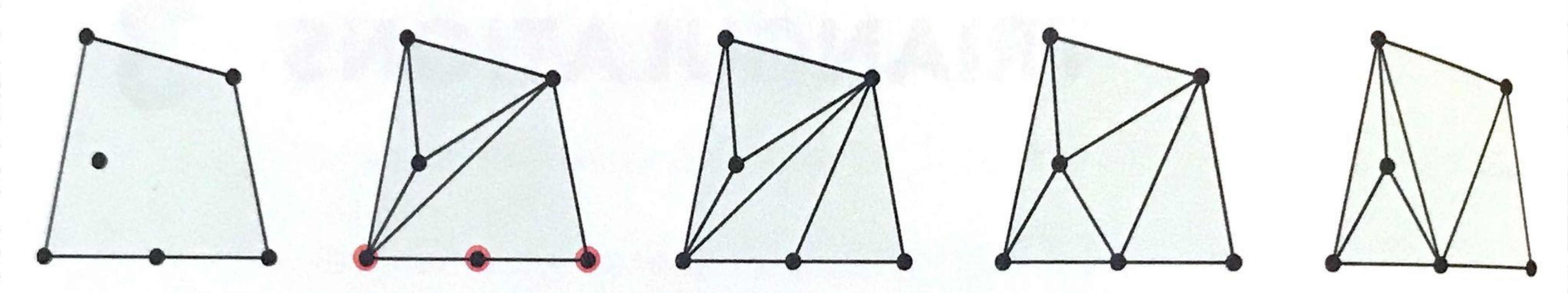
How many Different Triangulations?



a point set &
its convex hull

a (non-maximal)
planar subdivision

Several triangulations
of this point set

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Devadoss & O’Rourke,
Princeton University Press 2011,
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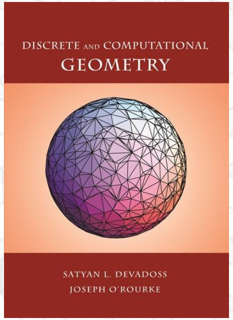
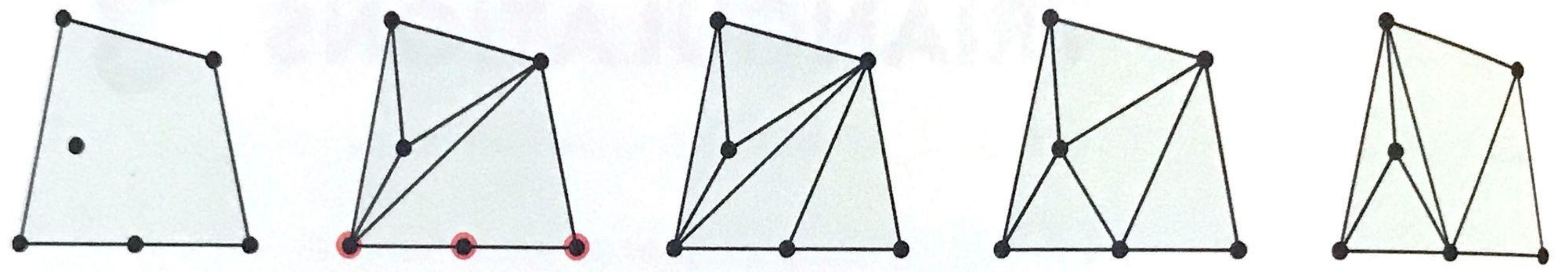


How many Different Triangulations?



a point set & its convex hull

a (non-maximal) planar subdivision

Several triangulations of this point set

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- Actually, counting the number of triangulations is hard!
- For n points, the number of triangulations is exponential
- Open Problem: Can we count the number of triangulations in $O(n)$ time?

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Construction by Point Insertion

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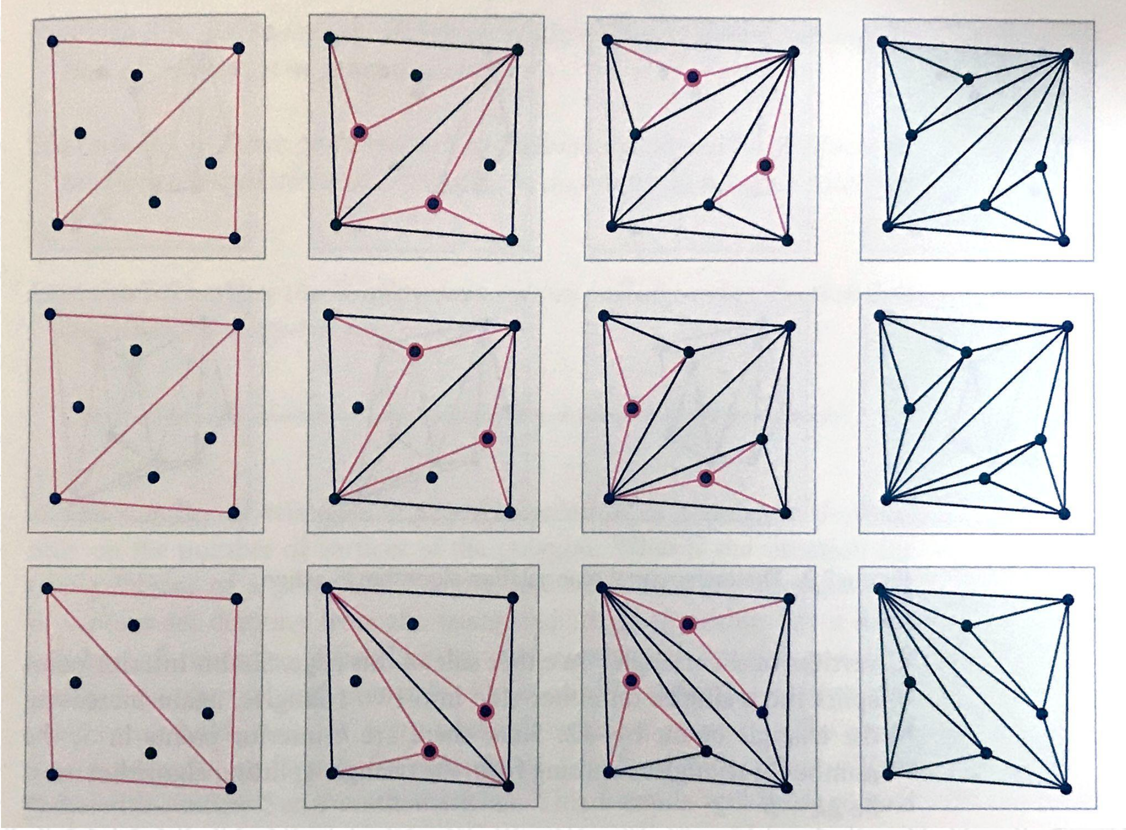
- Start with convex hull
  - Triangulate it
  - *k*-2 triangles
- For some ordering of the other points
  - Determine which triangle the point lies inside of
  - Replace that triangle with 3 triangles
  - (*n* - *k*) * 2 additional triangles
- 2**n* - *k* - 2 total triangles!



Construction by Point Insertion

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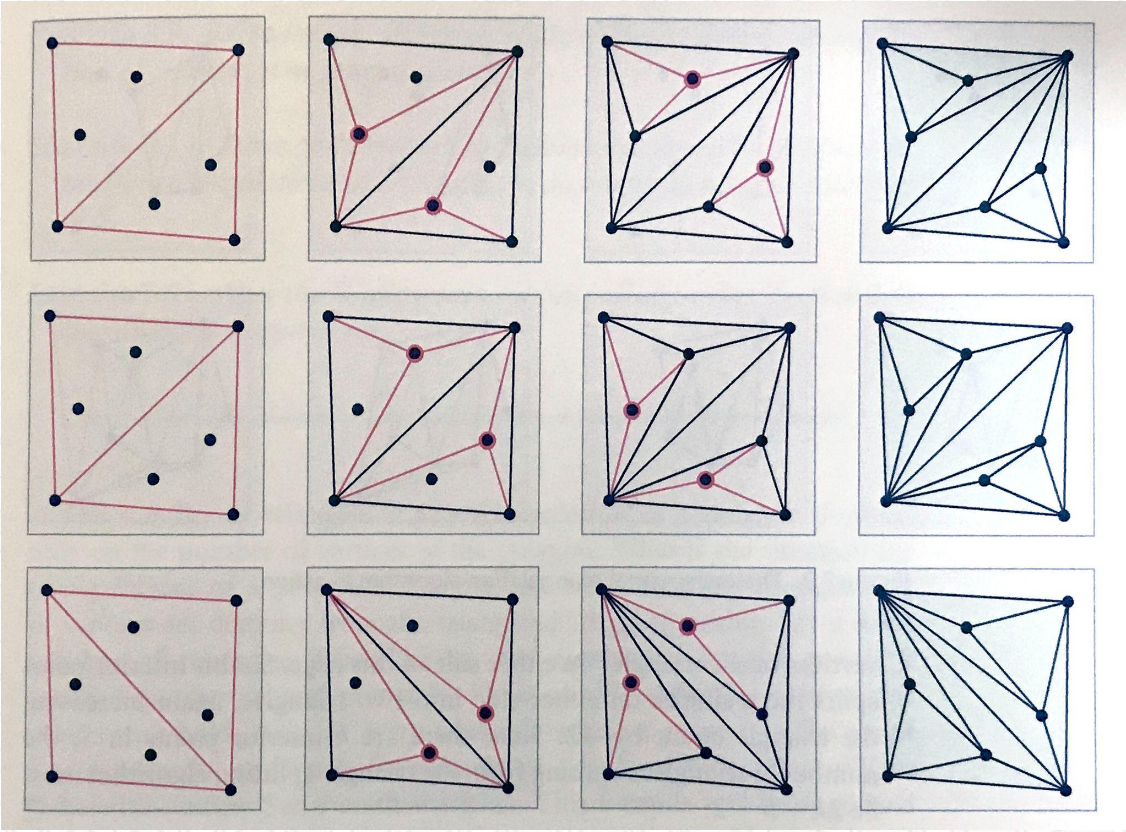
- Every solution has $2*n - k - 2$ total triangles!
- If we enumerate every triangulation of the hull and every sequence of point intersections…
- *Can we guarantee to generate every triangulation of these points?*



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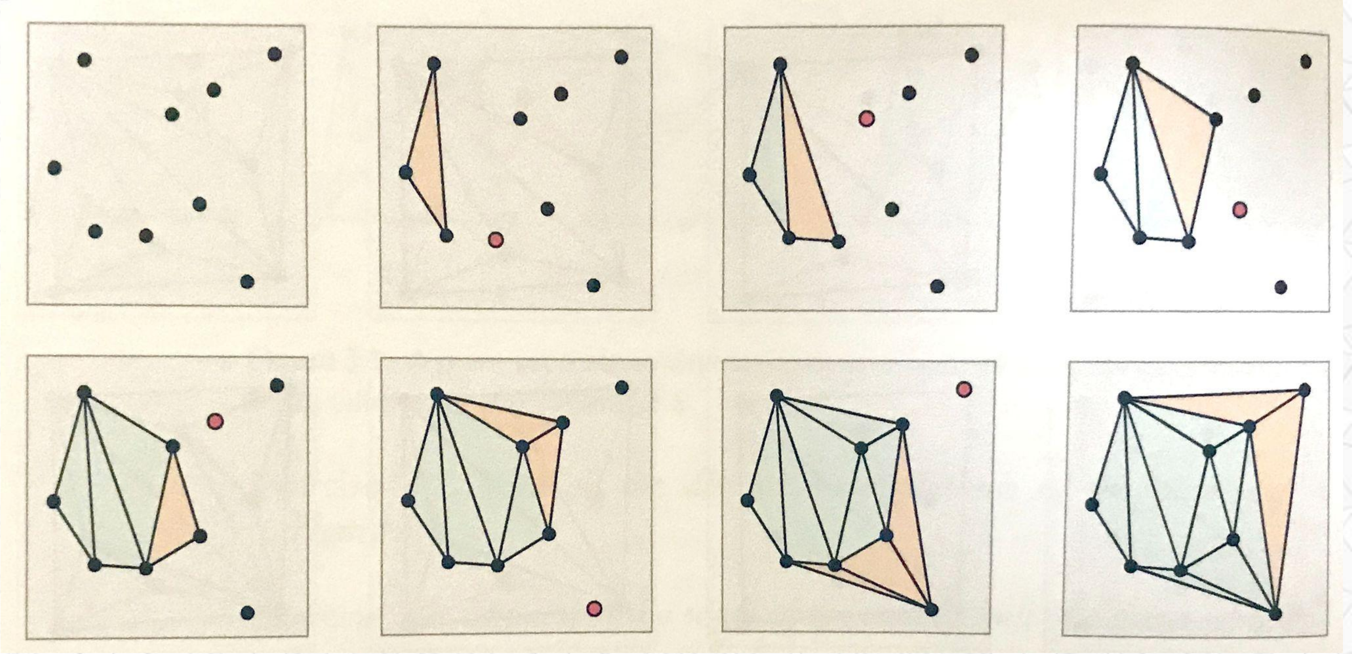
Construction by Line Sweep

- Sort the input points by x
- Form a triangle with the 3 leftmost points



- Add every other point from left to right
  - Determine which points on the current hull are visible from the new point
  - Add a fan of triangles connecting the new point to the visible hull points

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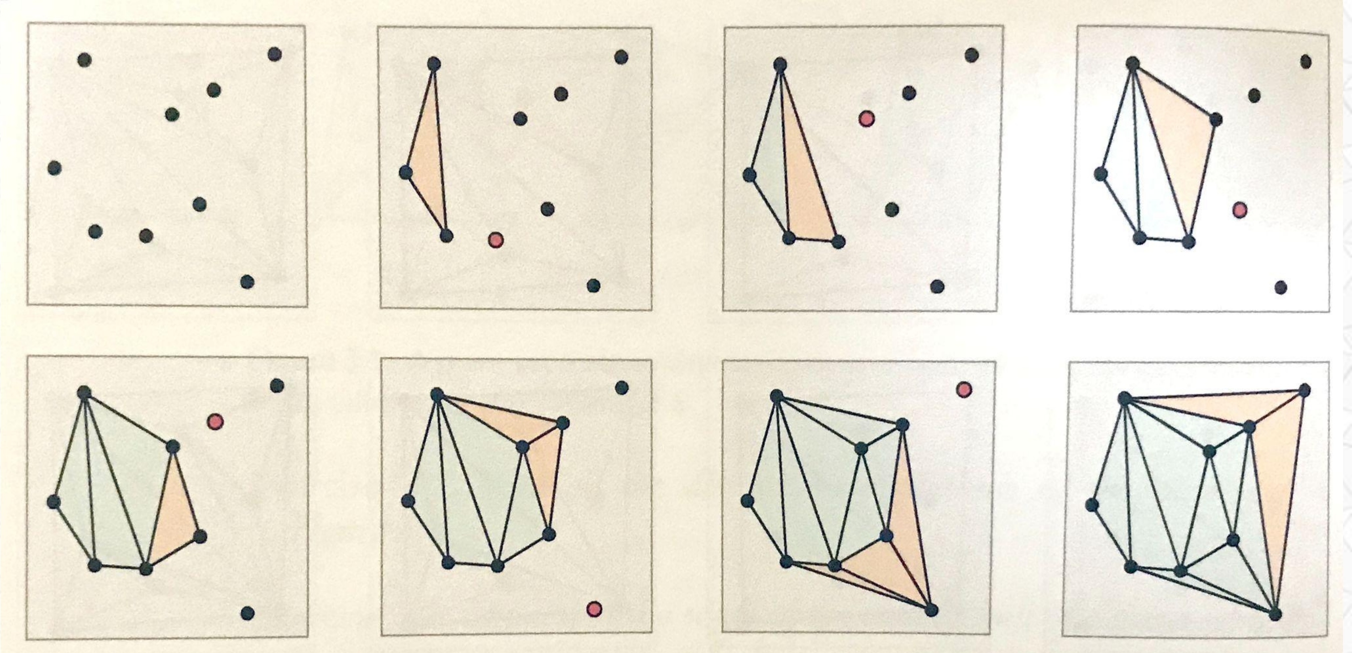
Construction by Line Sweep

- If we enumerate every sweep orientation (rotate the coordinate system)...

- *Can we generate every triangulation?*



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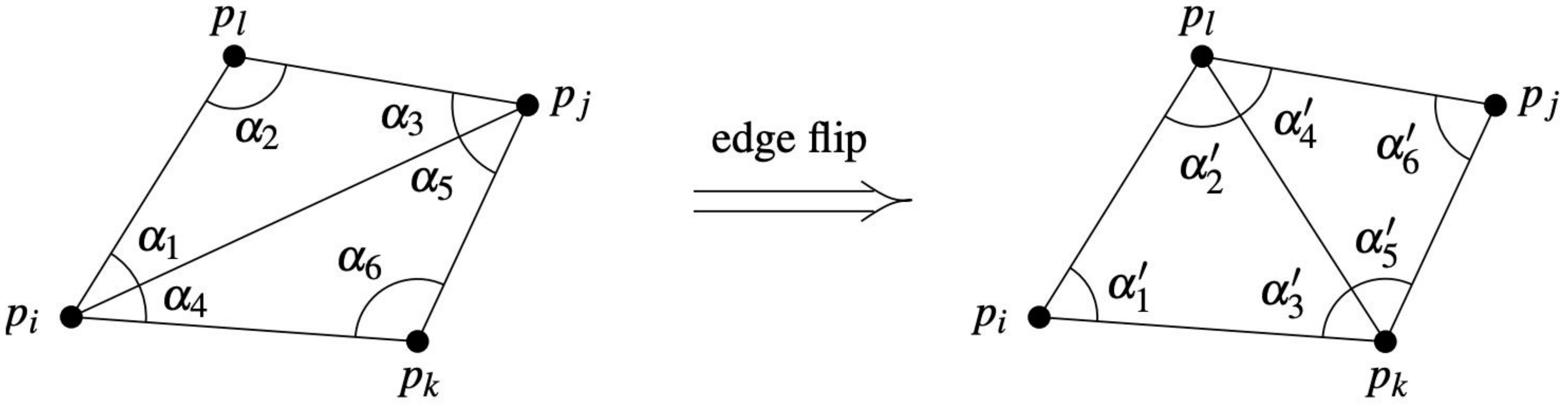
Triangle Swap (a.k.a. Flip the Edge)

- Replace any edge between two triangles with
  the edge connecting the other vertices of those two triangles
  (only possible if the combined area of the two triangles is convex)



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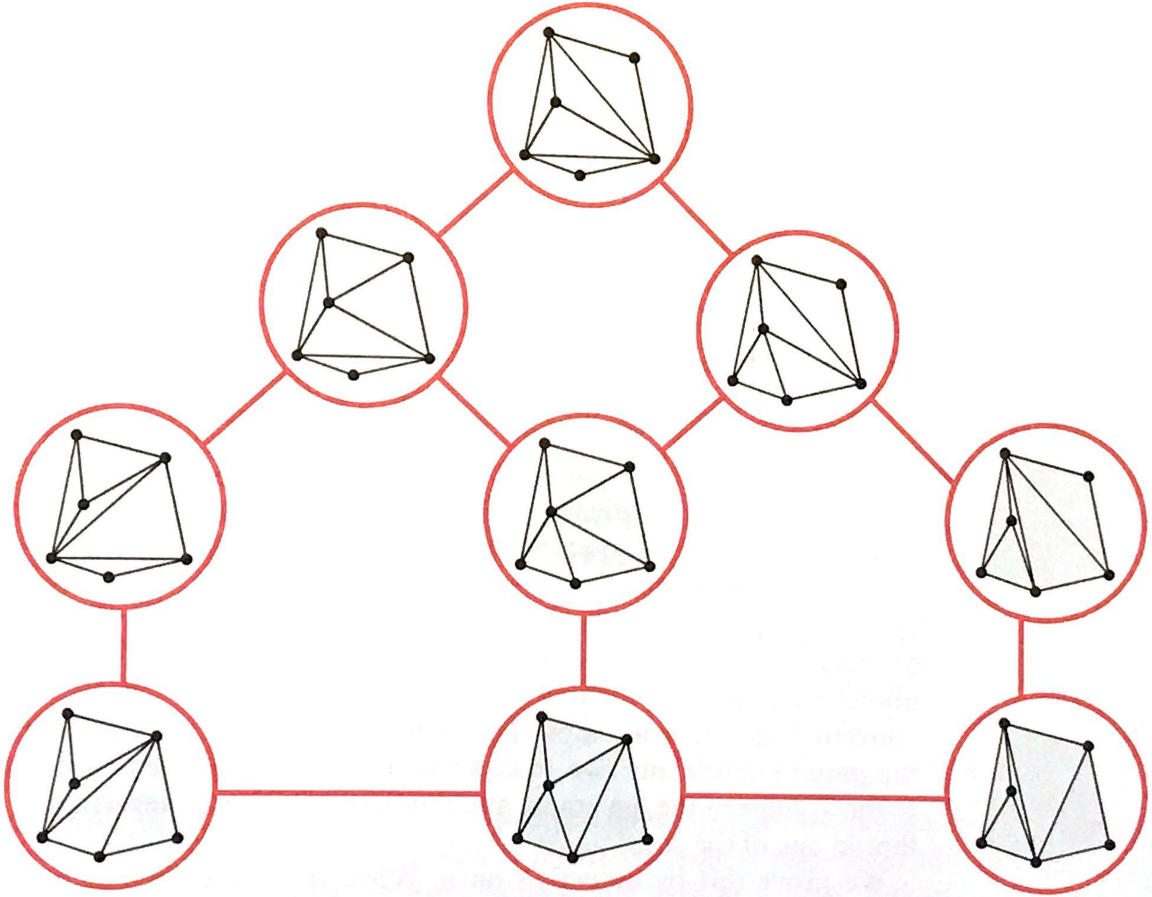
The Flip Graph

- If we did generate every triangulation…

- Let's organize the triangulations as nodes in a graph
- We'll put an edge between two nodes if flipping a single edge converts one triangulation into the other triangulation



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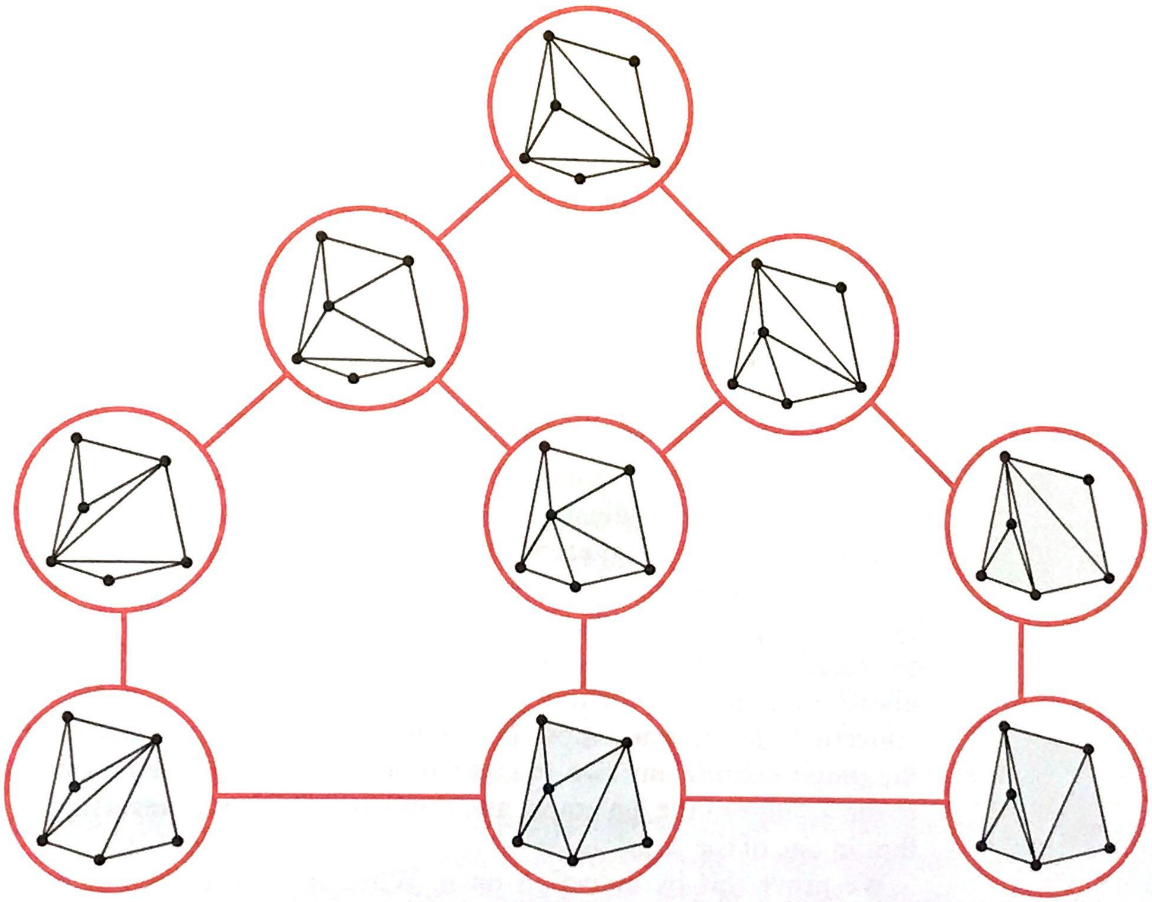
The Flip Graph

- Is this graph guaranteed to be connected?

- *Are we always able to find a sequence of edge flips that converts one triangulation into another triangulation?*



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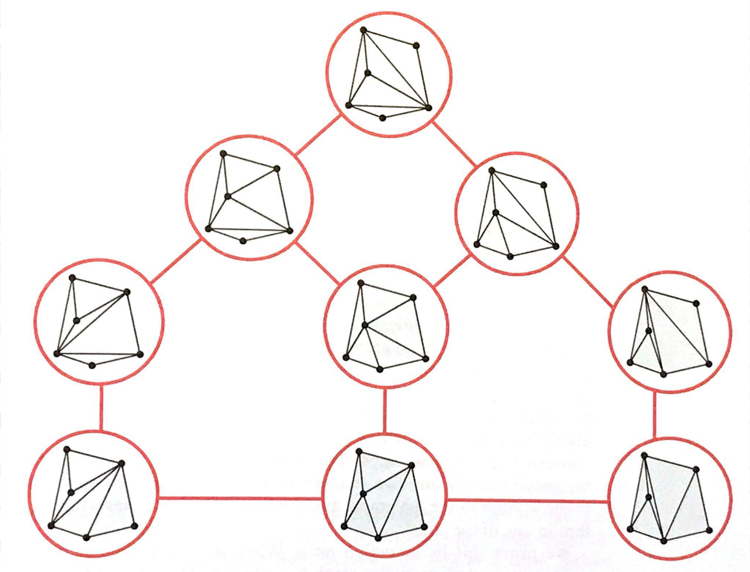
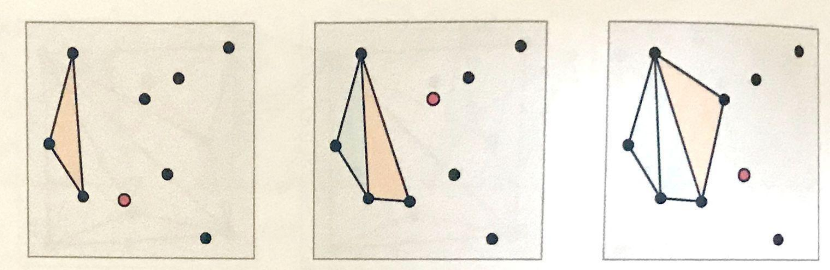
The Flip Graph *is Connected*

- Let’s show that every triangulation can be converted by edge flips to the triangulation that results from the *x* axis sweep line construction.

- Which will prove the flip graph is connected.





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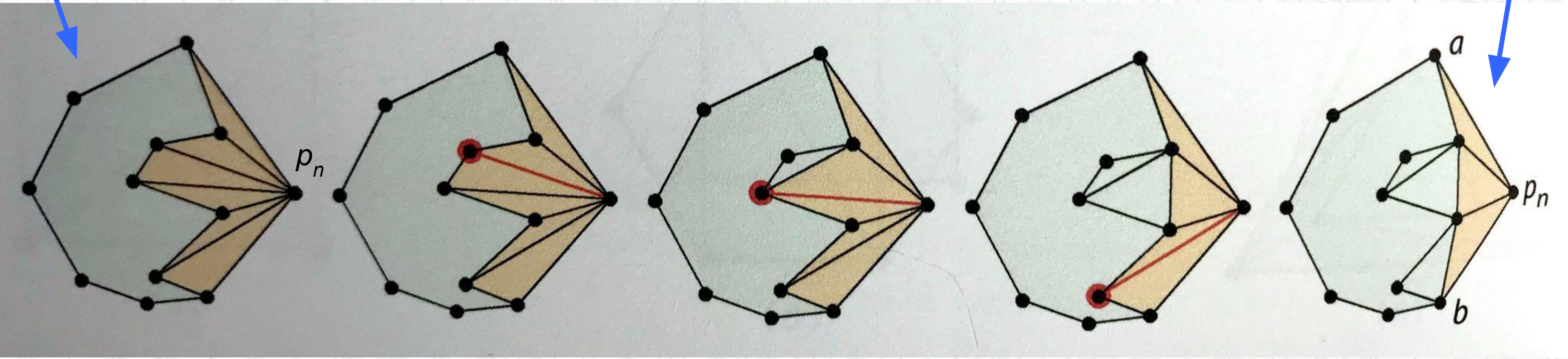
Proof by Induction & Construction in Reverse

- Given any target triangulation, let's deconstruct the triangulation by removing one vertex at a time, from right to left
- **Identify all triangles that touch the current rightmost vertex, p_n**

Our target triangulation

Triangulation constructed by Line Sweep



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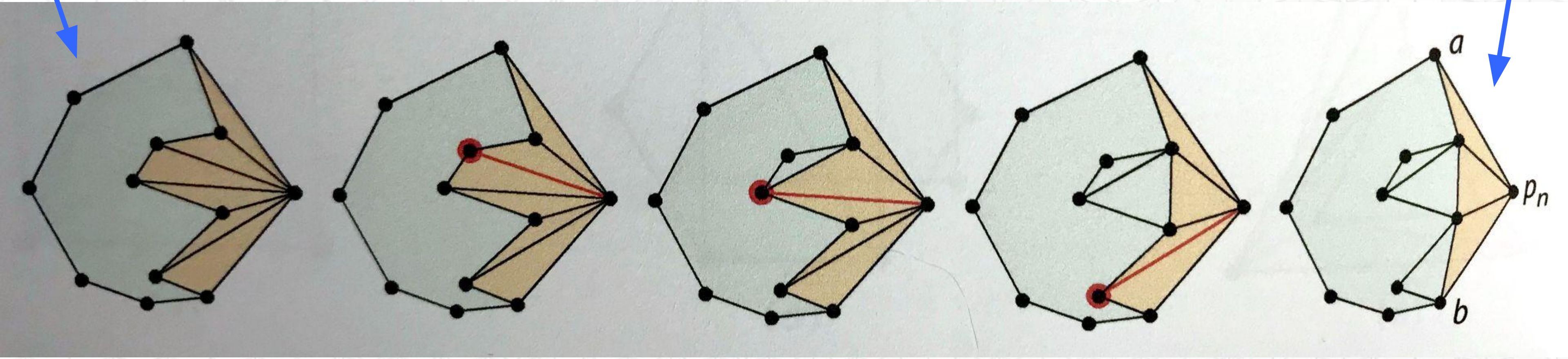
Proof by Induction & Construction in Reverse

- **Identify a vertex touching p_n that is not on the hull of the Line Sweep triangulation without p_n**
- *Flip that edge (if that quadrilateral is not convex, find one that is!)*

Our target triangulation

Triangulation constructed by Line Sweep



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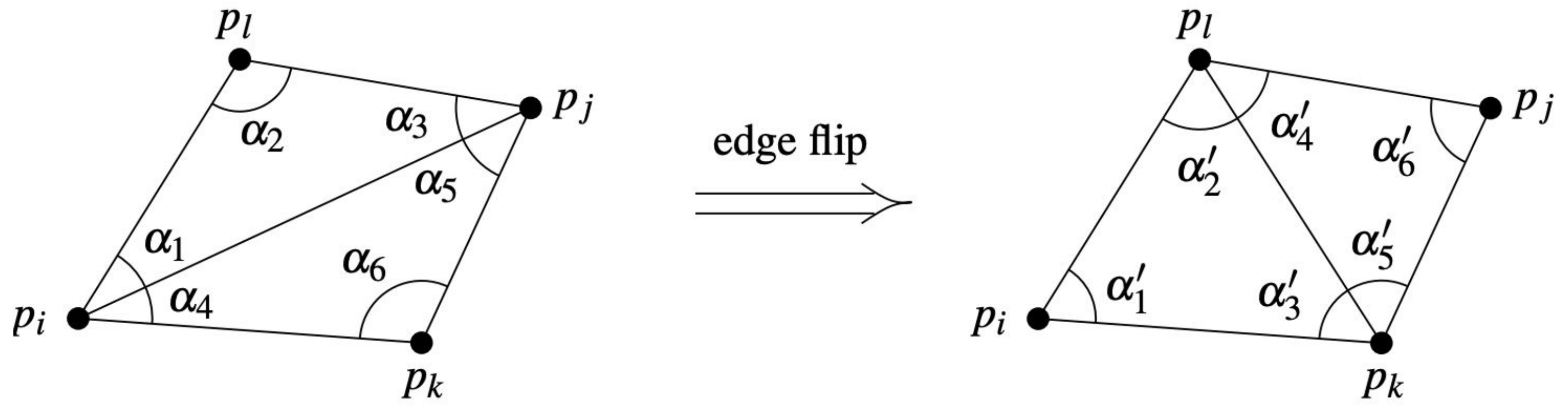
Definition: Angle-Optimal Triangulation

- We want to maximize the smallest angle
- Consider replacing each edge between two triangles with the edge connecting the other vertices of those two triangles (only possible if the combined area of the two triangles is convex)



- Edge $p_i p_j$ is said to be *illegal* if: $\min_{1 \leqslant i \leqslant 6} \alpha_i < \min_{1 \leqslant i \leqslant 6} \alpha'_i$

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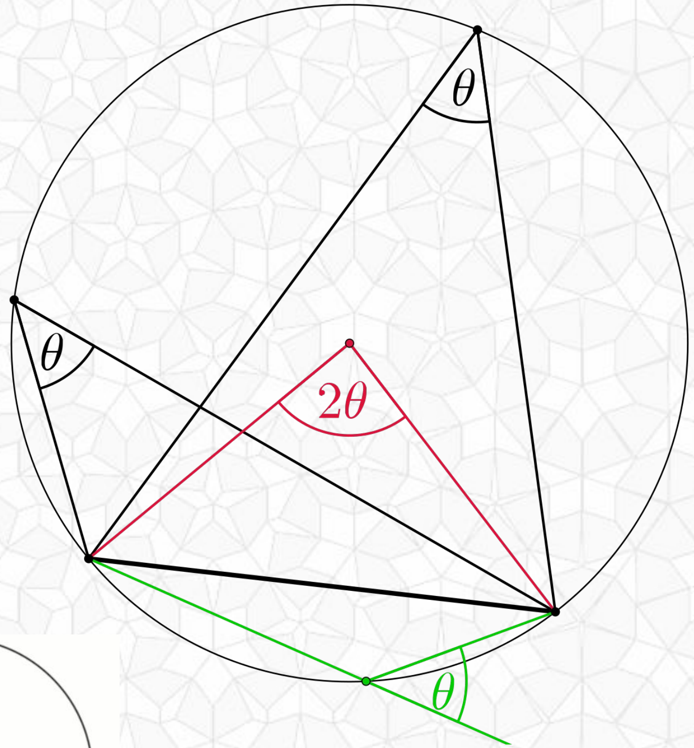
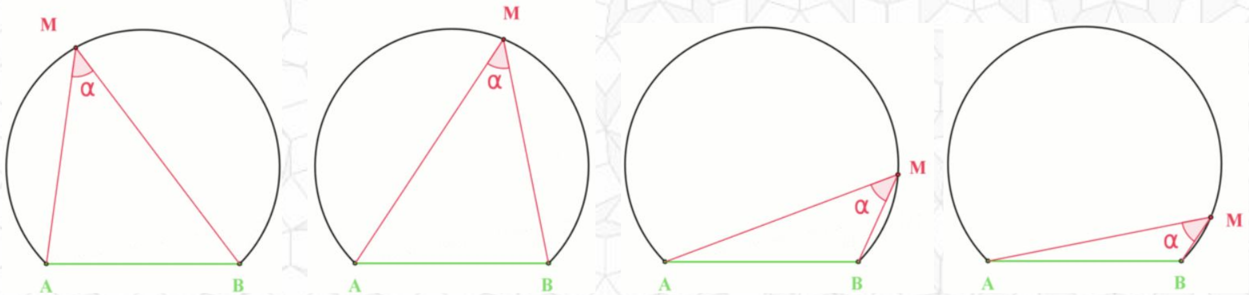
Inscribed Angle Theorem

The inscribed angle θ is half of the central angle 2θ that subtends the same arc on the circle. The angle θ does not change as its vertex is moved around on the circle.





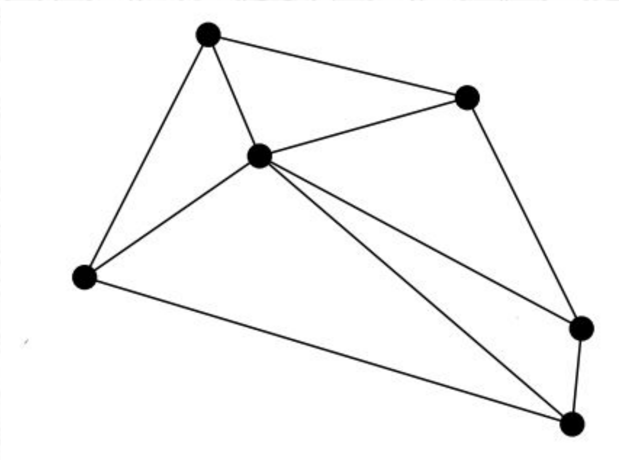
https://en.wikipedia.org/wiki/Inscribed_angle#Theorem

Constructing an Angle-Optimal Triangulation

- Brute Force

- Try all combinations of 3 vertices
- Construct the circumscribed circle
- If no other vertex is inside of that circle, keep it
- Only works if no more than 3 vertices are on the circle

- Analysis?



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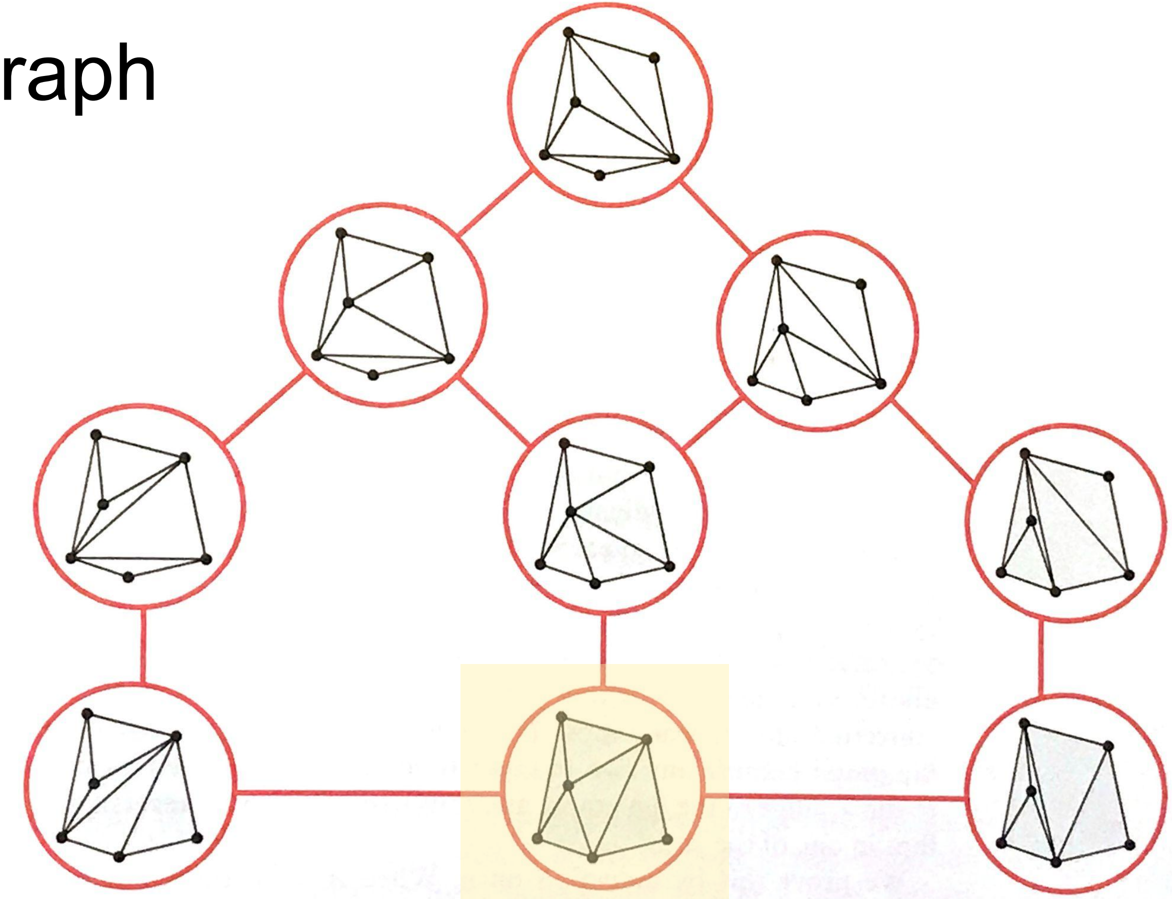
Walking the Flip Graph

- The Delaunay Triangulation is the Angle-Optimal Triangulation



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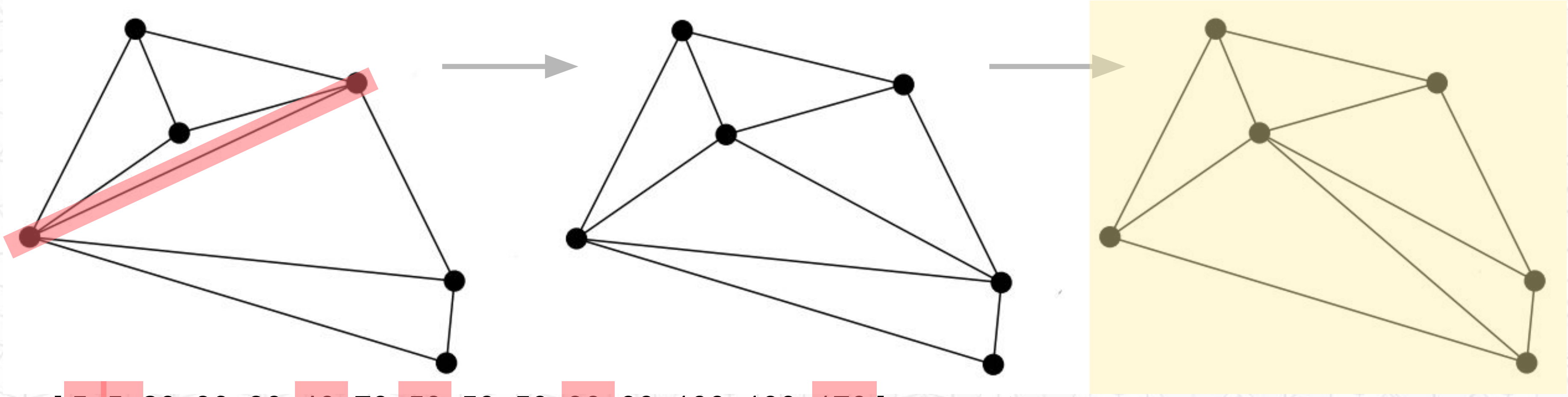
Guaranteed to Terminate? Yes!

- Create a sorted vector of all of the angles of every triangle
  vector length = 3 * # of triangles
- Each edge flip replaces one of the smaller angles
- New sorted vector representation is the same up to that angle..
  (it comes lexicographically after the previous vector representation)

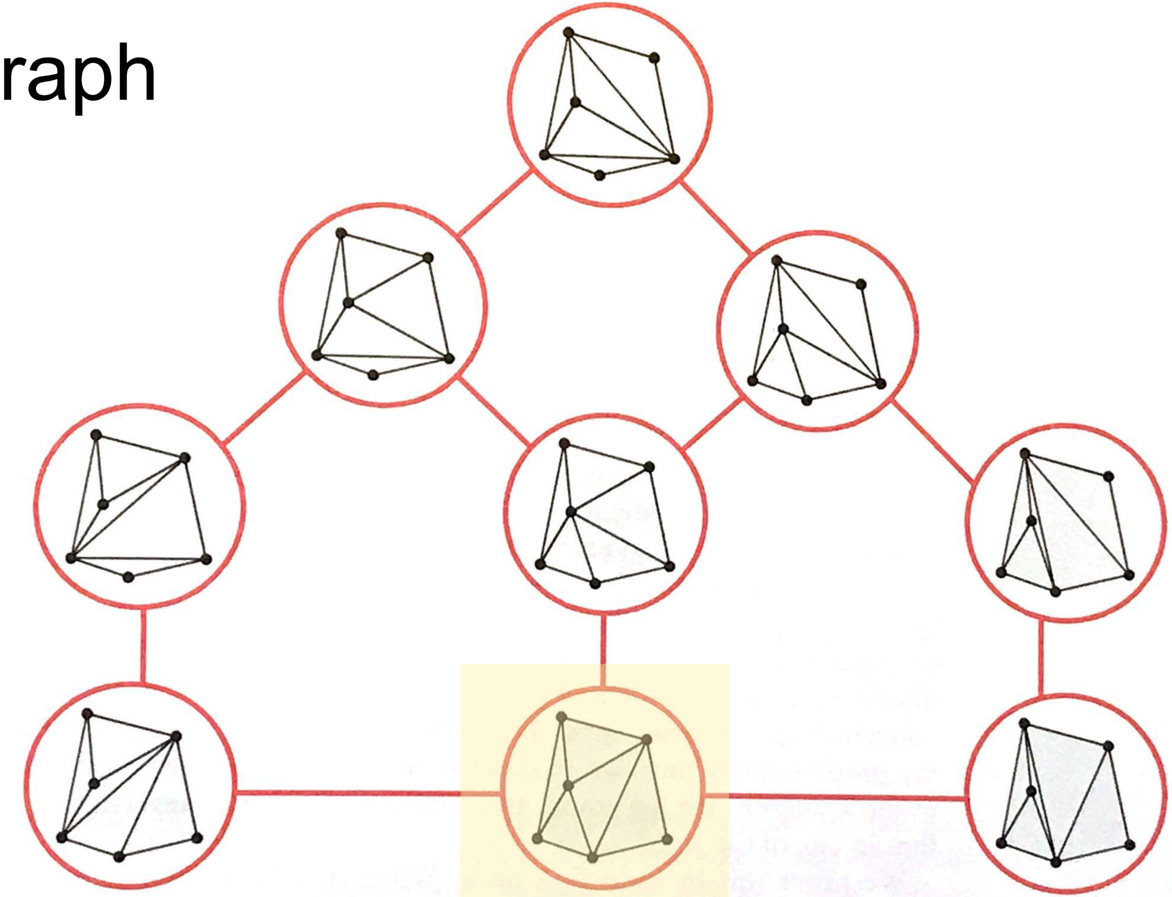


[5, 5, 20, 30, 30, 40, 70, 50, 50, 50, 90, 90, 100, 100, 170]

Walking the Flip Graph

- The Delaunay Triangulation is the Angle-Optimal Triangulation

- *How many flips are necessary to reach the Delaunay Triangulation?*



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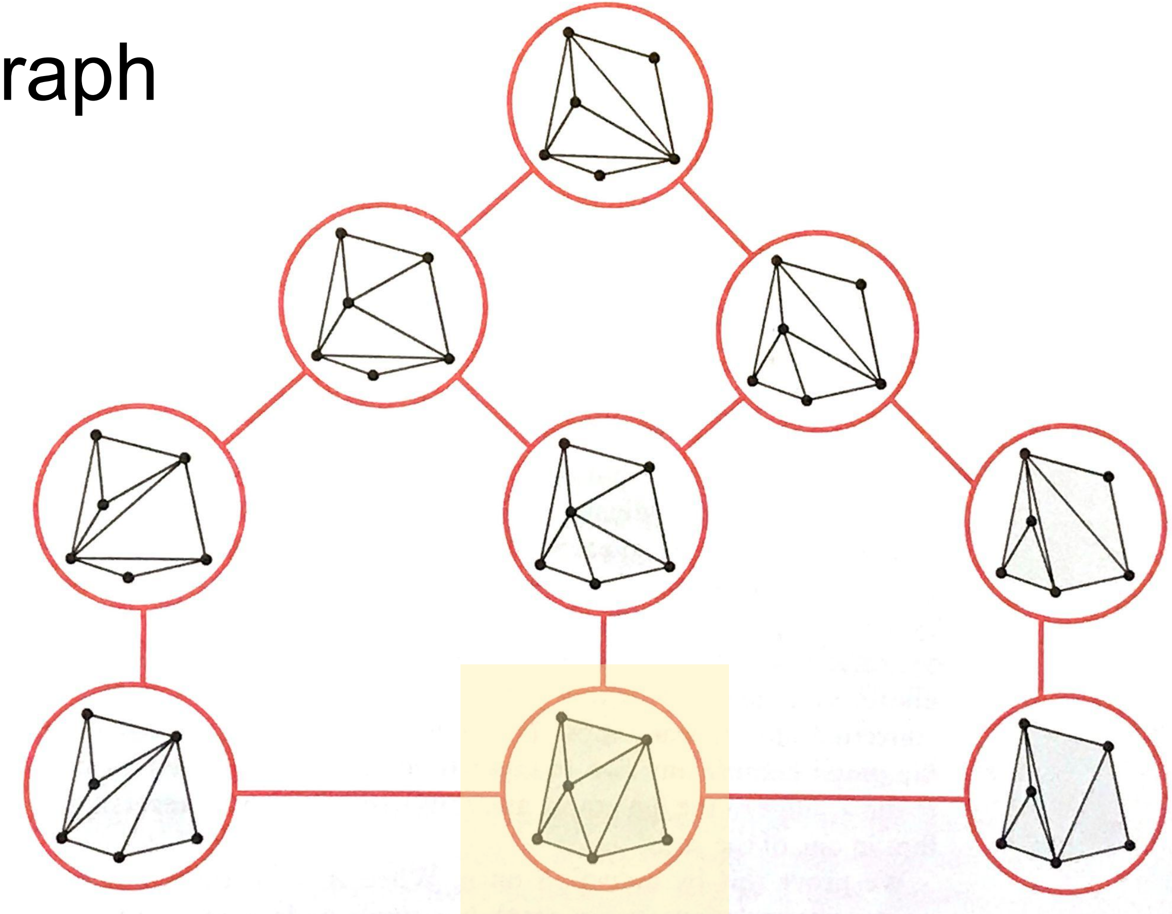
Walking the Flip Graph

- The Delaunay Triangulation is the Angle-Optimal Triangulation

- *How many flips are necessary to reach the Delaunay Triangulation?*
- What is the "*diameter*" of the flip graph?
  - At most (n-2) * (n-3)



See book for proof…

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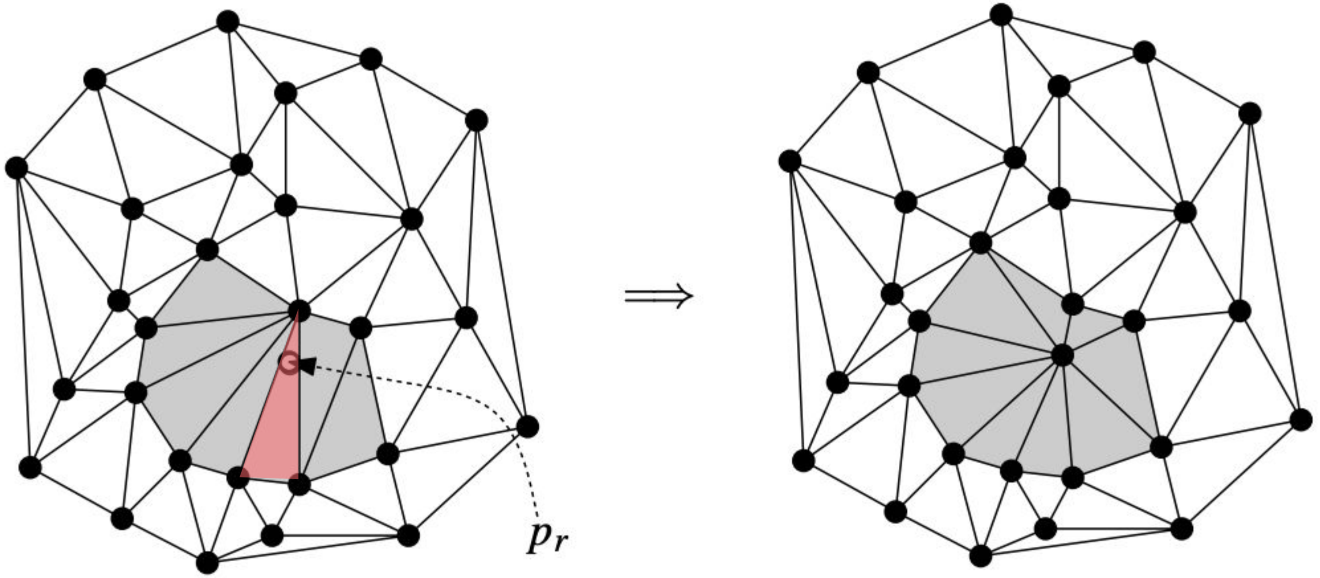
Randomized Incremental Construction of Delaunay Triangulation

- Randomize order of points and insert one at a time
- Identify which triangle contains p_r
- Split into 3 smaller triangles
- Flip neighboring edges as necessary



Hopefully the footprint of impact is small!

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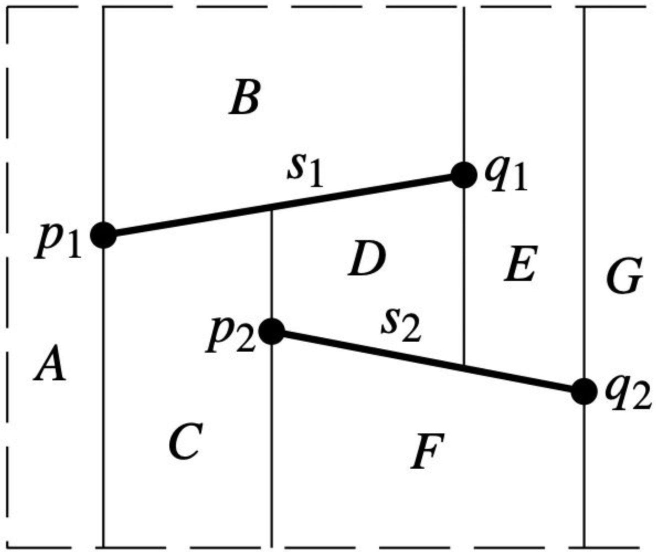
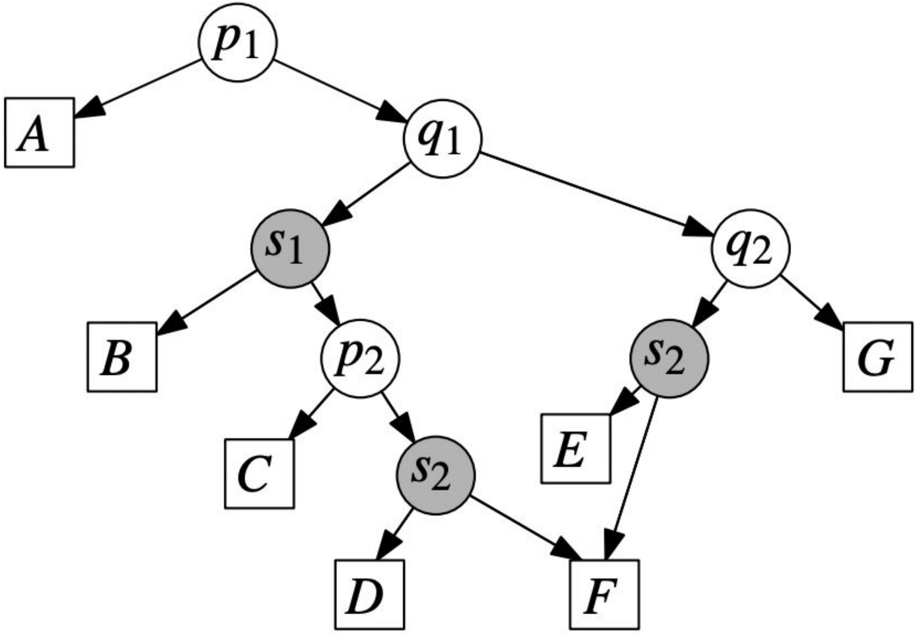
Lecture 9: Point Location by Directed Acyclic Graph

- Intermediate notes are vertices (vertical lines) and line segments
- The leaves are the trapezoidal regions (map back to original polygons)





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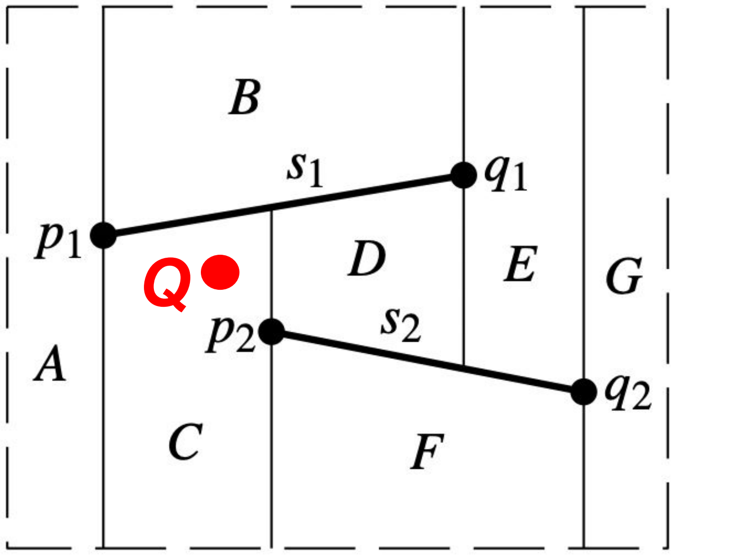

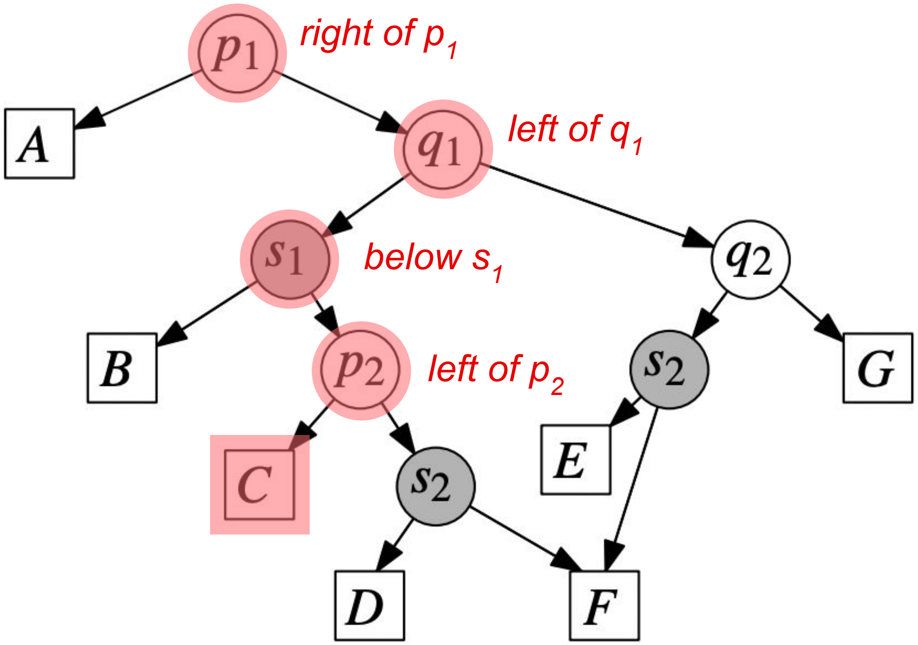

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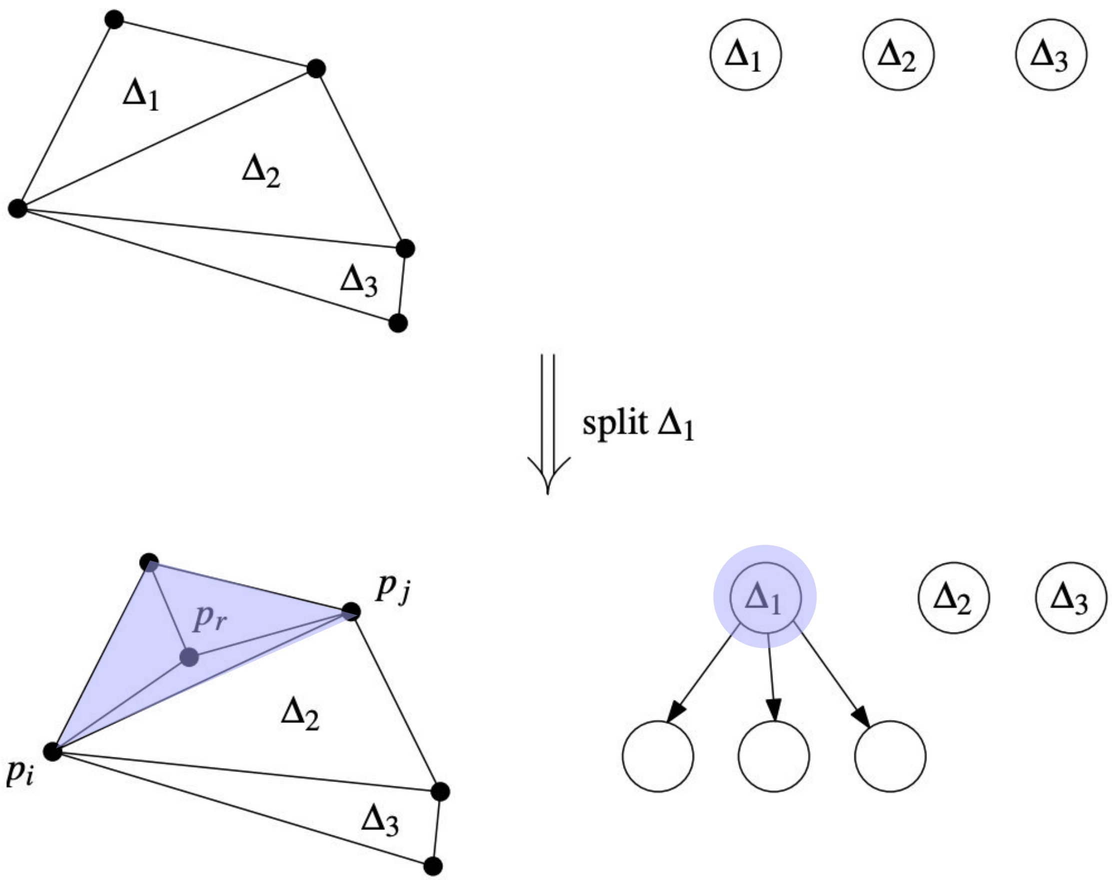
- Similarly…
  we'll construct a directed acyclic graph (DAG) of triangles
- The leaves will be the final triangulation
- We can use this to identify which triangle contains p_r
- And then split this triangle into 3 smaller triangles



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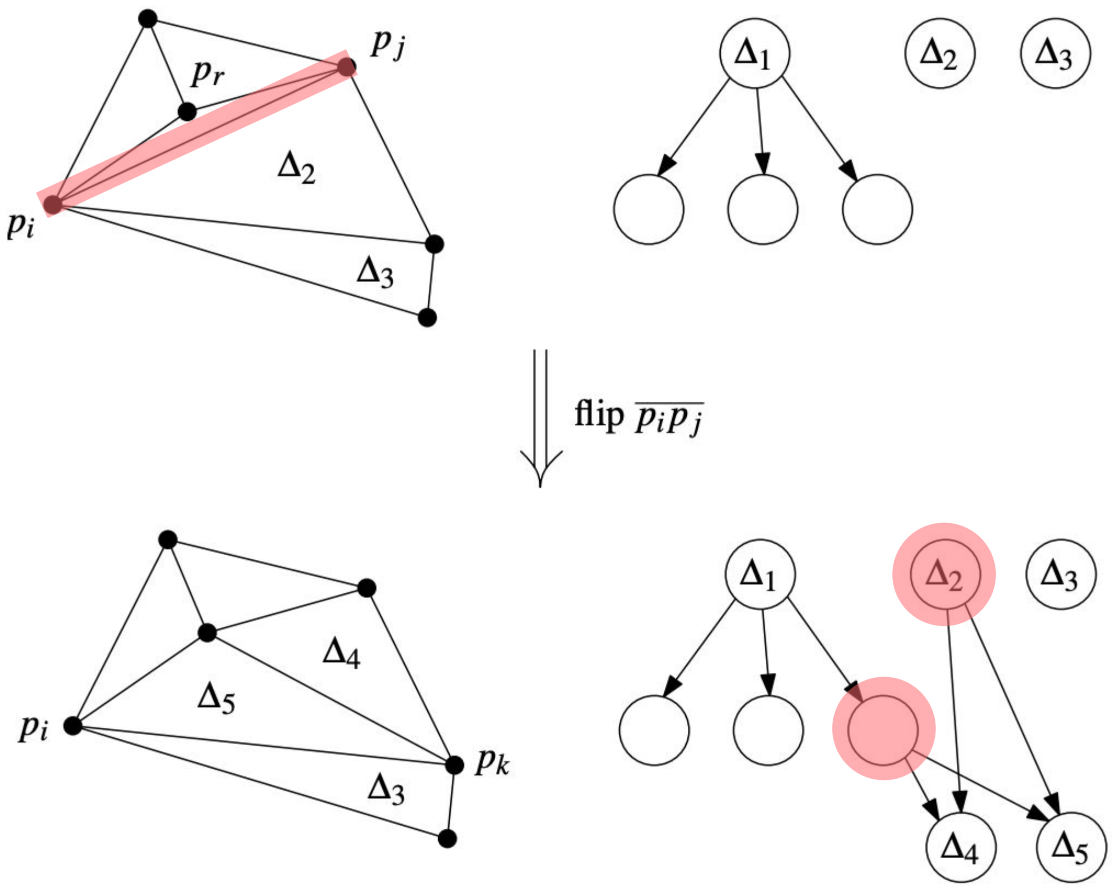
- Check the "legality" of the edges of the new triangles

- Flip edges if necessary
- Add new triangles to DAG

- & recurse



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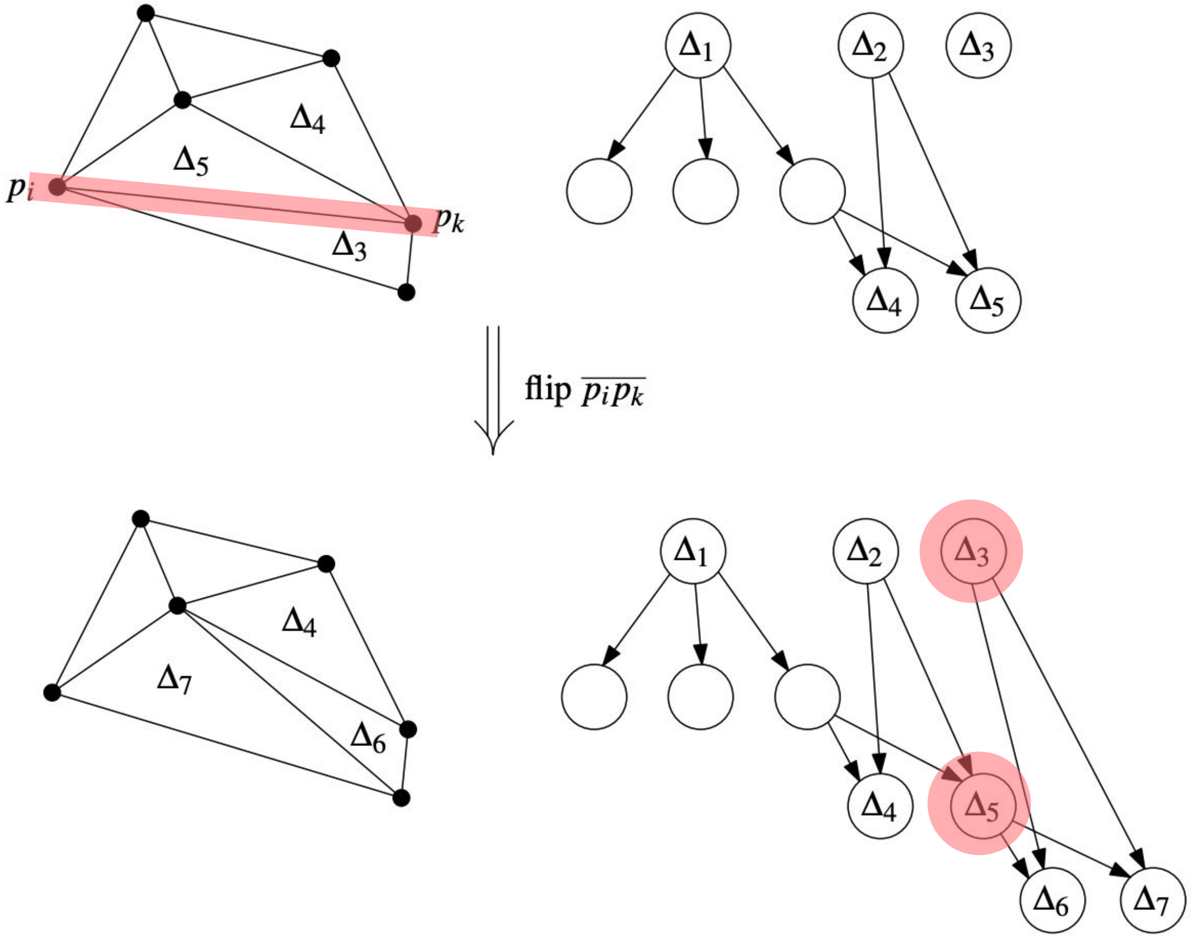

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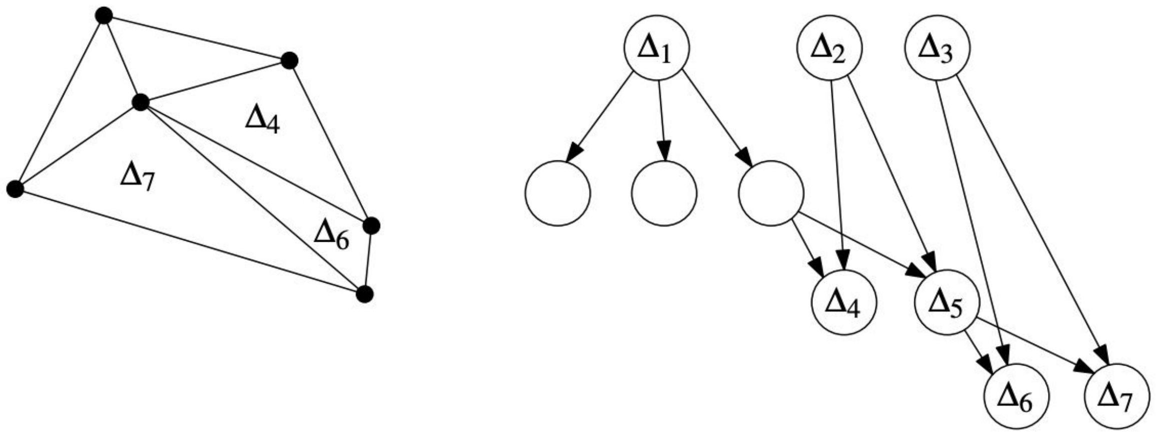
Randomized Incremental Construction of Delaunay Triangulation Analysis

- For *n* points, inserted one at a time
- Point location in DAG

- Split triangle

- Check edge legality
  Do edge flips
  & Recurse

- Overall



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Randomized Incremental Construction of Delaunay Triangulation Analysis

- For n points, inserted one at a time
- Point location in DAG

  → $O(\log n)$
- Split triangle

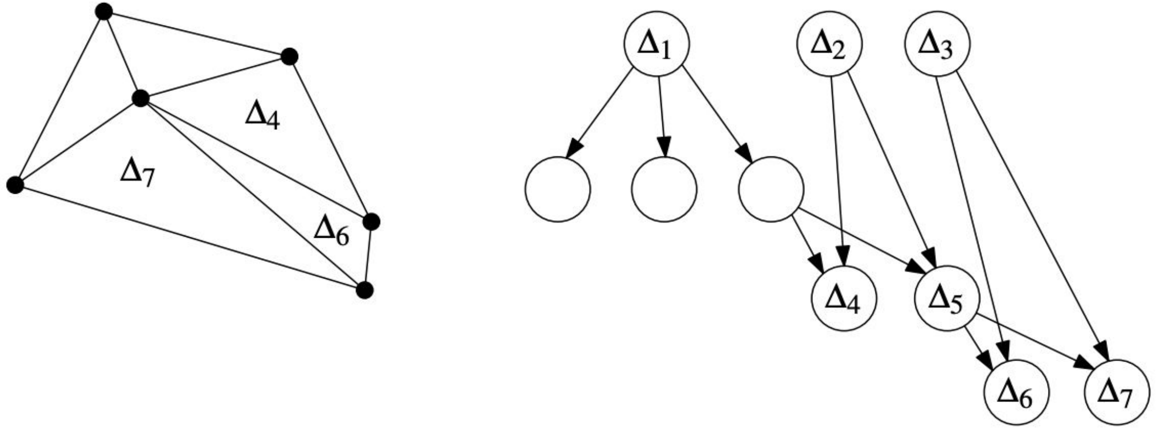
  → $O(1)$
- Check edge legality

  Do edge flips

  & Recurse

  → $O(1)$ *expected*
- Overall

  → $O(n \log n)$



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Outline for Today

- Final Project: Brainstorming Ideas & Partner Matching
- Motivation Review: Terrain Height Maps
- Polygon Triangulation vs. Point Set Triangulation
- Counting the Number of Triangulations
- Incremental Triangulation by Point Insertion
- Incremental Triangulation by Line Sweep
- Flip Graph & Connectedness of all Triangulations
- Delaunay Construction by Edge Flips
- Randomized Incremental Delaunay Triangulation
- Delaunay Triangulation Construction Analysis Summary

Delaunay Construction Analysis Summary

- Brute force (enumerate all triangles, construct circles, reject…)

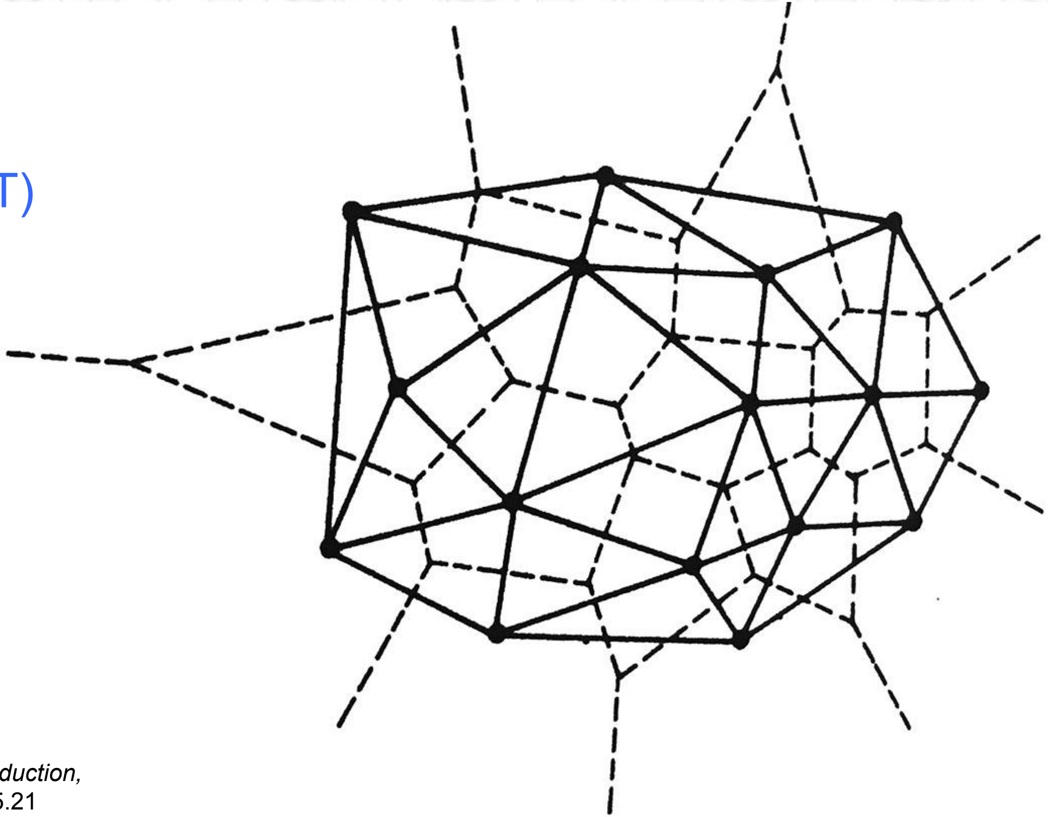
- Construct any triangulation & Flip until all edges are legal

- Randomized Incremental Construction

- By duality, reduce to problem of Constructing the Voronoi Diagram

Dual: Voronoi Diagram & Delaunay Triangulation

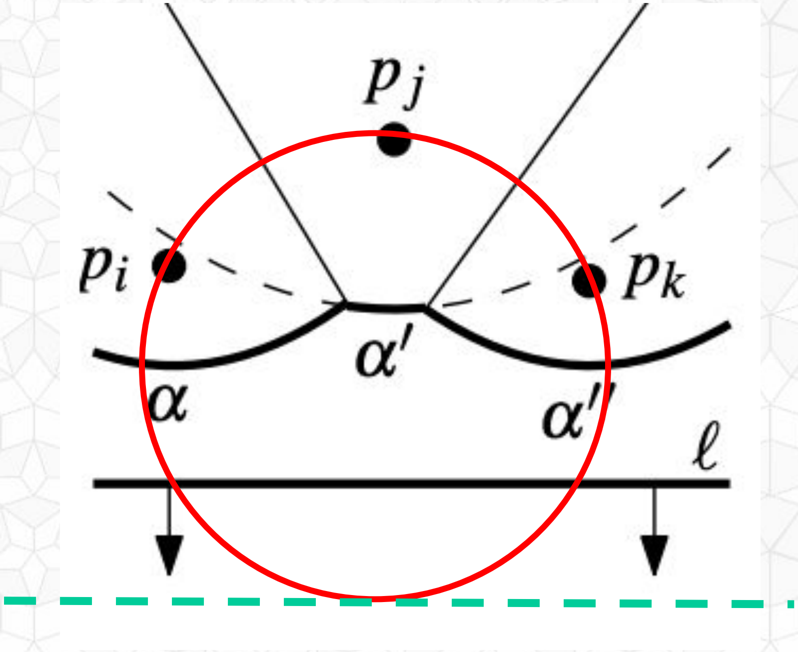
- The Voronoi Diagram (VD) *is the dual of the* Delaunay Triangulation (DT)
- Every Voronoi Site is a face in Voronoi Diagram and a vertex in the DT
- Every Voronoi Edge is an edge in the DT
- Every Voronoi Vertex is a triangle in the DT



Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.21

Lecture 10: Voronoi Sweep Line Algorithm

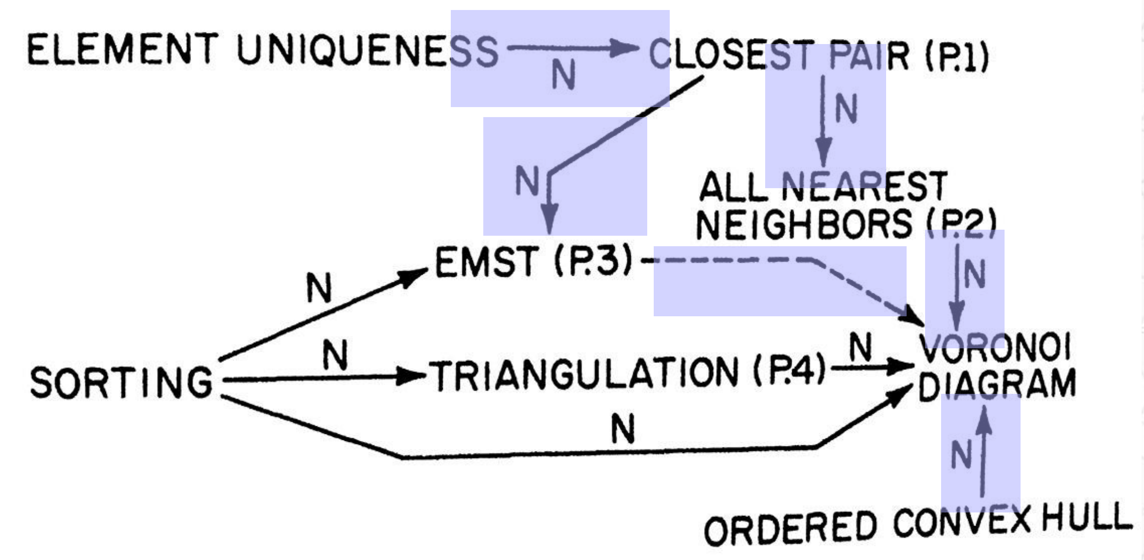
- For *n* Voronoi sites
- New Arc Events: Sort Voronoi sites vertically → *O(n log n)*
- Keep a horizontal sorted ordering of the parabolic arcs on the current beachline. 2*n* arcs maximum
- (Potential) Arc Absorption Events: For each triple of neighboring arcs α, α', α'' on the beachline, compute the **circle**, and **tangent sweep line** → *O(n)* Voronoi vertices
- Move sweep line to the next event…
- Overall: → *O(n log n)*



Computational Geometry Algorithms and Applications, de Berg, Cheong, van Kreveld and Overmars, Chapter 7

Lecture 12: Problems that Reduce to Voronoi Diagram

- We can compute the Voronoi Diagram of *n* points in *O(n log n)* time and *O(n)* space.
- These other problems can be computed in *O(n)* additional time if given the Voronoi Diagram.



Computational Geometry: An Introduction,
Preparata & Shamos, Figure 5.30

- *Therefore they are also O(n log n) time and O(n) space.*

Delaunay Construction Analysis Summary

- Brute force (enumerate all triangles, construct circles, reject…)

  → $O(n^3 * n) = O(n^4)$

- Construct any triangulation & Flip until all edges are legal

  → $O(n^2)$

- Randomized Incremental Construction

  → $O(n \log n)$

- By duality, reduce to problem of Constructing the Voronoi Diagram

  → $O(n \log n)$