CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/S22/

Lecture 16: Windowing, Interval & Segment Trees

Outline for Today

- Review from Last Time: Delaunay Triangulations
- Motivation: Cartography Windowing & Data Selection
- Lecture 8 Review: Points in k-D trees
- 1D Interval Tree
- 1D Interval Tree Analysis
- 2D Interval Tree + Range Tree
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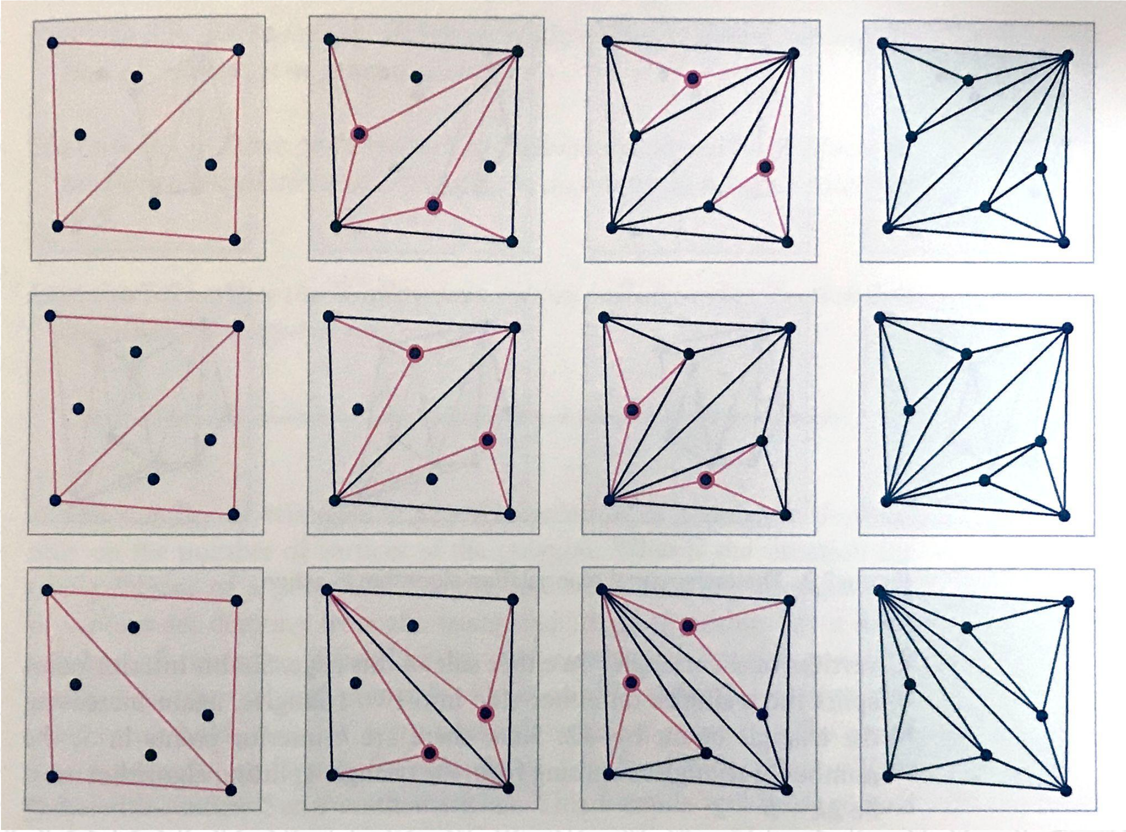
Construction by Point Insertion

"Discrete and Computational Geometry", Devadoss & O'Rourke, Princeton University Press 2011, Chapter 3

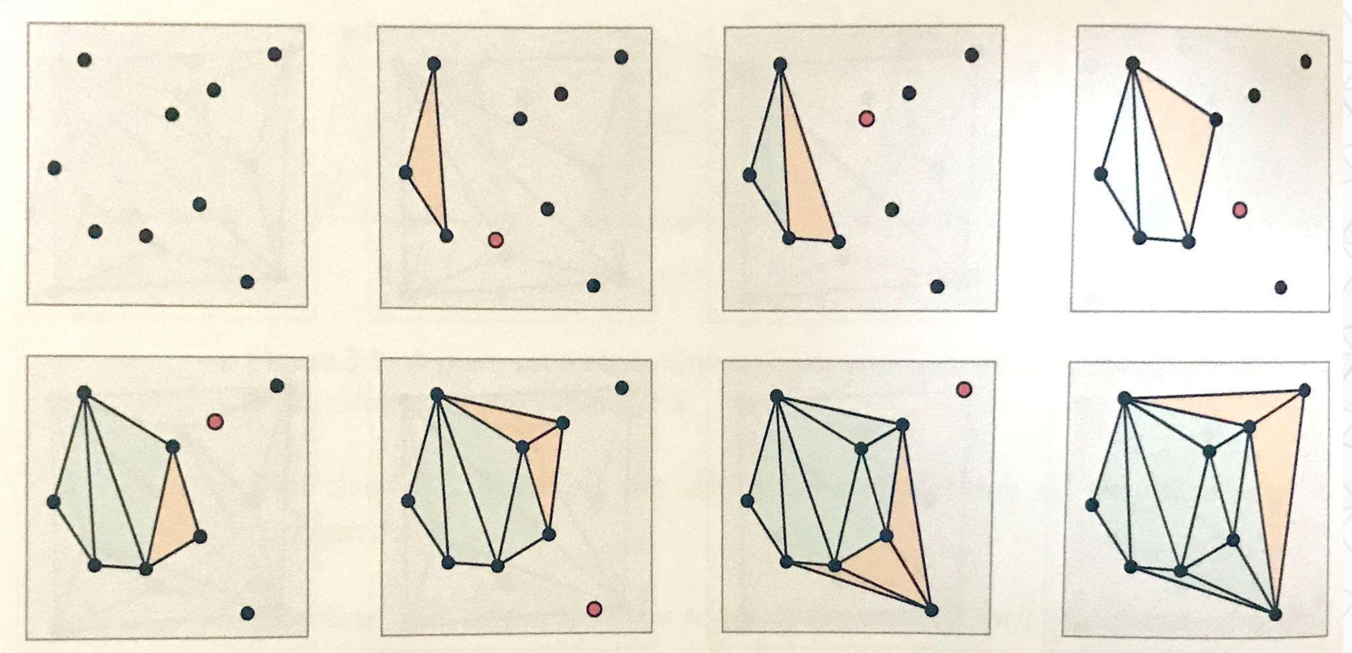
- Start with convex hull
  - Triangulate it
  - *k*-2 triangles
- For some ordering of the other points
  - Determine which triangle the point lies inside of
  - Replace that triangle with 3 triangles
  - (*n* - *k*) * 2 additional triangles
- 2**n* - *k* - 2 total triangles!



Construction by Line Sweep

"Discrete and Computational Geometry", Devadoss & O'Rourke, Princeton University Press 2011, Chapter 3

- Sort the input points by x
- Form a triangle with the 3 leftmost points

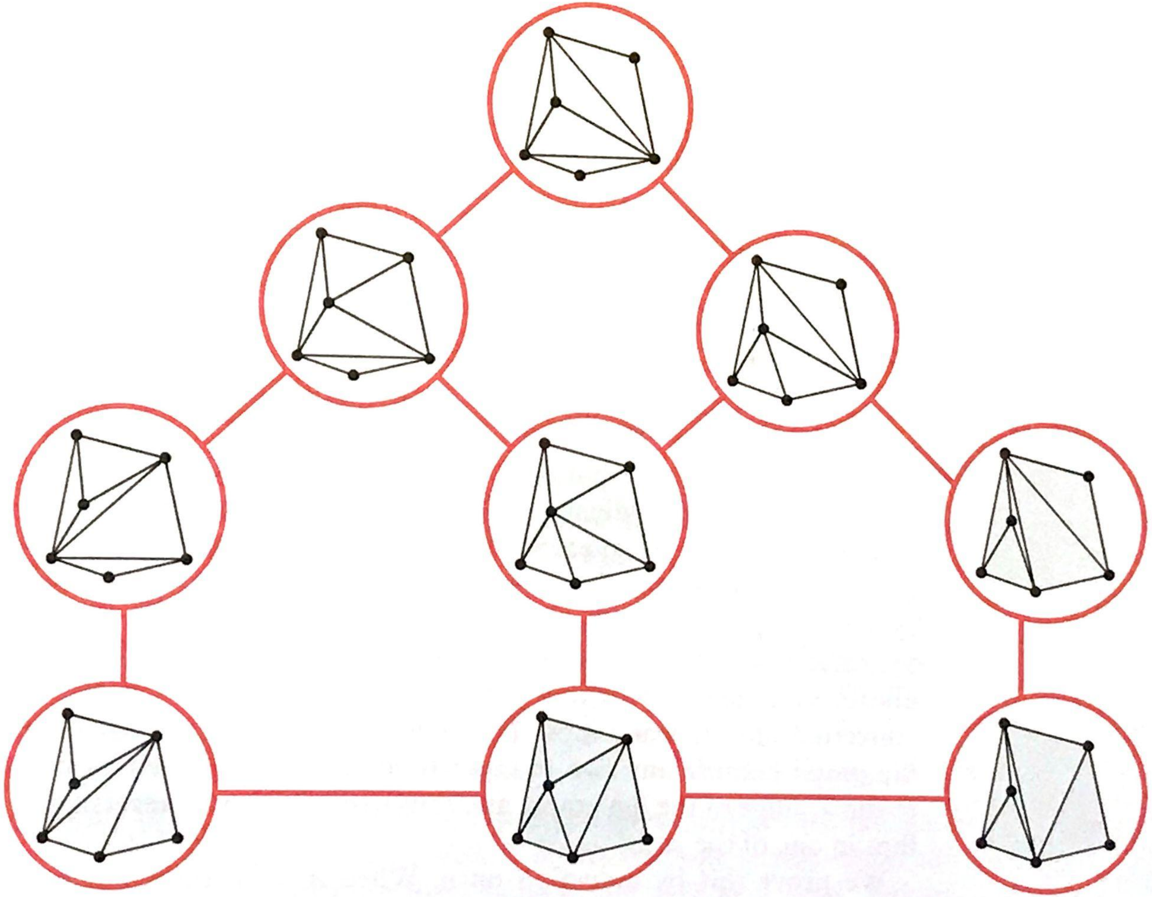


- Add every other point from left to right
  - Determine which points on the current hull are visible from the new point
  - Add a fan of triangles connecting the new point to the visible hull points

The Flip Graph

- If we did generate every triangulation…

- Let's organize the triangulations as nodes in a graph
- We'll put an edge between two nodes if flipping a single edge converts one triangulation into the other triangulation



"Discrete and Computational Geometry", Devadoss & O'Rourke, Princeton University Press 2011, Chapter 3

Delaunay Construction Analysis Summary

- Brute force (enumerate all triangles, construct circles, reject…)

  → $O(n^3 * n) = O(n^4)$

- Construct any triangulation & Flip until all edges are legal

  → $O(n^2)$

- Randomized Incremental Construction

  → $O(n \log n)$

- By duality, reduce to problem of Constructing the Voronoi Diagram

  → $O(n \log n)$

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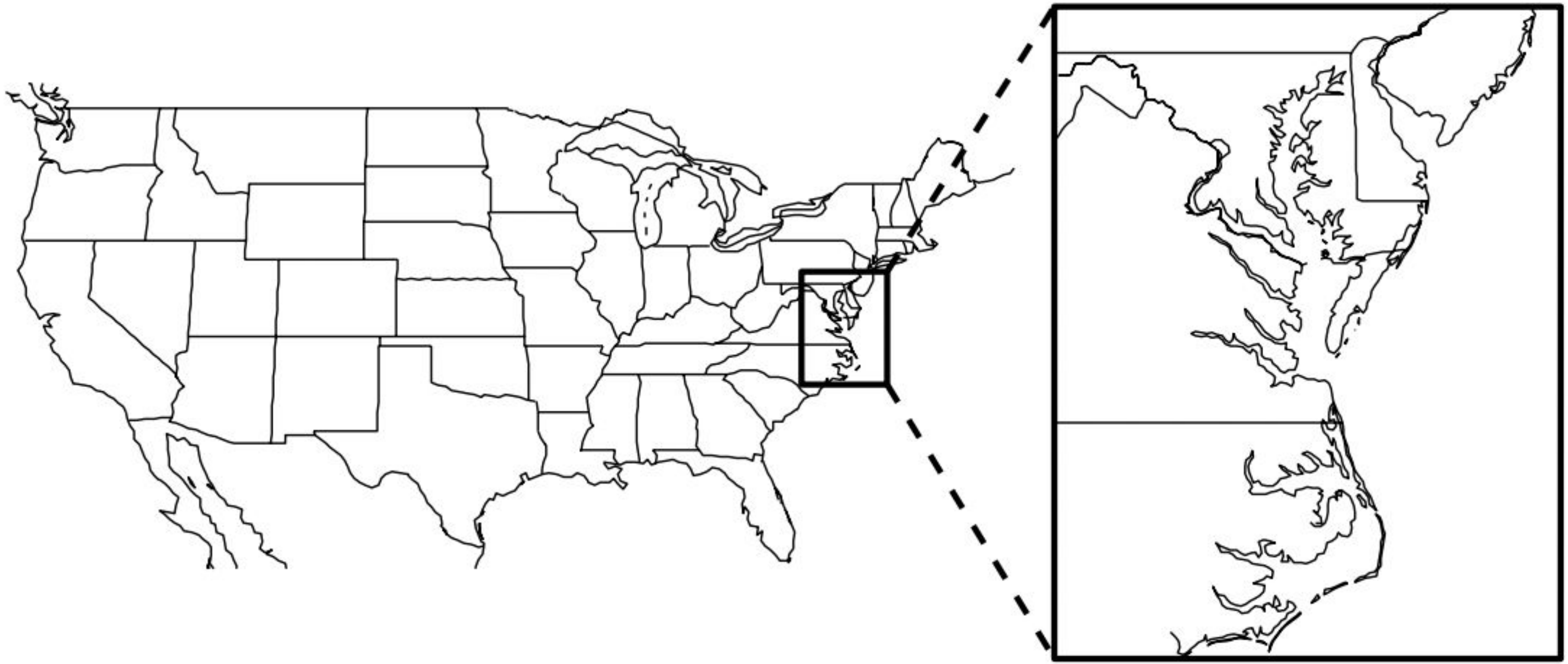
Motivation: Cartography (Map-Making)

- Select a small rectangular region to display in a window at larger scale



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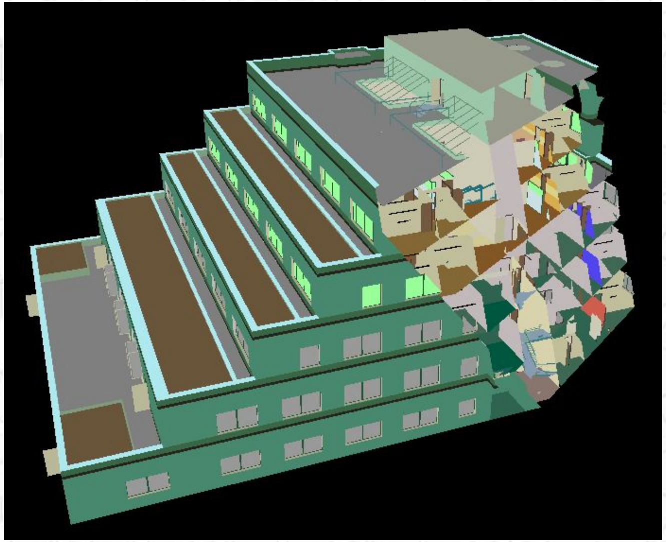
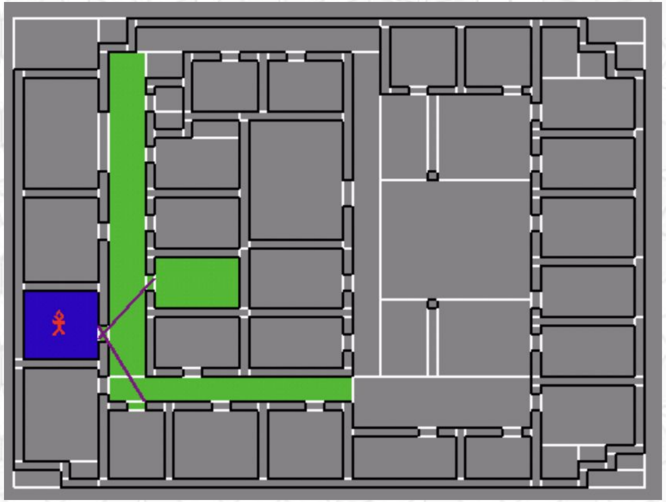
Motivation: Visibility

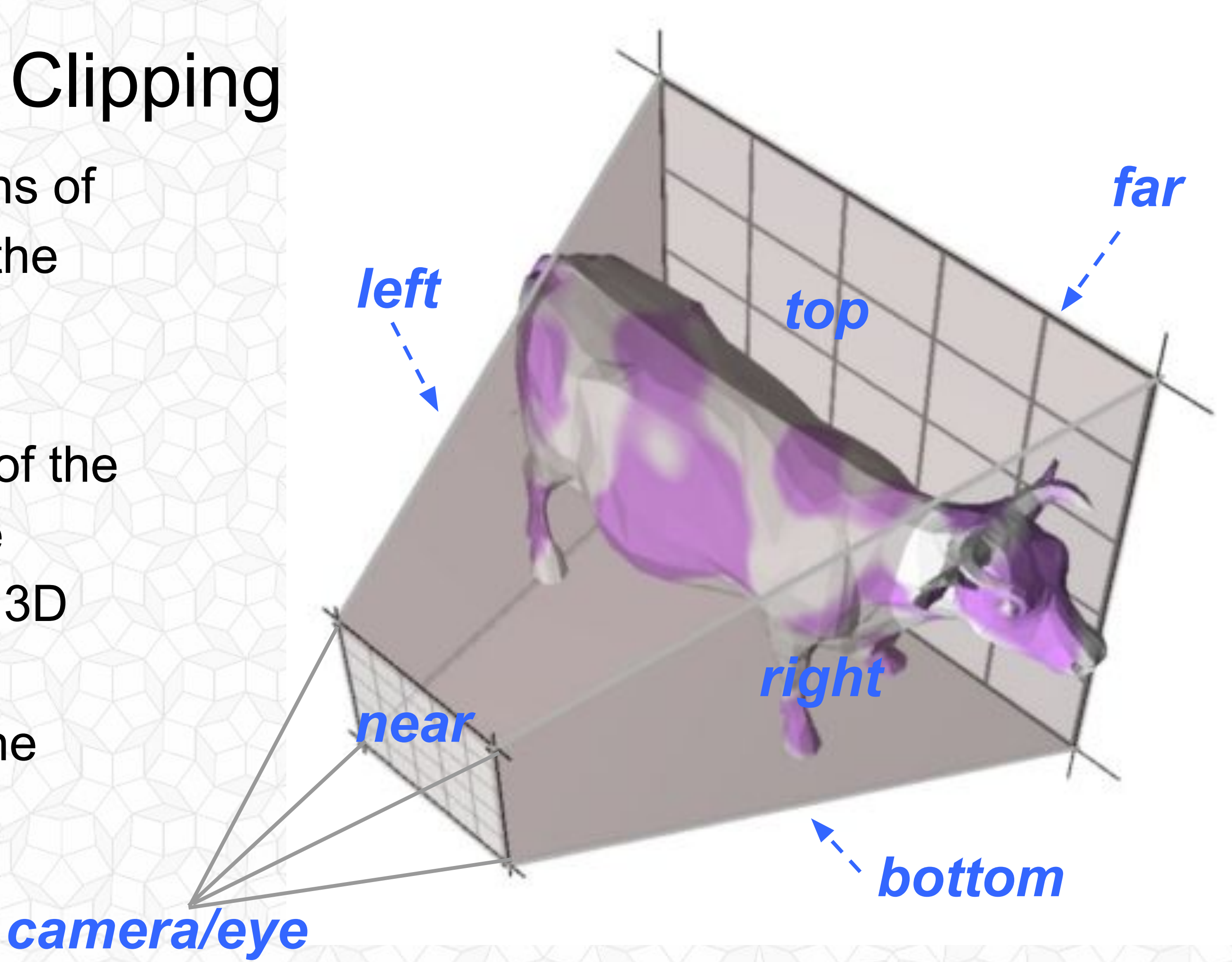




Seth Teller, PhD thesis, 1992,
Berkeley Soda Hall walkthrough

Graphics: 3D Clipping

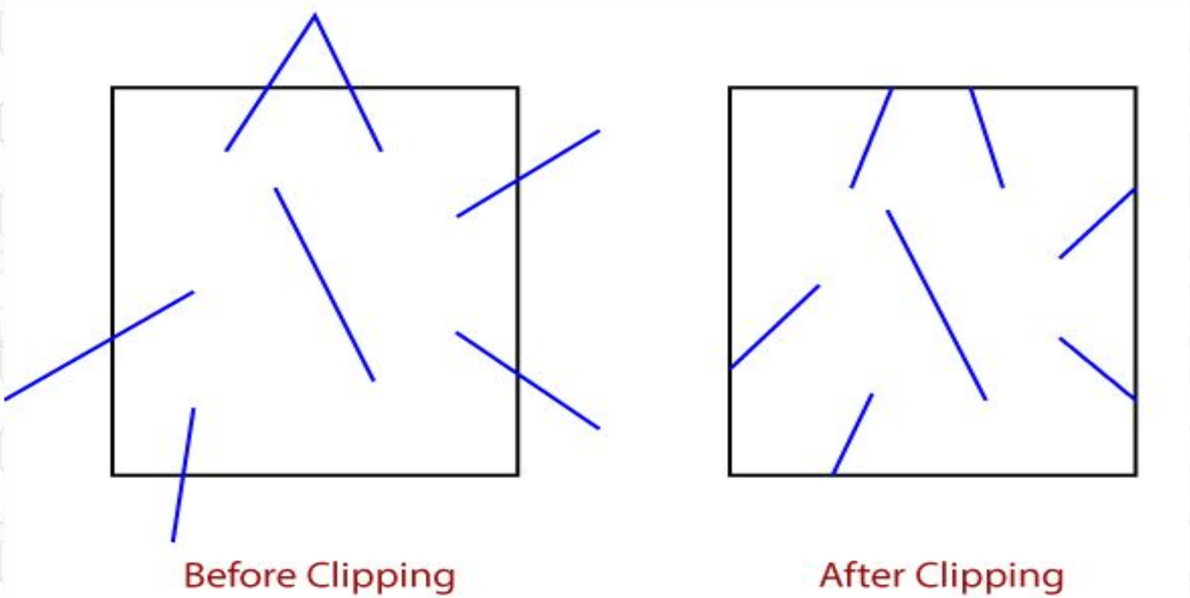
- Eliminate portions of objects outside the viewing frustum
- View Frustum
  - boundaries of the image plane projected in 3D
  - a near & far clipping plane



Graphics: 2D Clipping

Why do it?

- Reduce amount of geometry going through graphics pipeline
- Prevent rendering bugs from overflow, wraparound, things behind the camera, etc.



https://www.tutorialandexample.com/clipping-in-computer-graphics

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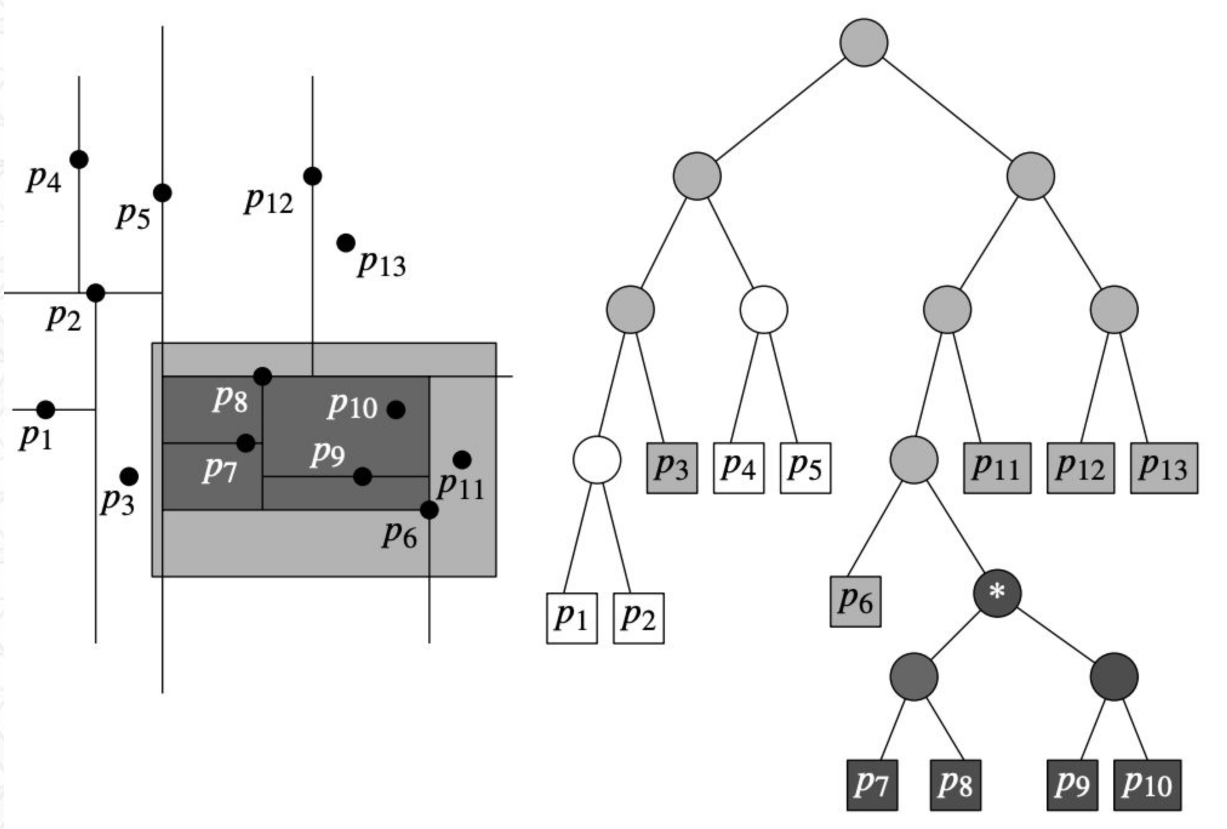
Review from Lecture 8: 2D k-d Tree

- Used to store points
- Alternate splitting horizontally & vertically
- If data is available for preprocess, the structure is easy to balance
- Point data is only stored at the leaves



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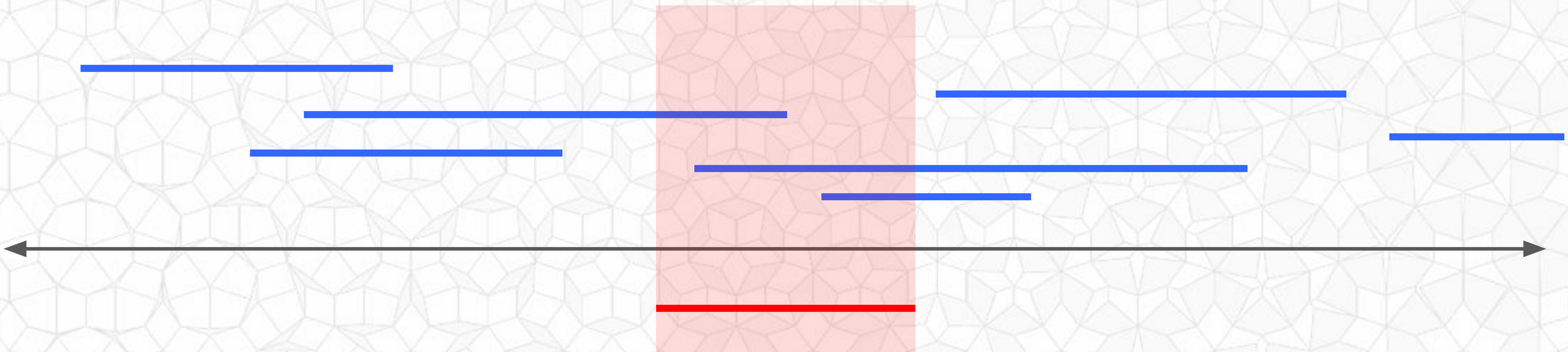
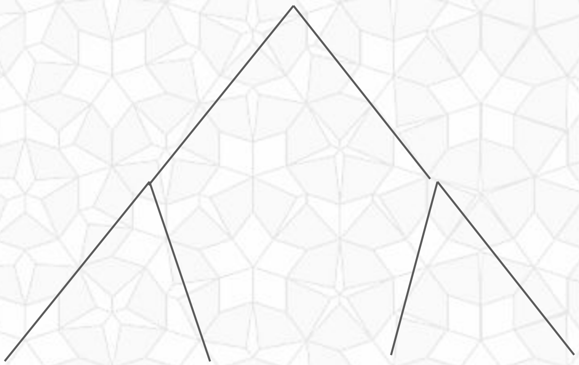
What about Segments? Let's Tackle 1D First…

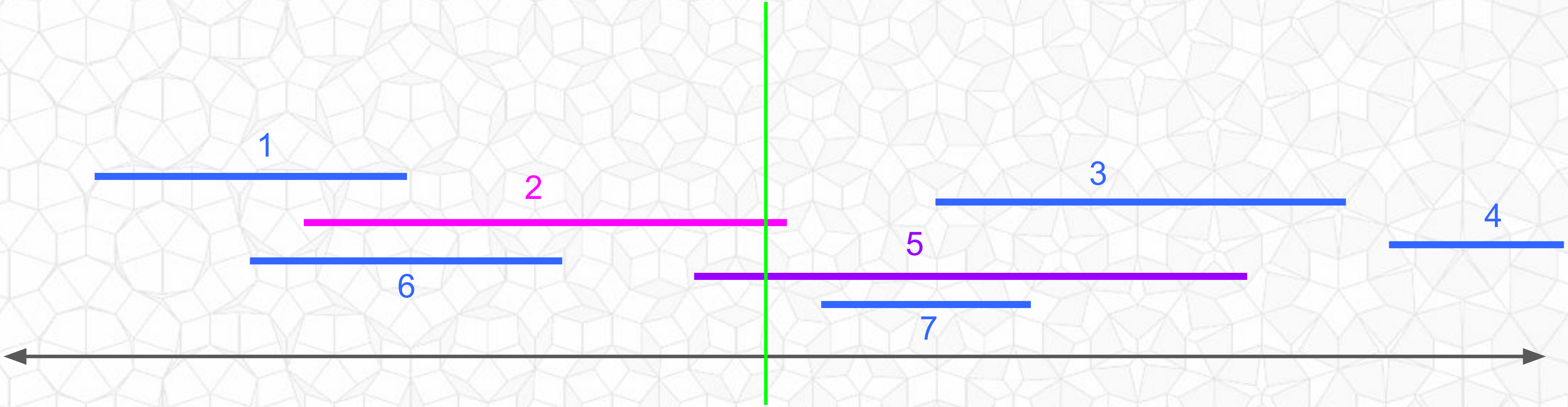
- Input: A collection of *n* line segments on the x-axis
- For a query interval, return all line segments that overlap the query interval



Traditional Binary Search Tree

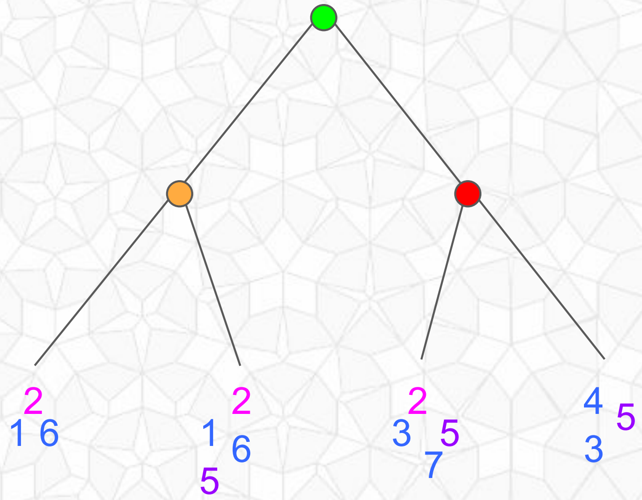


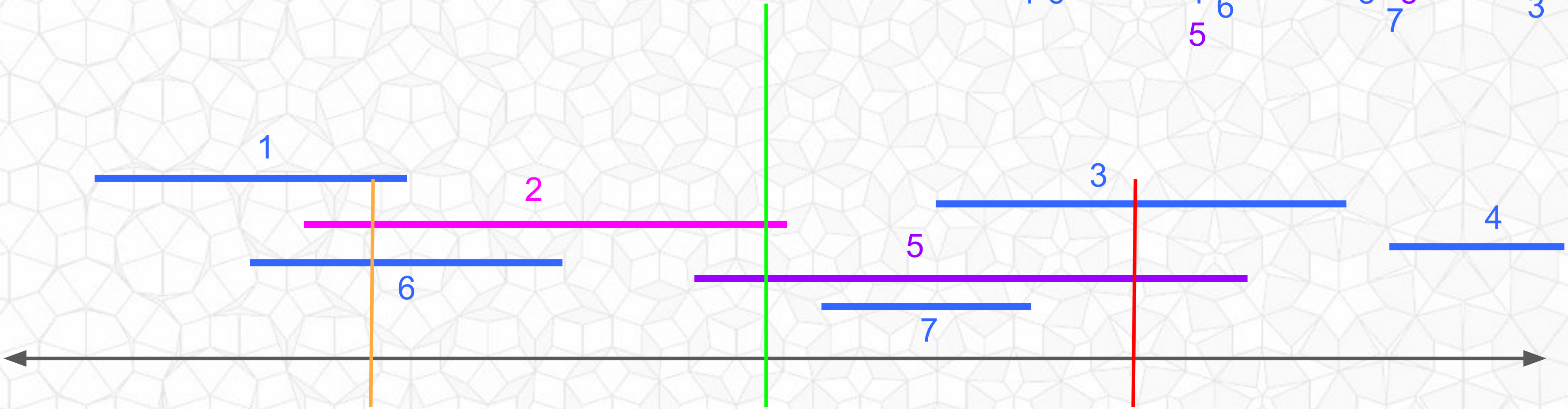
- Select split point near middle of data
- What about segments that overlap the split?



Traditional Binary Search Tree

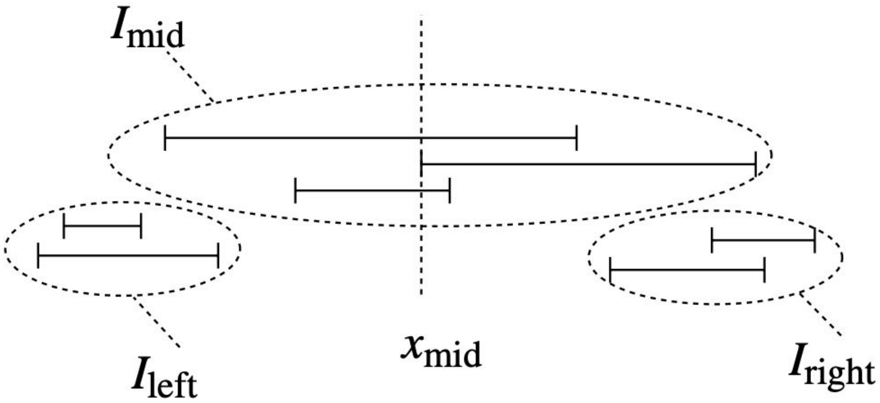
- Select split point near middle of data
- What about segments that overlap the split?
- Should we store them on both sides?
  - Uses extra memory
  - We may lose our O(log n) performance!

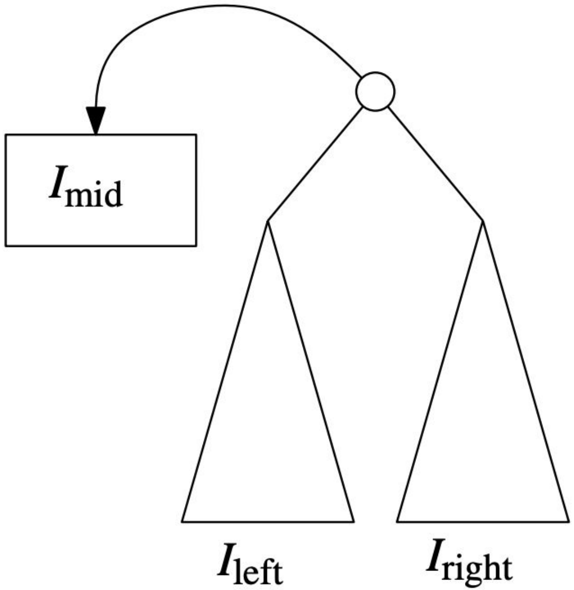




Interval Tree

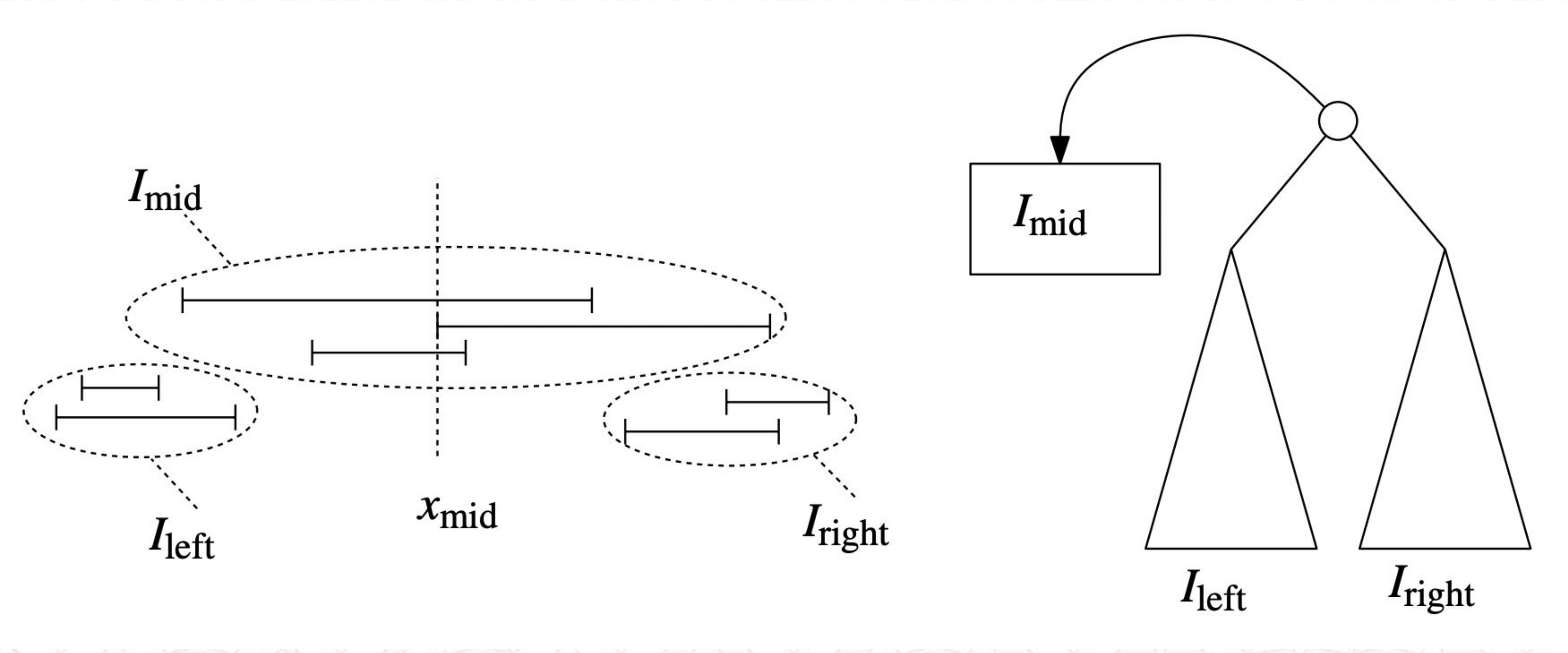
- Chose a split point and make 3 groups:
  - I_{mid} = Segments that overlap the split
  - I_{left} = Segments completely to the left
  - I_{right} = Segments completely to the right





Interval Tree

- Recurse down the tree only with items that DO NOT overlap the split point.

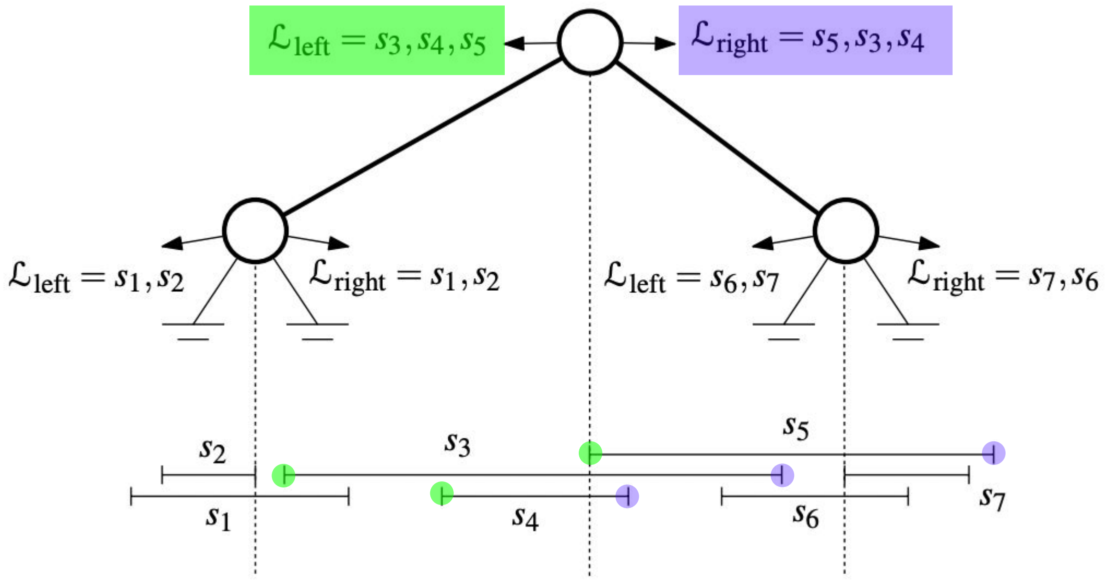


Interval Tree

- Items in I_{mid} group will stay at the current node

- Each node stores two two sorted lists:

- $L_{left} = I_{mid}$ sorted by left endpoint (increasing)
- $L_{right} = I_{mid}$ sorted by right endpoint (decreasing)



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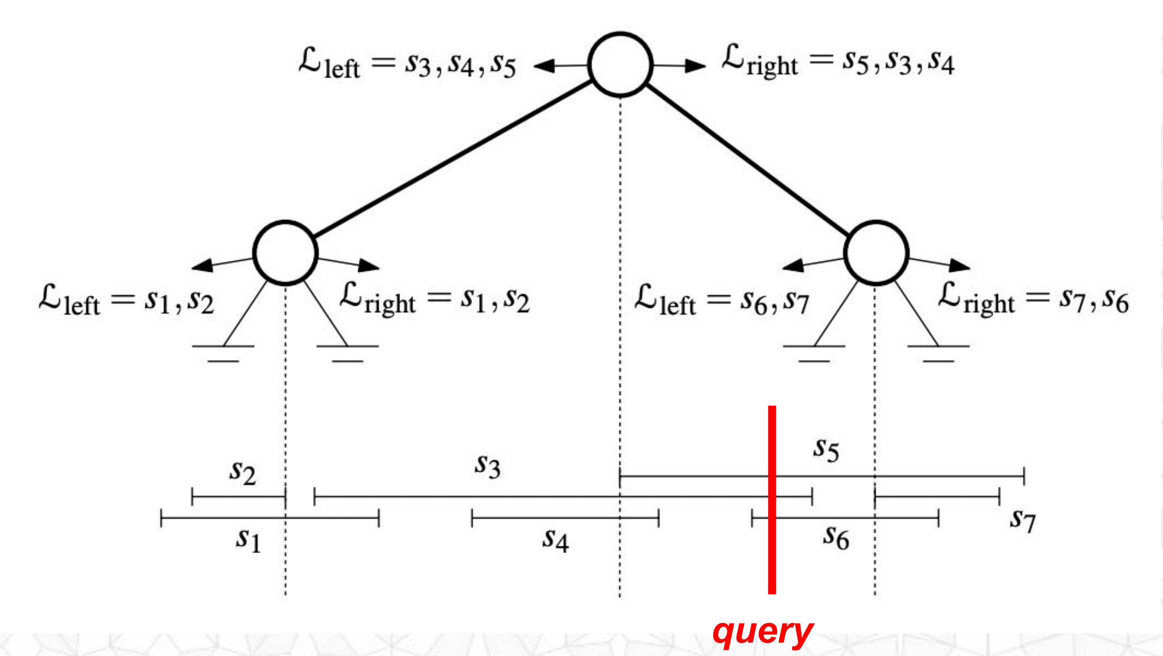
Interval Tree

- For a specific *query*



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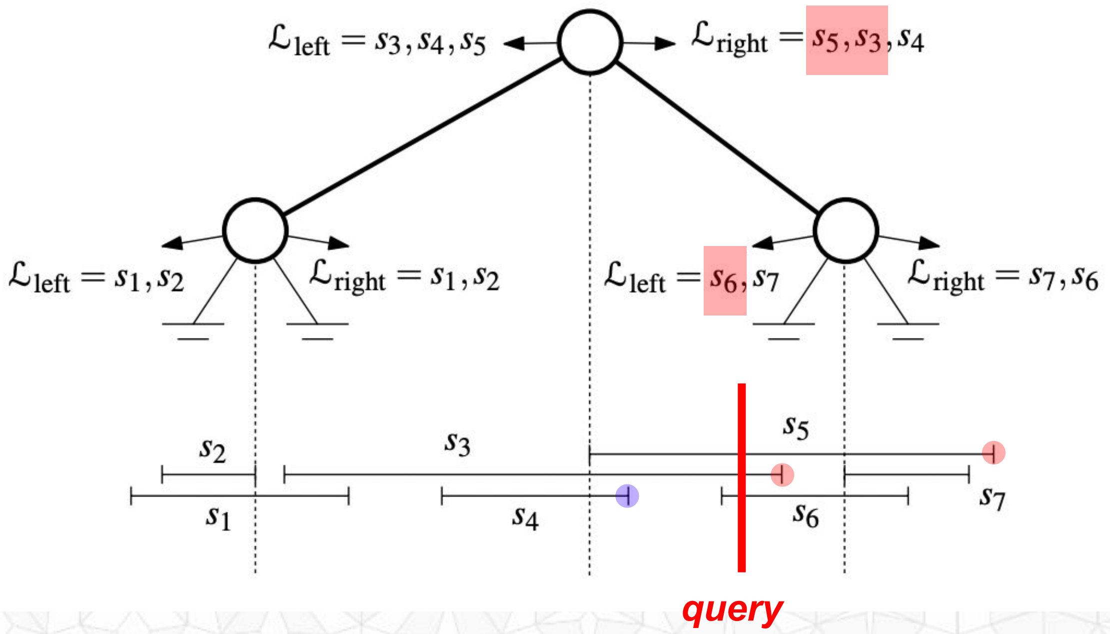
Interval Tree

- For a specific *query*
- Determine if the query is to the right (or left) of the current node
- ***Binary search*** within the L_{right} list (or L_{left} list) by right (or left) endpoint
  - Return all segments with endpoint further away from the query
- And recurse down the right (or left)



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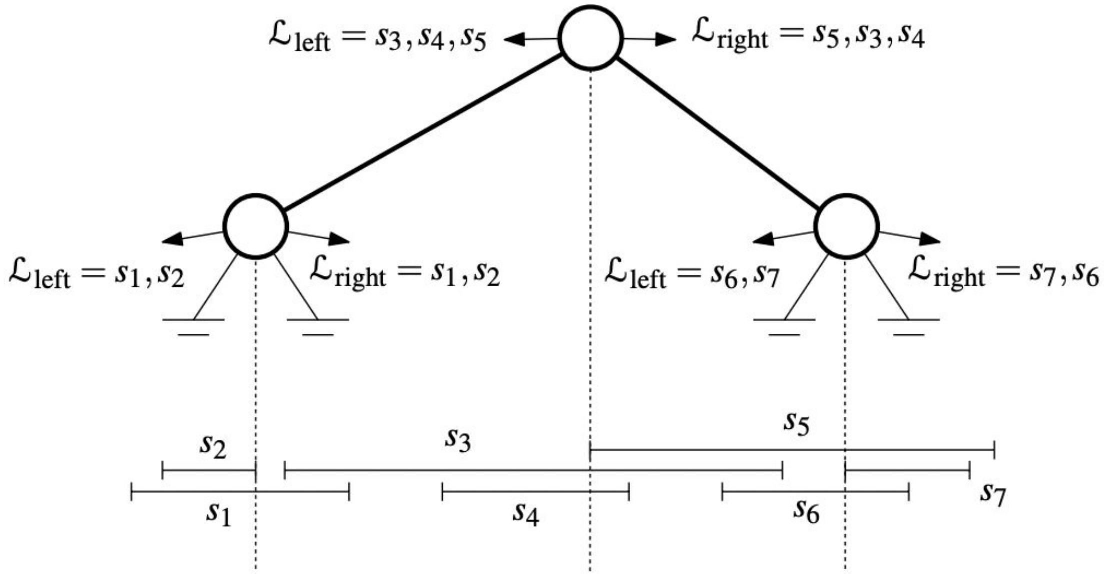
1D Interval Tree Analysis

- For *n* input segments and a query that will return *k* items
- Memory Usage:
- Construction Time:
- Query Time:



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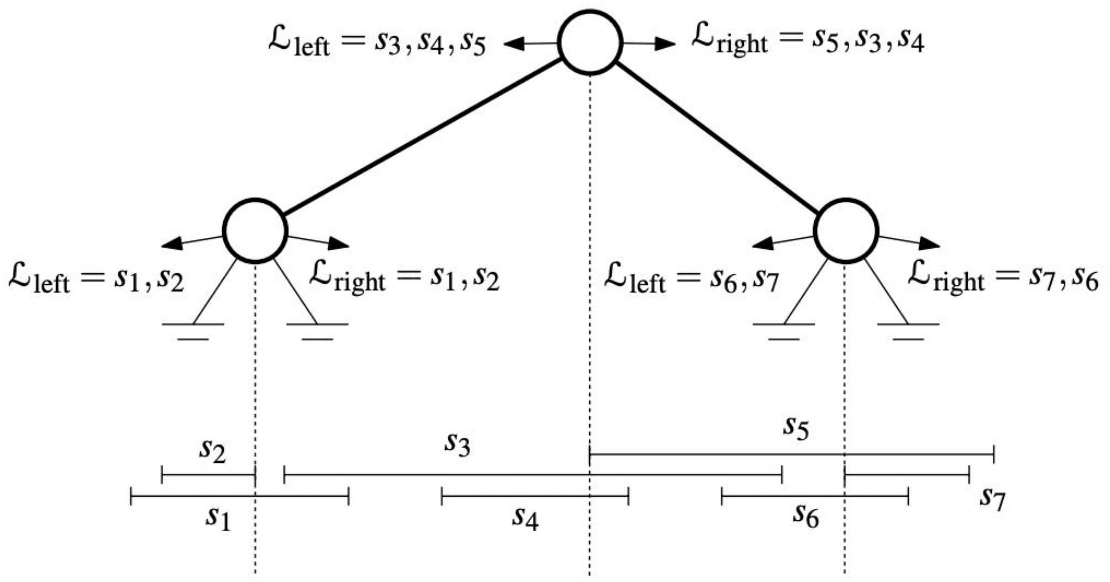
1D Interval Tree Analysis

- For n input segments and a query that will return k items
- Memory Usage:
  → $O(n)$
- Construction Time:
  → $O(n \log n)$
- Query Time:
  → $O(\log n + k)$



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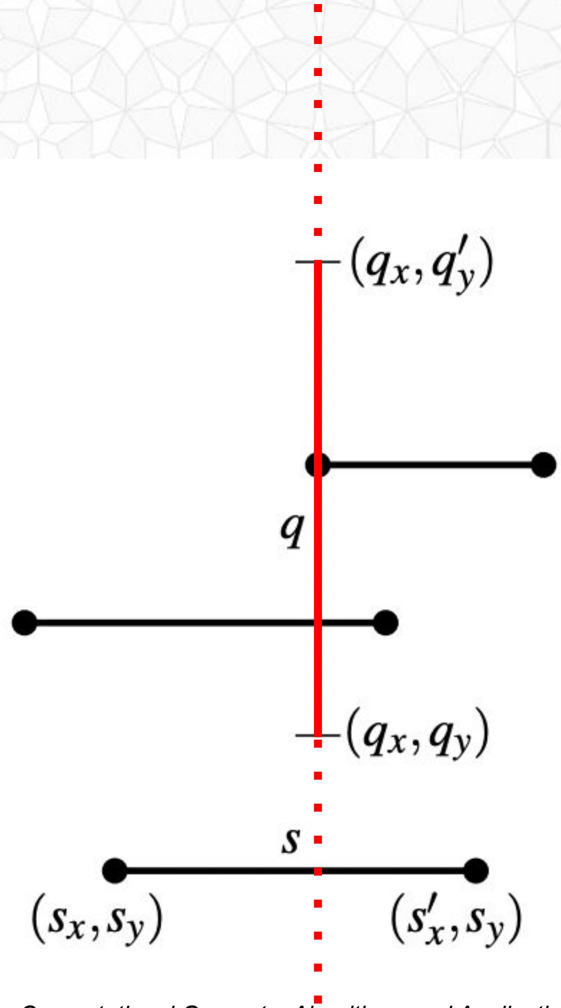
How do we Extend to 2D?

- First consider only horizontal input line segments
- And instead of a query line, we'll have a ***query line segment***



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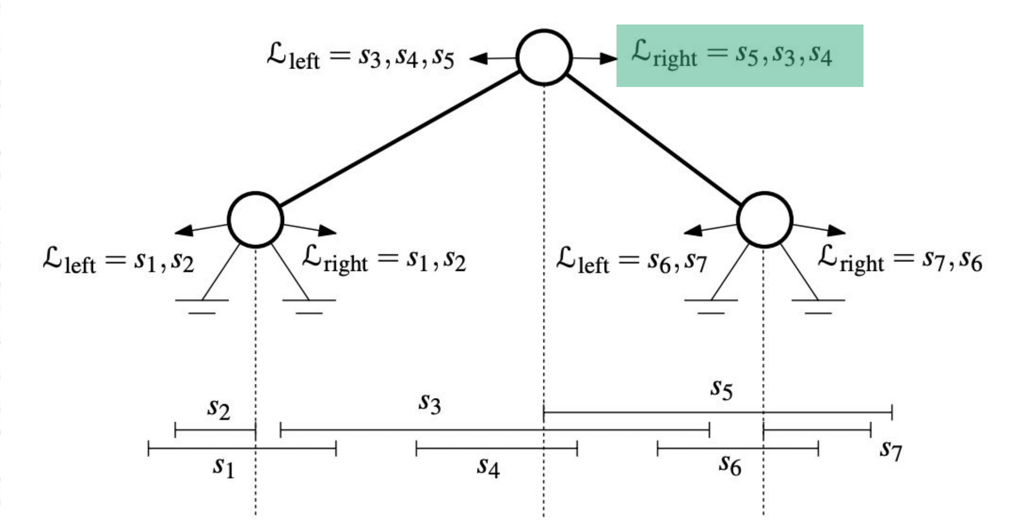
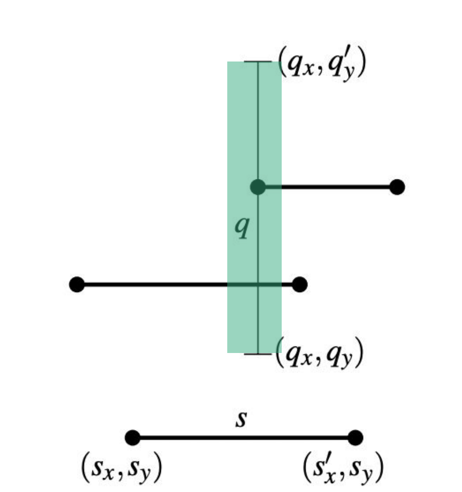
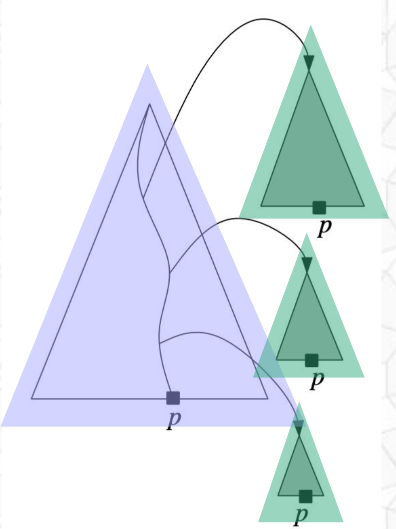
How do we Extend to 2D?

- We'll replace the sorted lists of the interval tree with a 2D range query (Lecture 8)
- This will require O(log n) additional memory…







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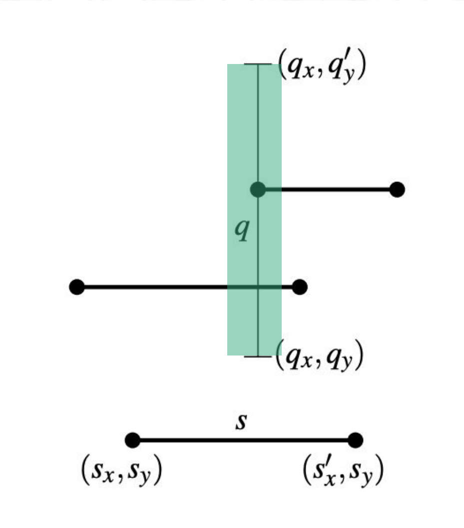
2D Interval Tree + Range Tree Analysis

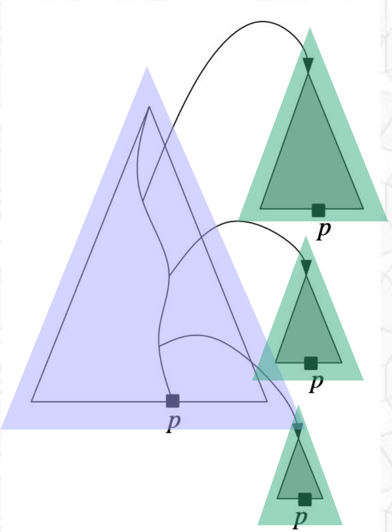
- For n horizontal input segments and a query segment that will return k items

- Memory Usage:

- Construction Time:

- Query Time:





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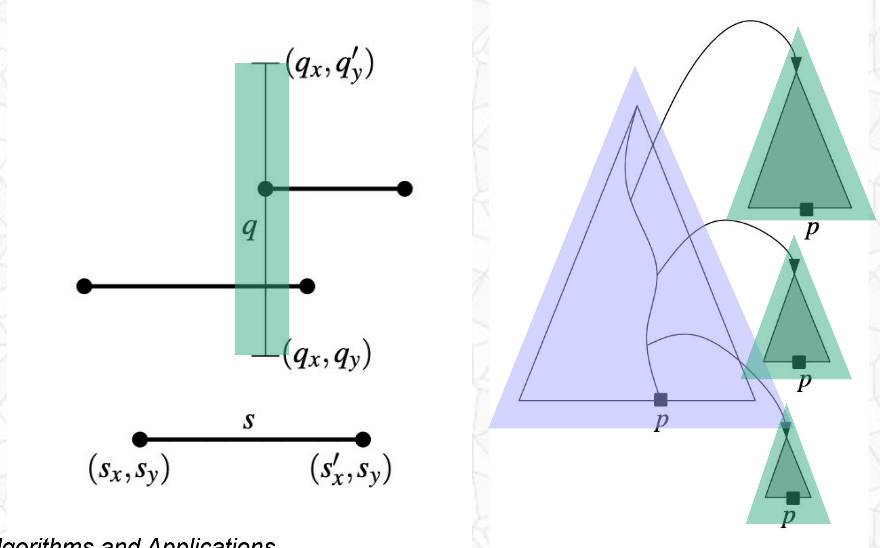
2D Interval Tree + Range Tree Analysis

- For n horizontal input segments and a query segment that will return k items

- Memory Usage:
  → $O(n \log n)$

- Construction Time:
  → $O(n \log n)$

- Query Time:
  → $O(\log n + k)$



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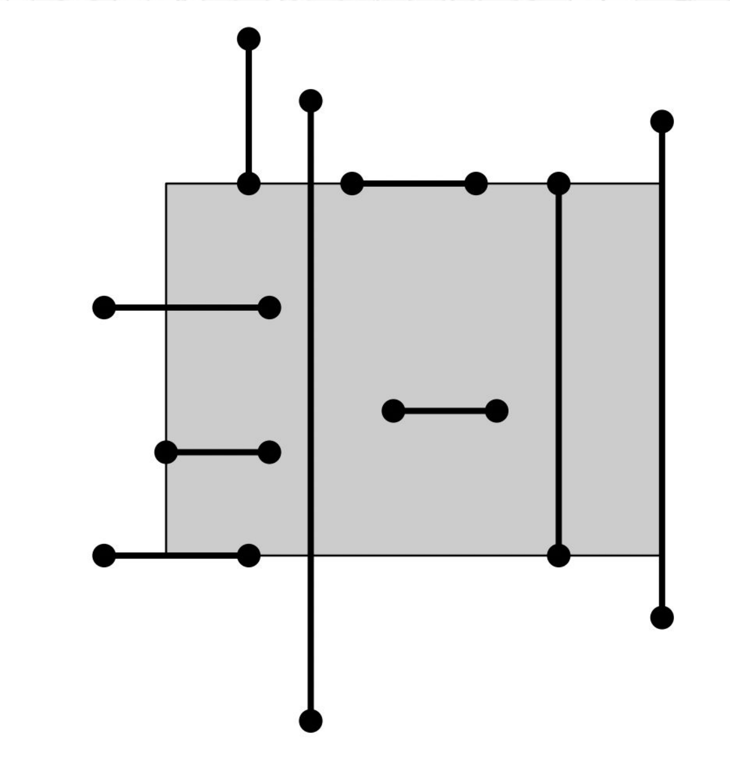
How to handle a 2D Axis-Aligned Query Box?

- Initially, let's restrict to horizontal & vertical segments



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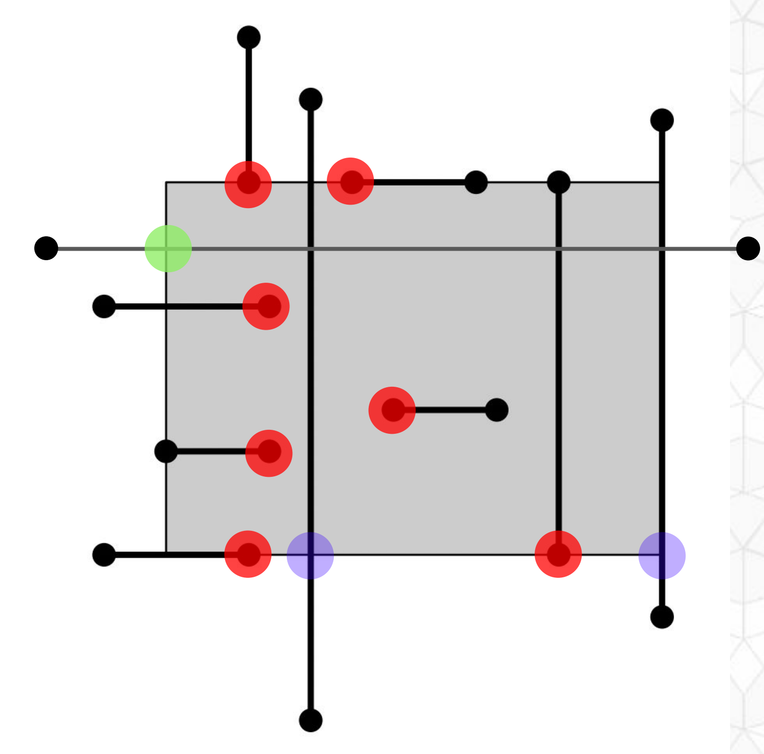
How to handle a 2D Axis-Aligned Query Box?

- Initially, let's restrict to horizontal & vertical segments

- Case Analysis:
  Segments that touch the query box will:
  - Have one endpoint inside the box,
    OR
  - Will have both endpoints outside the box
    AND
    - Will be a horizontal segment
      that overlaps the left edge of the box
      OR
    - Will be a vertical segment
      that overlaps the bottom edge of the box



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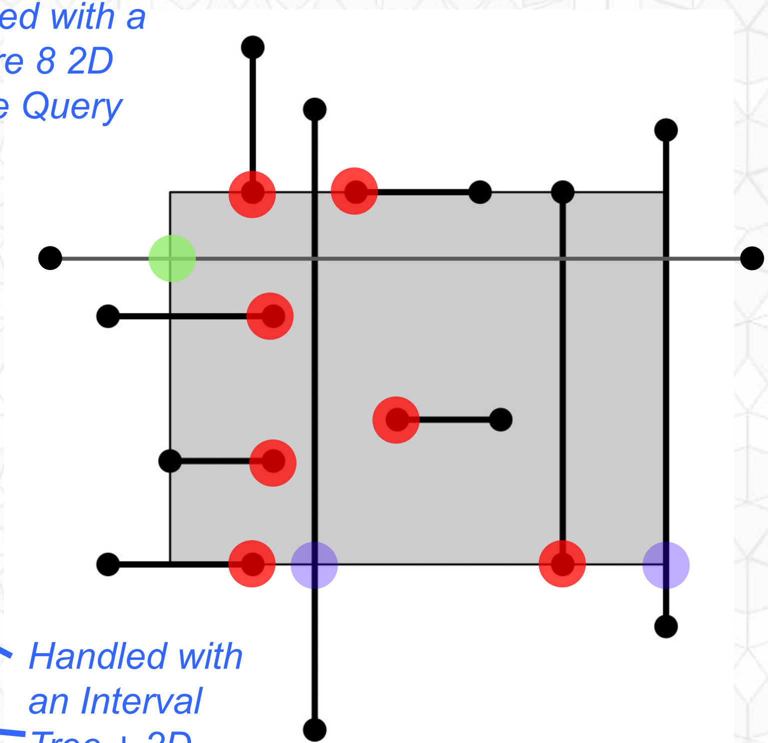
Handled with a Lecture 8 2D Range Query

Handled with an Interval Tree + 2D Range Query



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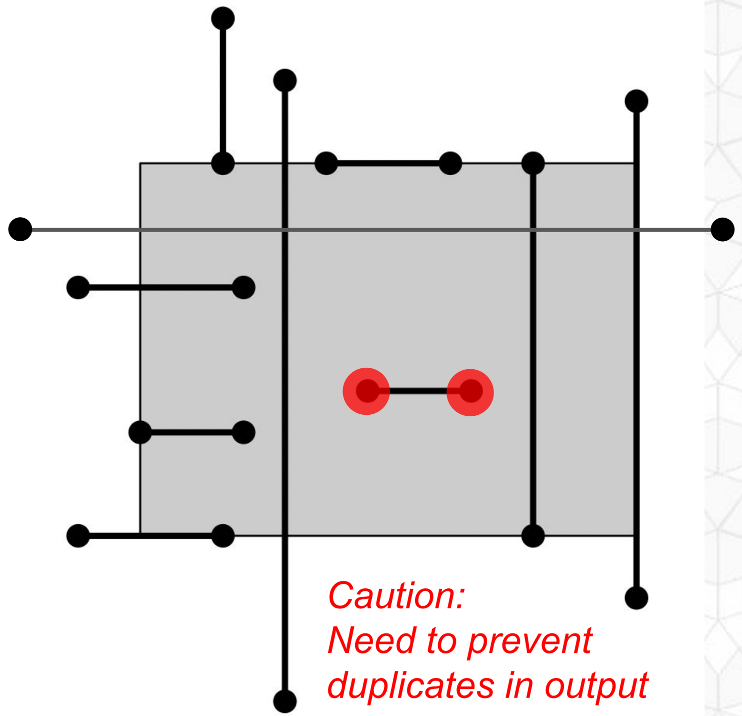

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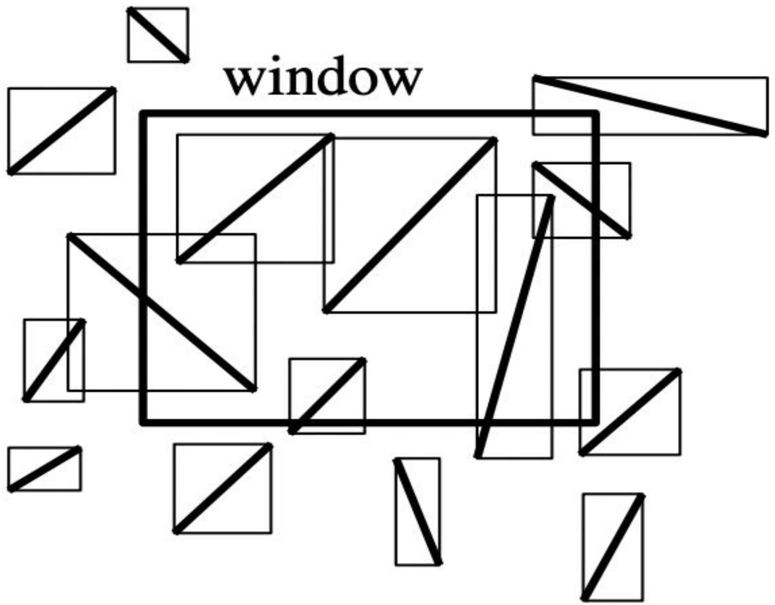
How do we handle General 2D Segments?

- Not restricted to horizontal & vertical segments!
- (Note: We will later insist that the segments do not cross…)

How do we handle General 2D Segments?

- *Do the (sloppy?) Computer Graphics thing…*
  Output the segment if its bounding overlaps the axis-aligned query box



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How do we handle General 2D Segments?

- *Do the (sloppy?) Computer Graphics thing…*
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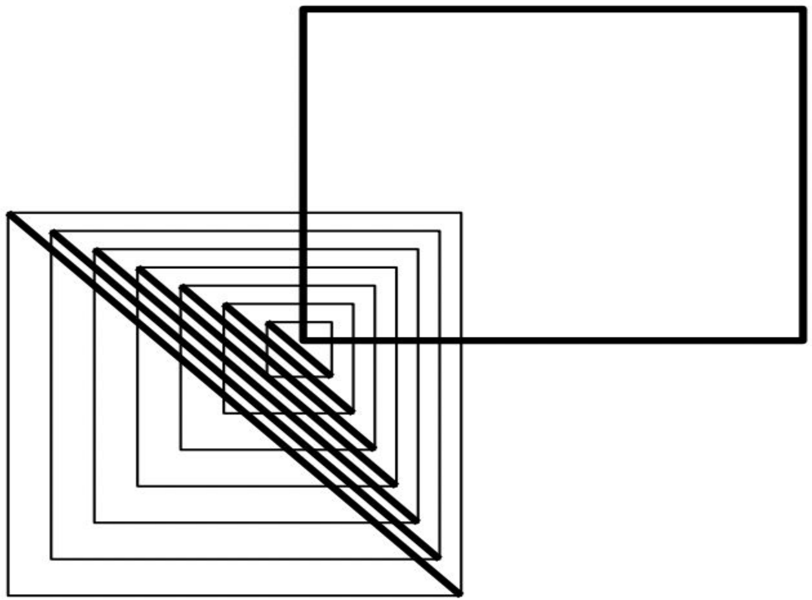
- ***We might have LOTS of false positives!***



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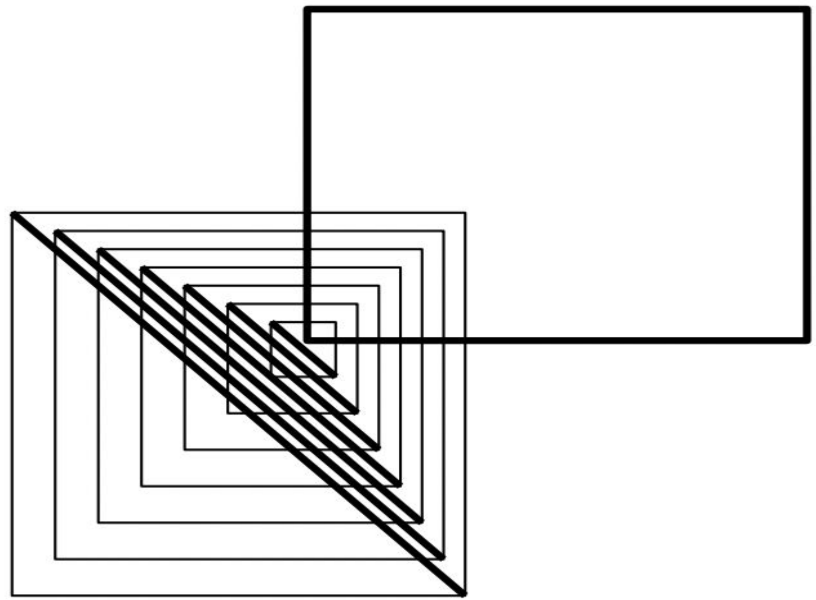
How do we handle General 2D Segments?

- *Do the (sloppy?) Computer Graphics thing…*
  Output the segment if its bounding overlaps the axis-aligned query box

- ***We might have LOTS of false positives!***

- Can we do better?
  - Ensure good (output sensitive) Performance AND
  - Avoid false positives?



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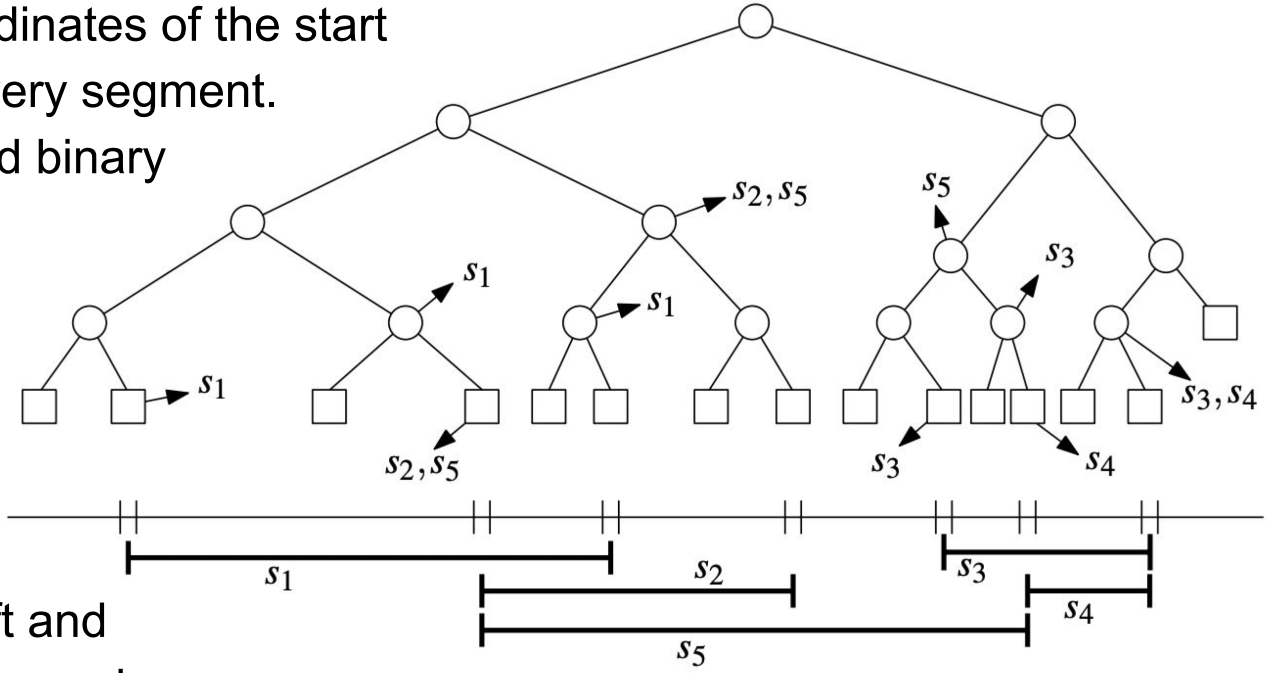
Segment Tree - First Dimension (*x*)

- First, sort the *x* coordinates of the start and end points of every segment.
- Construct a balanced binary search tree with these *x* values.
- Insert every segment into the structure
- If a segment overlaps both the left and right subranges of the node store it at the node (do not recurse)



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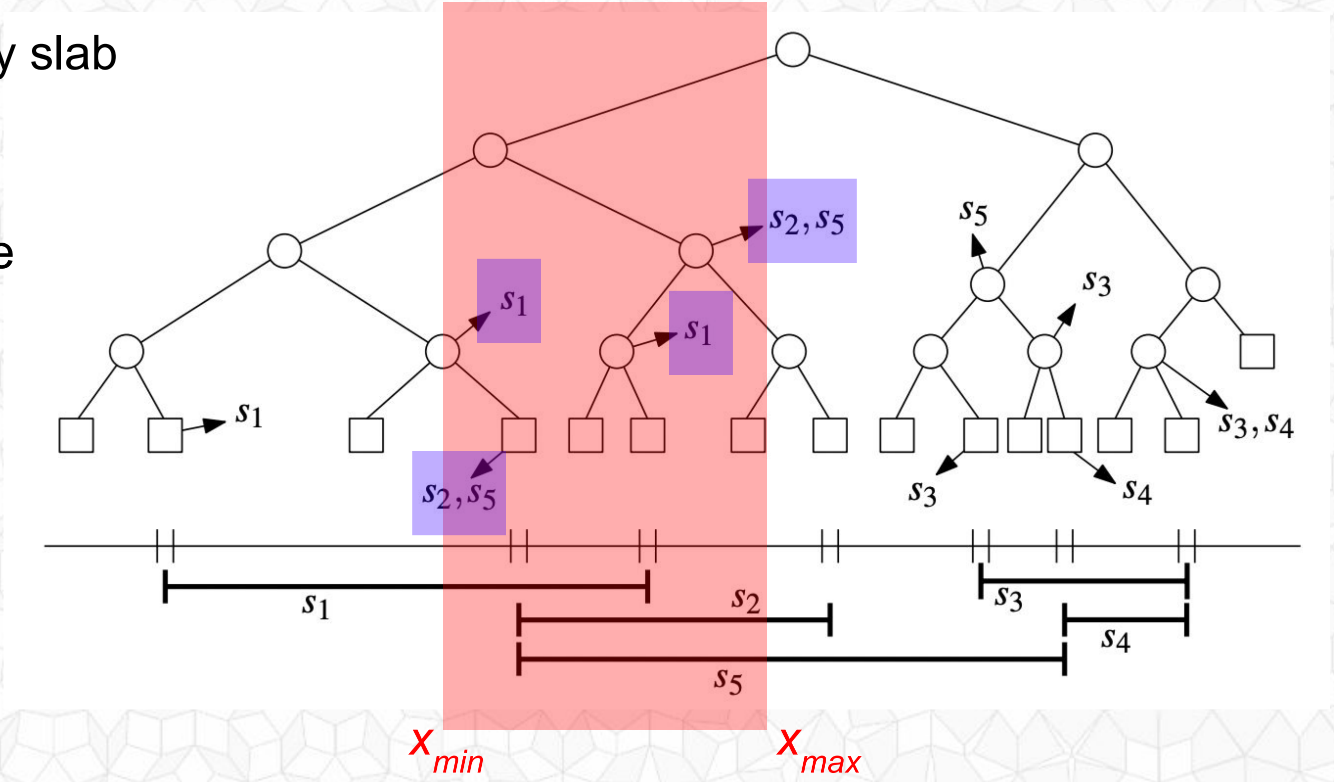
Segment Tree - First Dimension (x)

- For a vertical query slab (x_{min}, x_{max})

- Walk down the tree
- If the node is in range, return all items at that node
- Recurse left and/or right as appropriate

- & filter duplicates…



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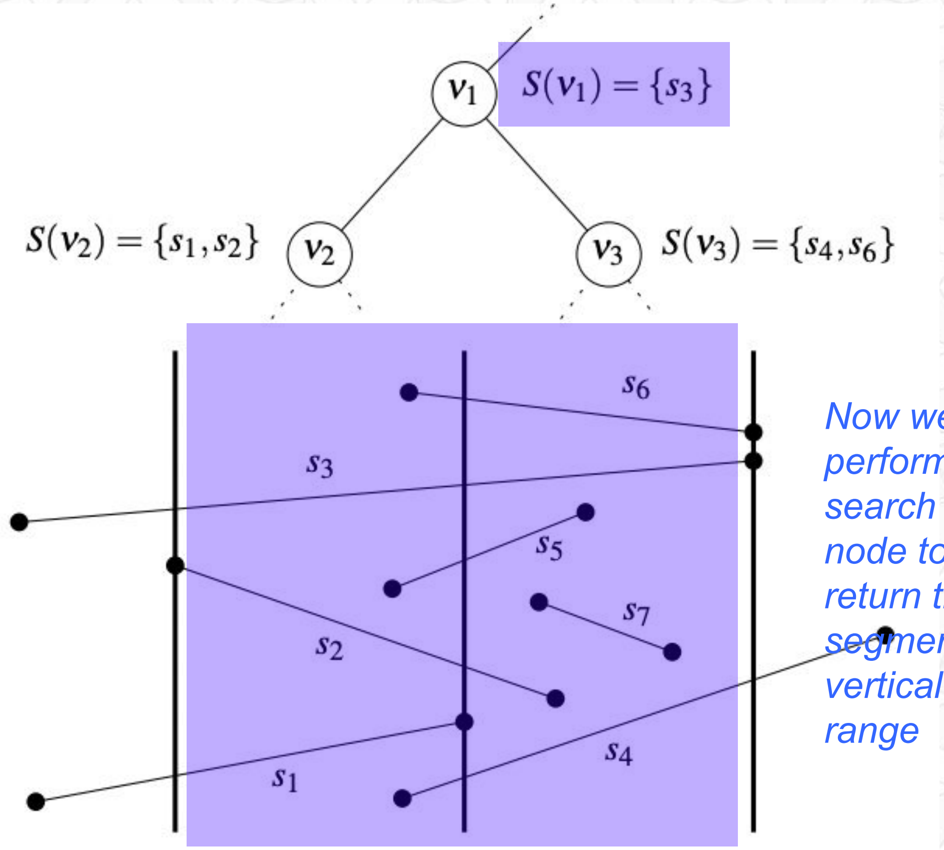
Segment Tree - Second Dimension (*y*)

- To efficiently query a vertical range in addition to the horizontal range:
- Sort the segments stored at each node by *y*
- *Remember: this is only the segments that completely overlaps the node's range*
- Note: this is why we require no crossings in the input segments



Now we can perform binary search at each node to only return the segments in the vertical query range

Segment Tree - Second Dimension (*y*)

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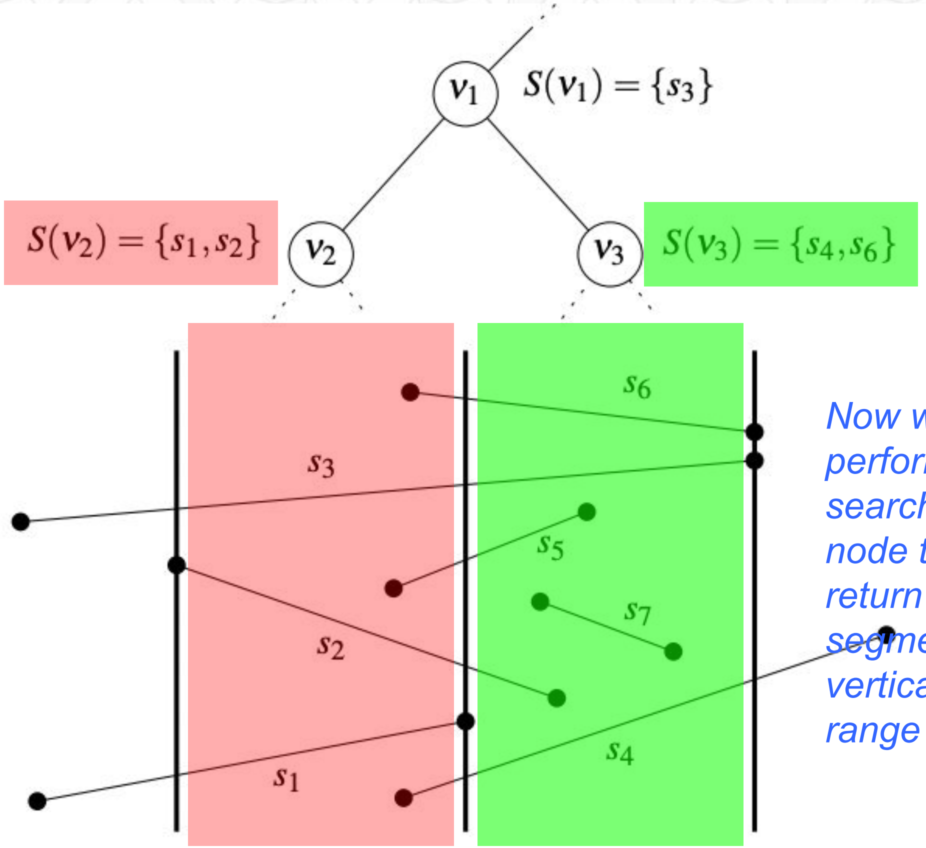


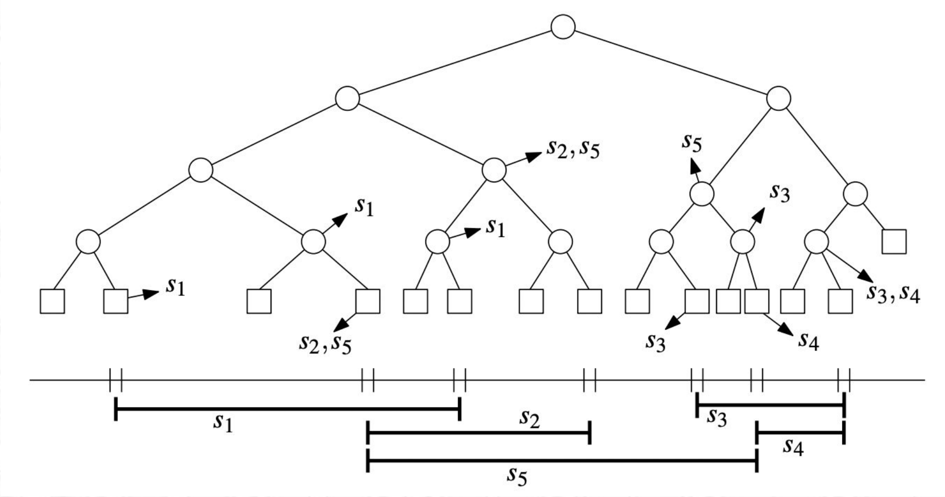
Now we can perform binary search at each node to only return the segments in the vertical query range

Outline for Today

- Review from Last Time: Delaunay Triangulations
- Motivation: Cartography Windowing & Data Selection
- Lecture 8 Review: Points in k-D trees
- 1D Interval Tree
- 1D Interval Tree Analysis
- 2D Interval Tree + Range Tree
- 2D Interval Tree + Range Tree Analysis
- 2D Axis Aligned Segment Query
- Segment Tree for General 2D Segment Query
- Segment Tree Analysis

Segment Tree - Analysis

- For *n* input segments, for a query that will return *k* segments
- Memory:
  Each segment is stored in
  at most 2 nodes per level
- Construction Time:
  Presort all endpoints
  by x & y *O(n log n)*
- Query Time:



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- Construction Time:
  Presort all endpoints
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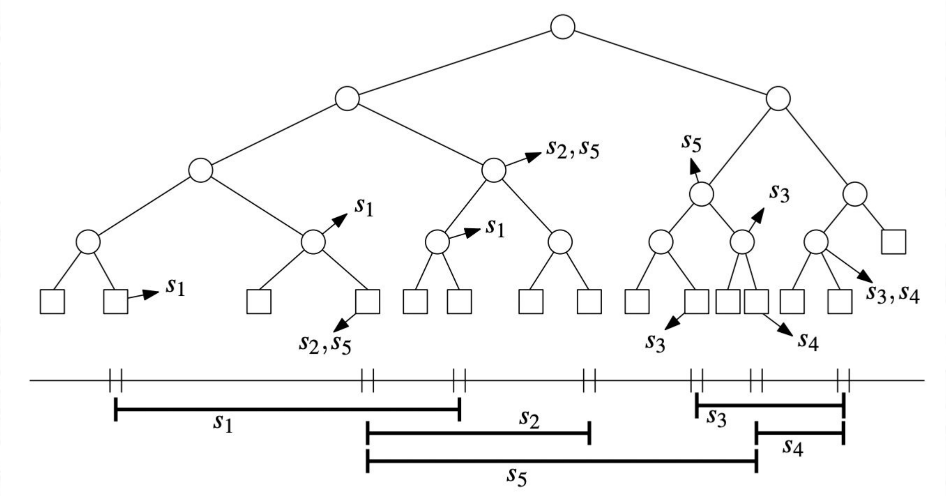
  → *O(n log n)*

- Query Time:
  → *O(log n * log n + k)*
  → *O*($\log^2 n + k$)



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