CSCI 4560/6560 Computational Geometry

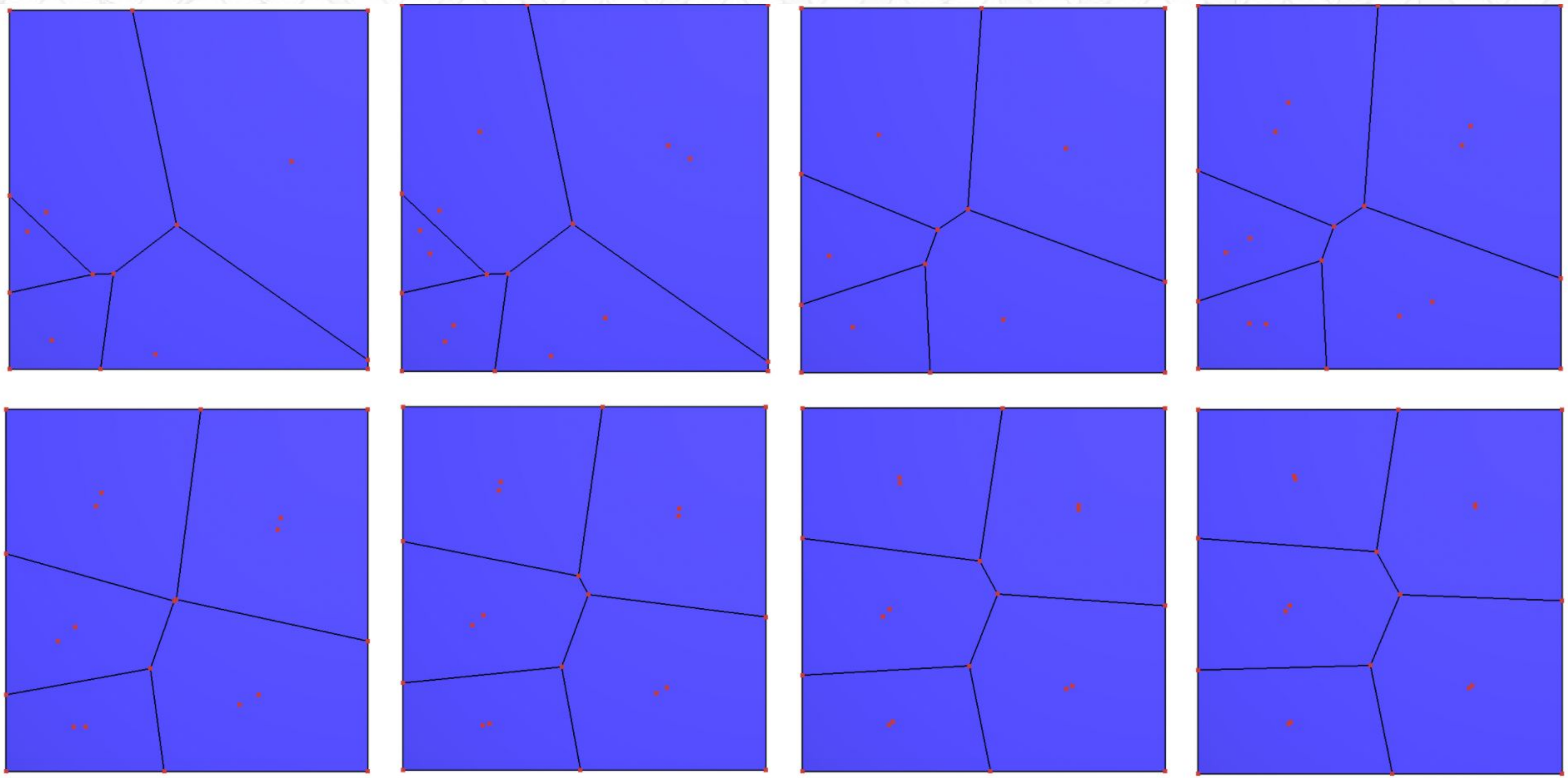
https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/S22/

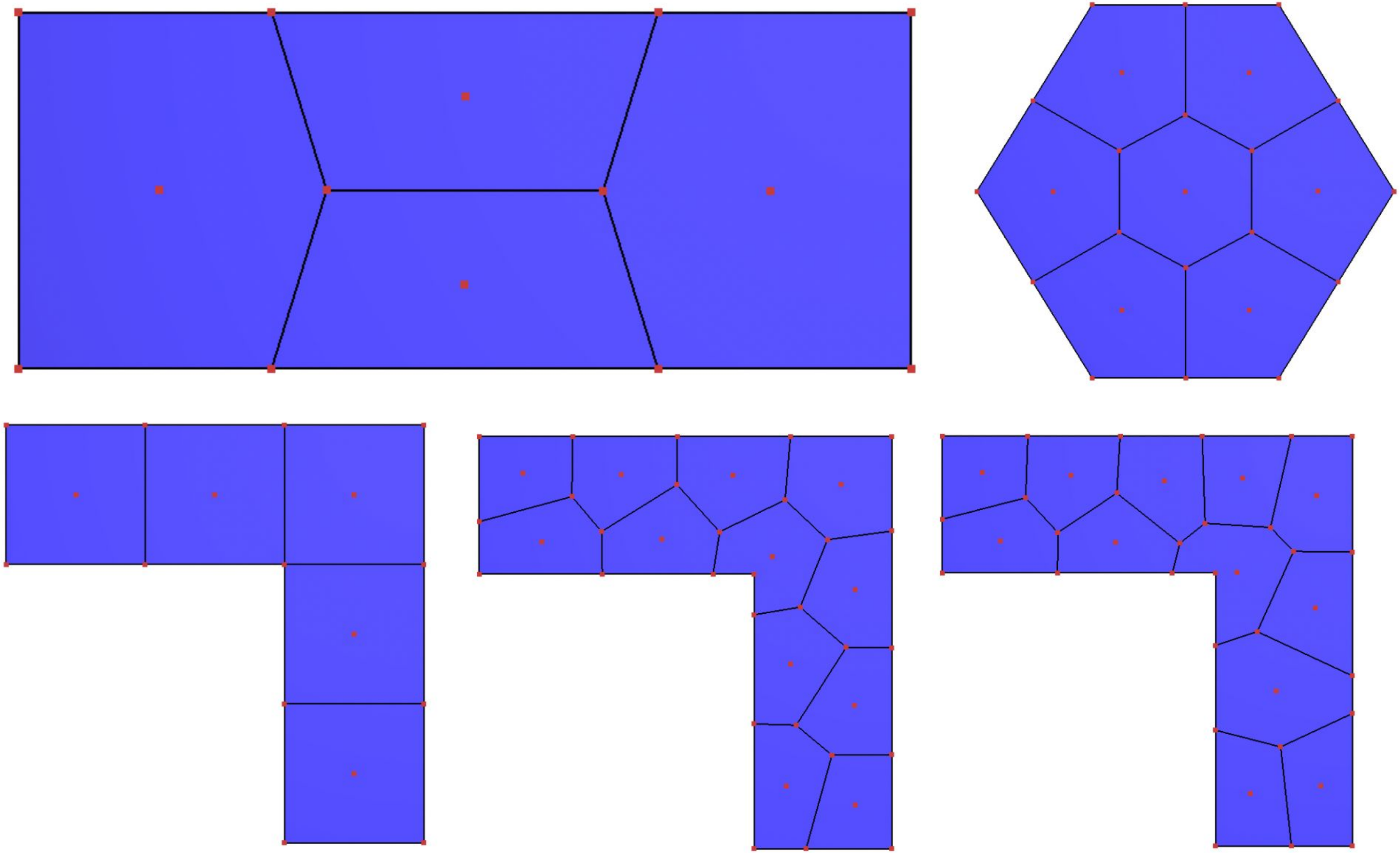
Lecture 19: Polyominoes & Tiling

Outline for Today

- Homework 5 Questions?
- Last Time: Signed Distance & Level Sets
- Polyominoes Terminology
- Counting Polyominoes
- Tiling / Packing Polyominoes
- Polyomino Themed Puzzles
- Next Time: More Tiling!

Homework 5 Questions?





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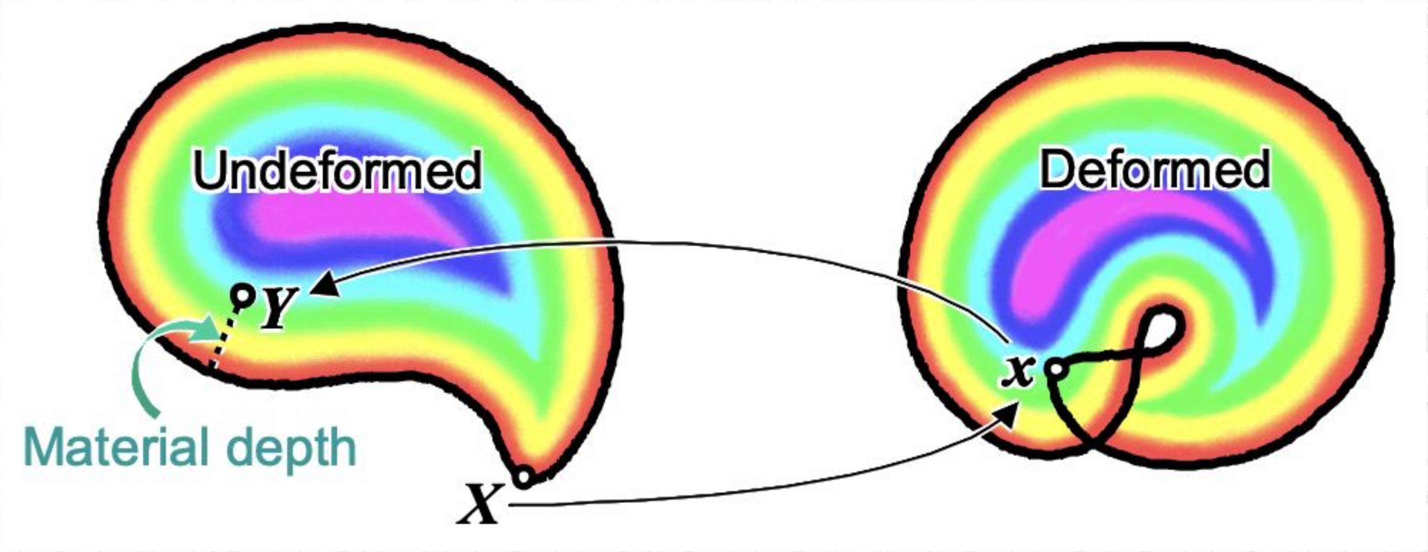
Motivation: Collision Detection

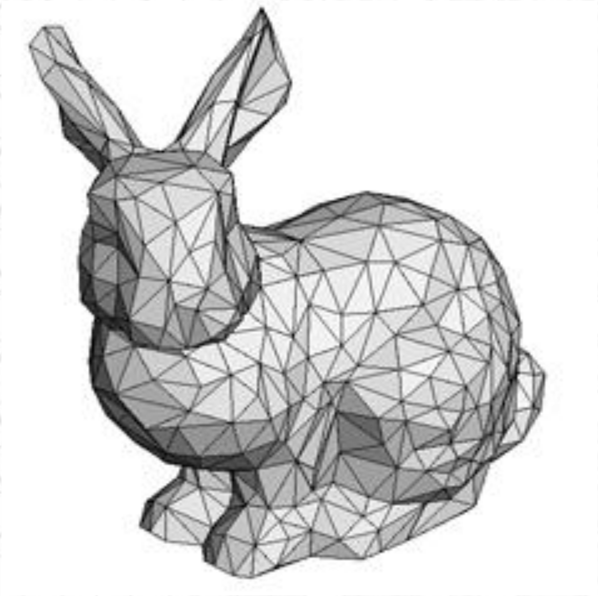
- Detect the intersection
- Depth of intersection penetration
- Gradient & normal of closest surface – Determine penalty force to resolve collision

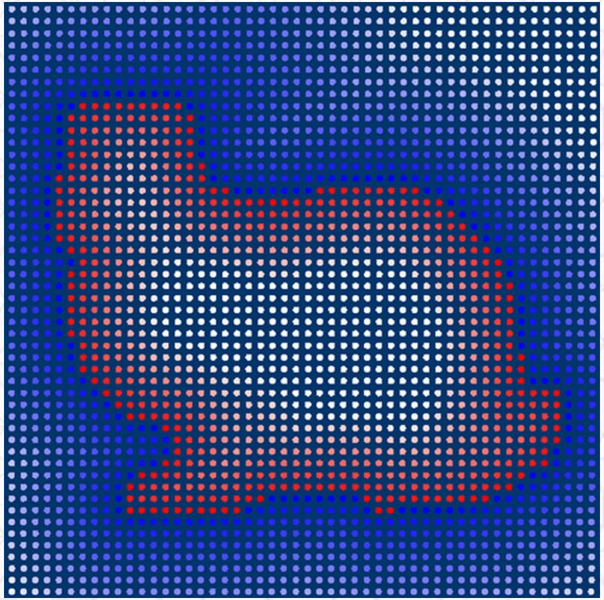
"An Implicit Finite Element Method for Elastic Solids in Contact", Hirota, Fisher, State, Lee, & Fuchs, SCA 2001



Explicit vs. Implicit Surface Representations

- We may not be able to construct a compact mathematical function…
- But can we convert the bunny mesh into a signed distance field?

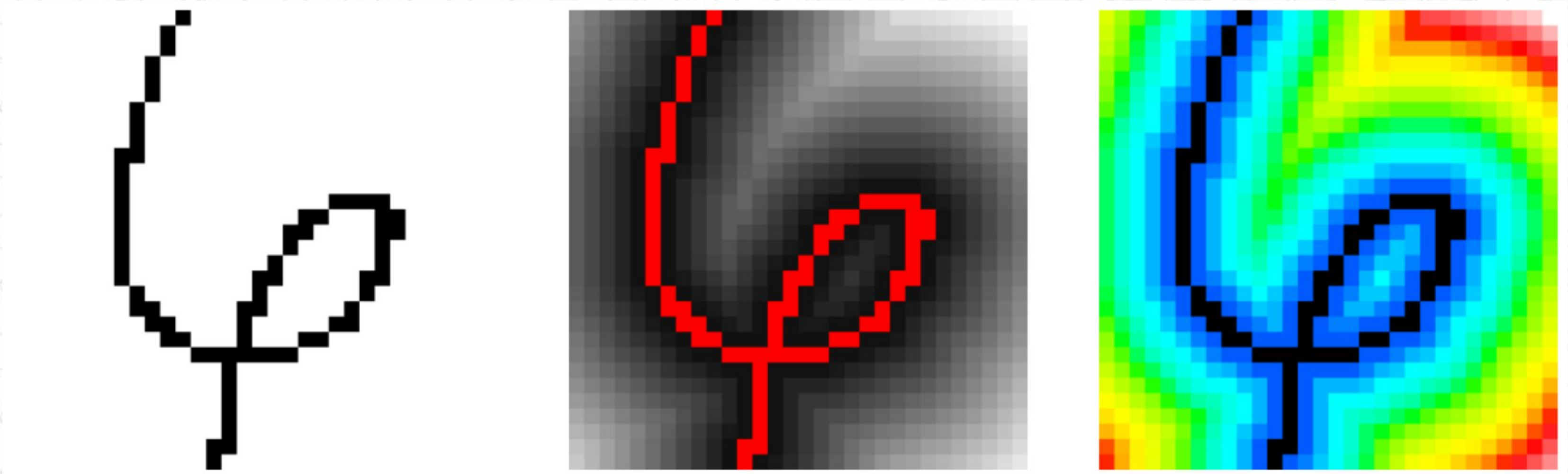




Computing a Signed Distance Field

- Given a shape/surface
- Cost to compute shortest distance to original shape for each point (on a grid) in the volume?

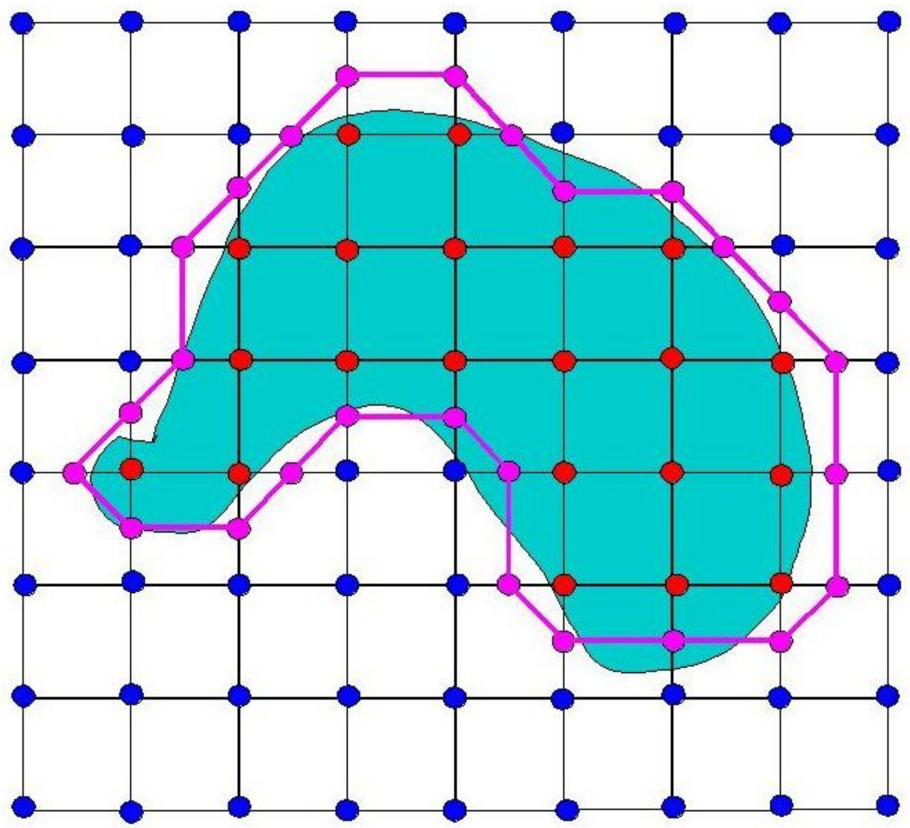
Naive: O(# of volume grid samples * # of surface elements) = O(w^2h^2)



Marching Cubes

- Each point in the 3D grid is labeled "inside" (red dots) or "outside" (blue dots) the unknown surface.
- Any cell in the grid that has at least one red vertex and at least one blue vertex, must be crossed by the unknown surface.

- We can piecewise construct an approximation of the surface.

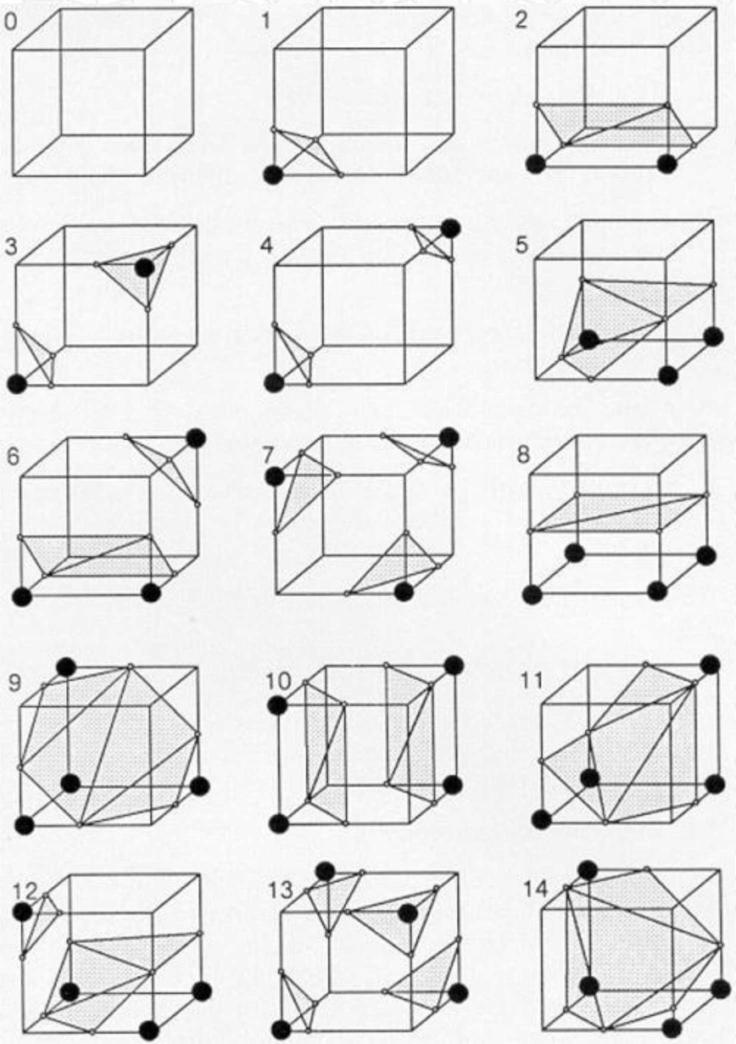


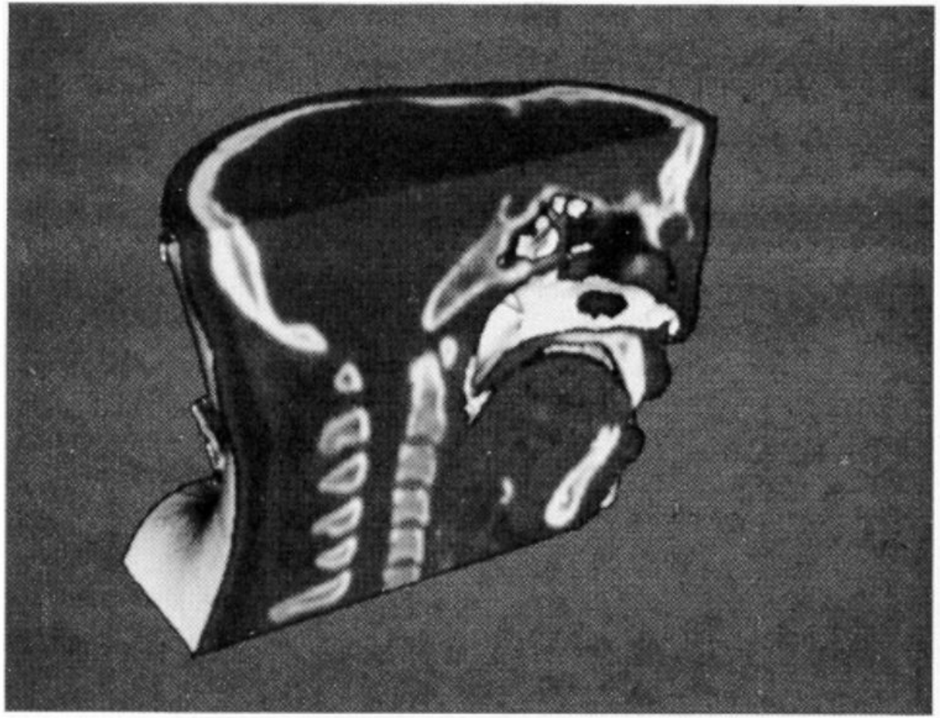
http://www.cs.carleton.edu/cs_comps/0405/shape/marching_cubes.html

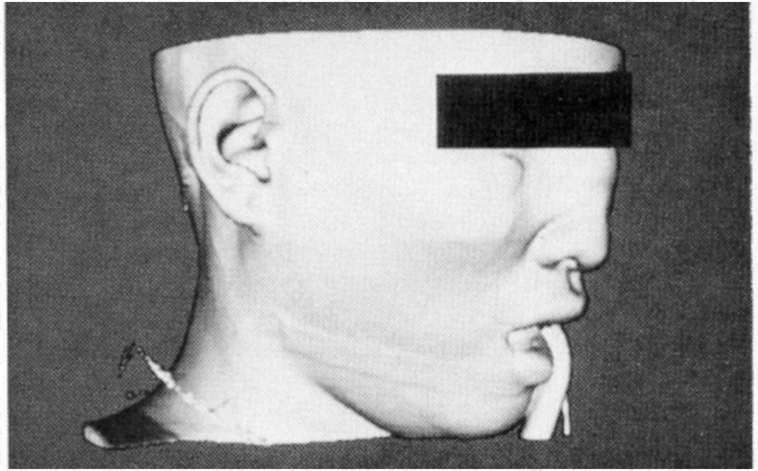
Marching Cubes

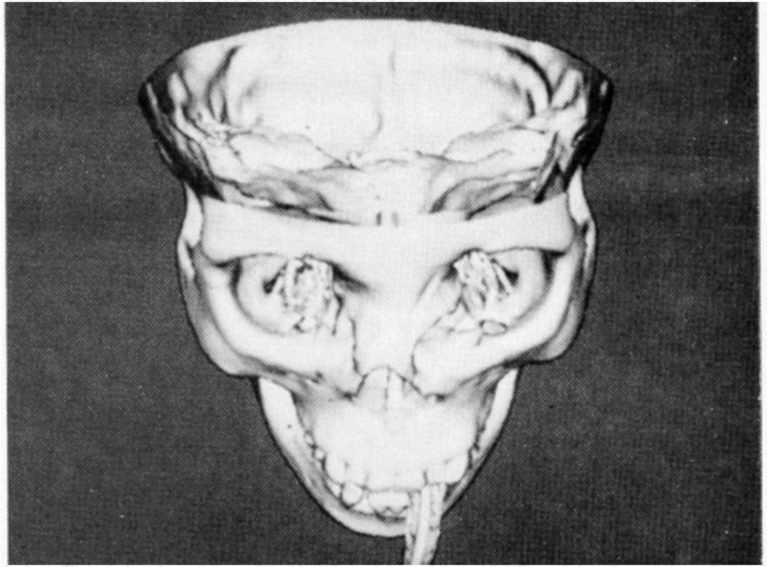
- 256 possible inside/outside labelings of each grid cube.
- Merging rotations…
  15 unique cases to implement

"Marching Cubes: A High Resolution 3D Surface Construction Algorithm", Lorensen and Cline, SIGGRAPH '87.









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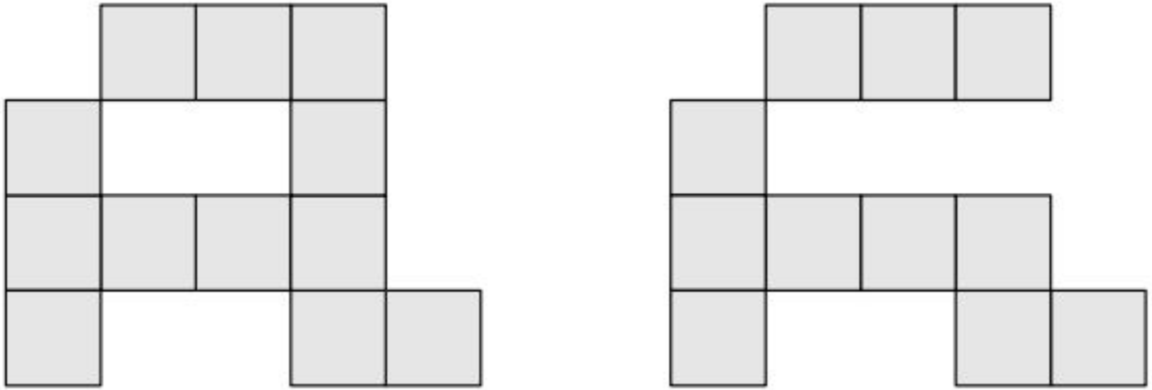
What is a Polyomino?

- An n-omino is a set of n cells on a square graph that is connected

is a polyomino

is NOT a polyomino

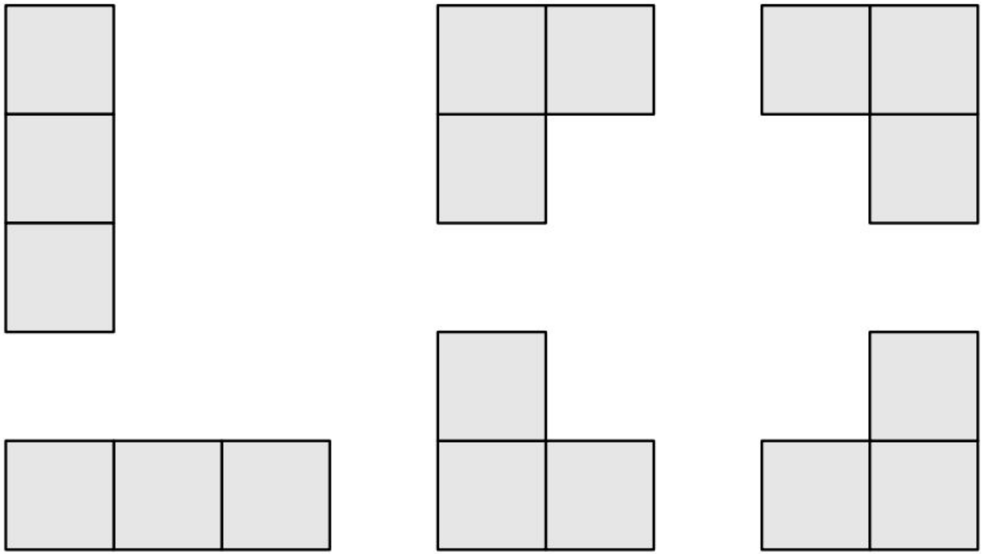


“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Translation-Equivalent / Fixed Polyomino

- Only left/right/up/down translation is allowed

- There are 6 unique Fixed 3-ominoes (a.k.a. trominoes):



“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Translation-Equivalent / Fixed Polyomino

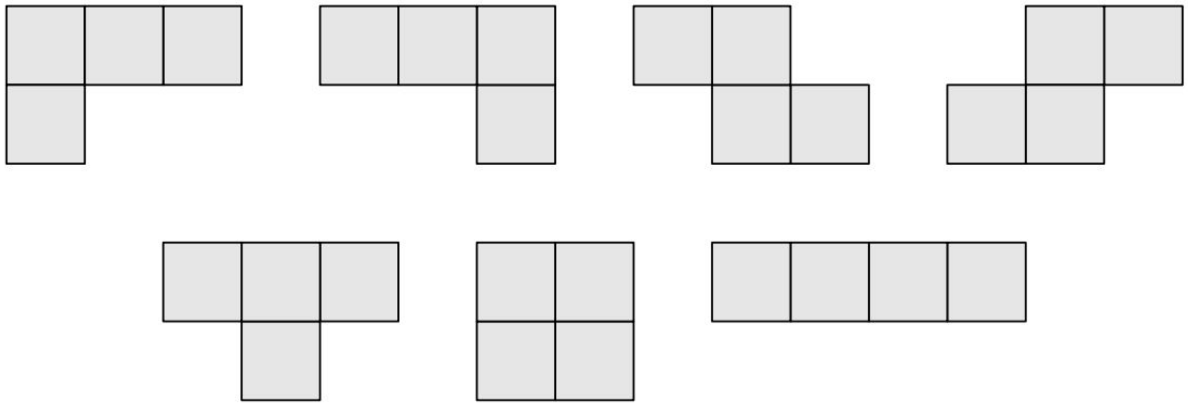
- Only left/right/up/down translation is allowed

- *How many fixed 2-ominoes (a.k.a. dominoes) are there?*
- *Draw them!*

Rotation-Equivalent / Chiral Polyomino

- left/right/up/down translation allowed
- 90°/180°/270° rotation allowed

- There are 7 unique chiral 4-ominoes (a.k.a. tetrominoes):



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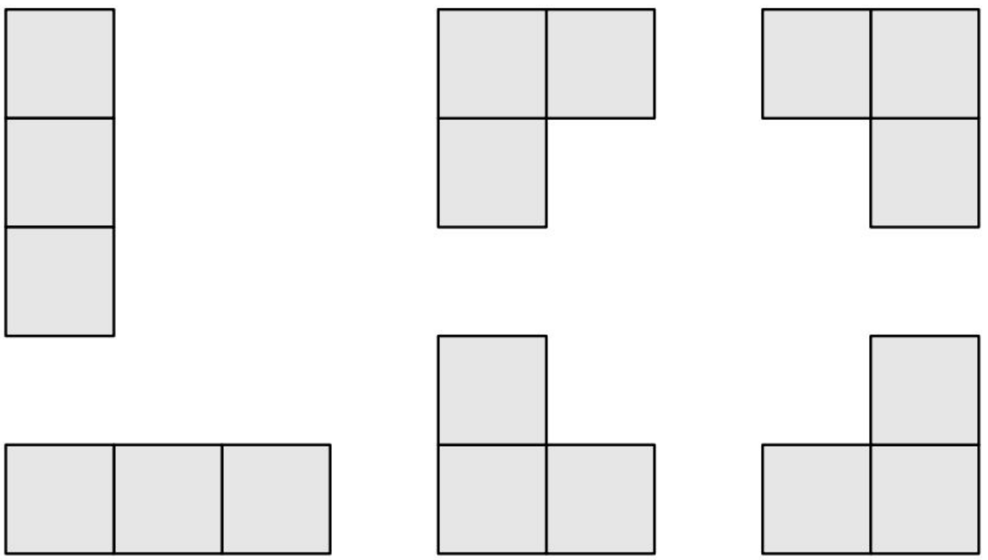
Rotation-Equivalent / Chiral Polyomino

- left/right/up/down translation allowed
- 90°/180°/270° rotation allowed

- *How many chiral 3-ominoes are there?*
- *Which of these shapes are rotationally-equivalent?*

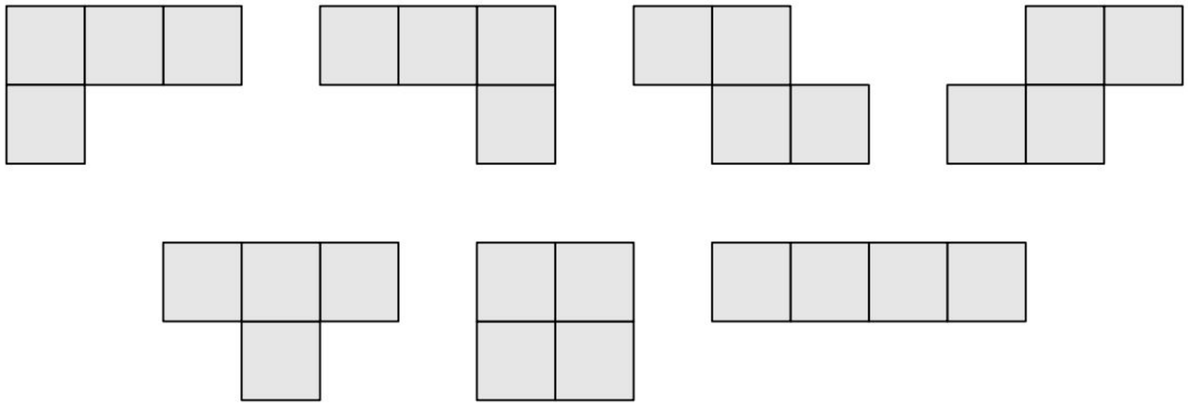


"Ch 14: Polyominoes", Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Translation-Equivalent / Fixed Polyomino

- Only left/right/up/down translation is allowed

- *How many fixed 4-ominoes are there?*
- *Which of these shapes are unique when rotated?*

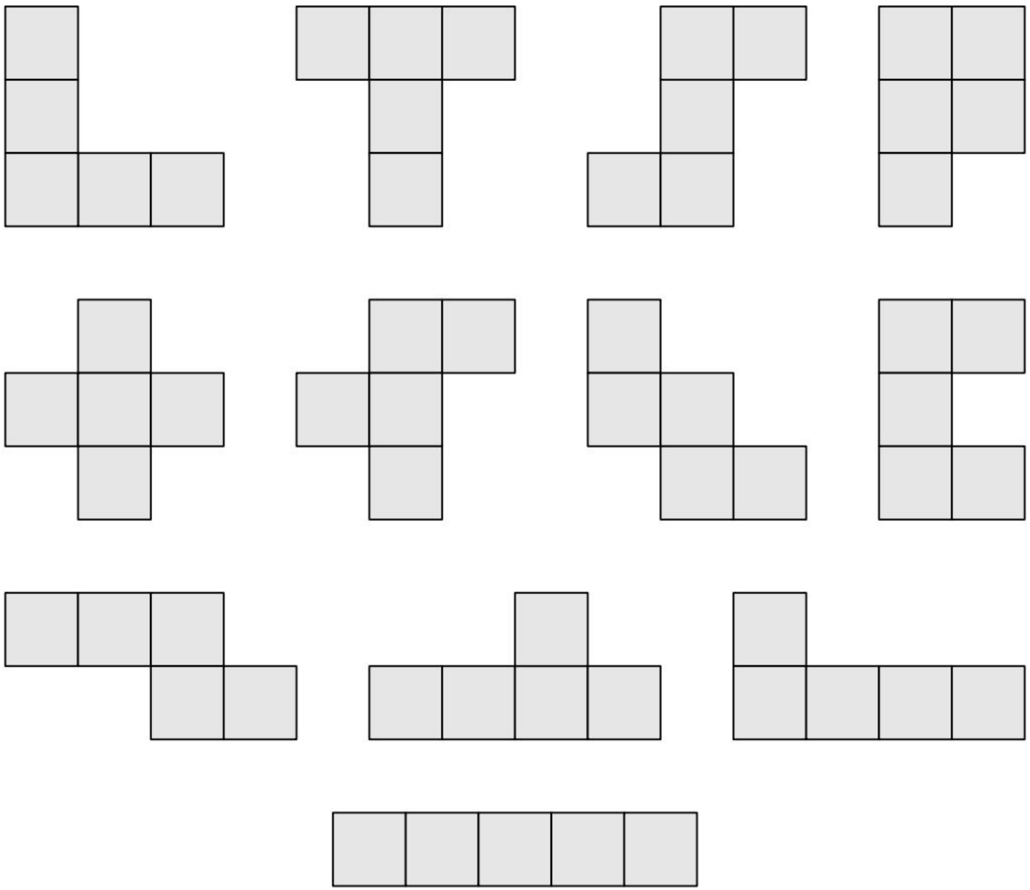


“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Free Polyomino

- Translation allowed
- Rotation allowed
- Reflection allowed

- There are 12 unique free 5-ominoes (a.k.a. pentominoes):



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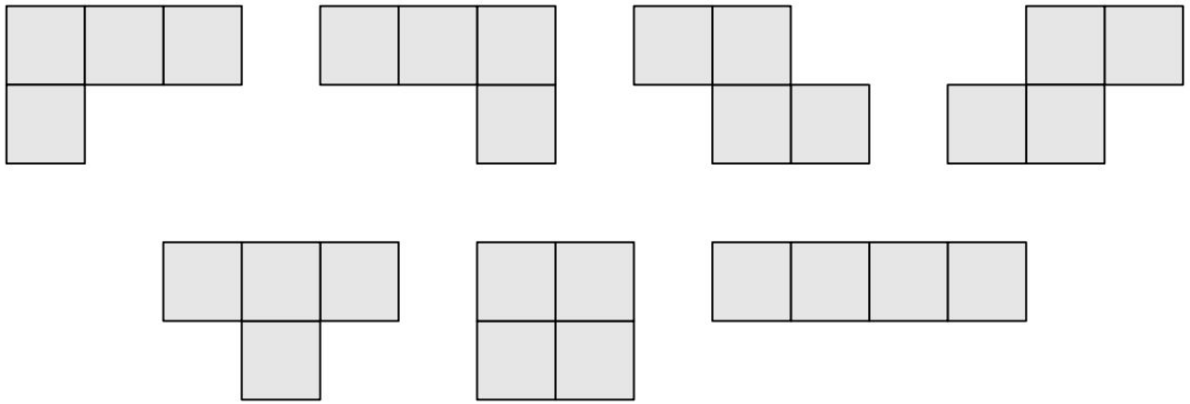
Congruent / Free Polyomino

- *How many free 4-ominoes are there?*
- *Which of these shapes are congruent? (duplicates when reflected)*



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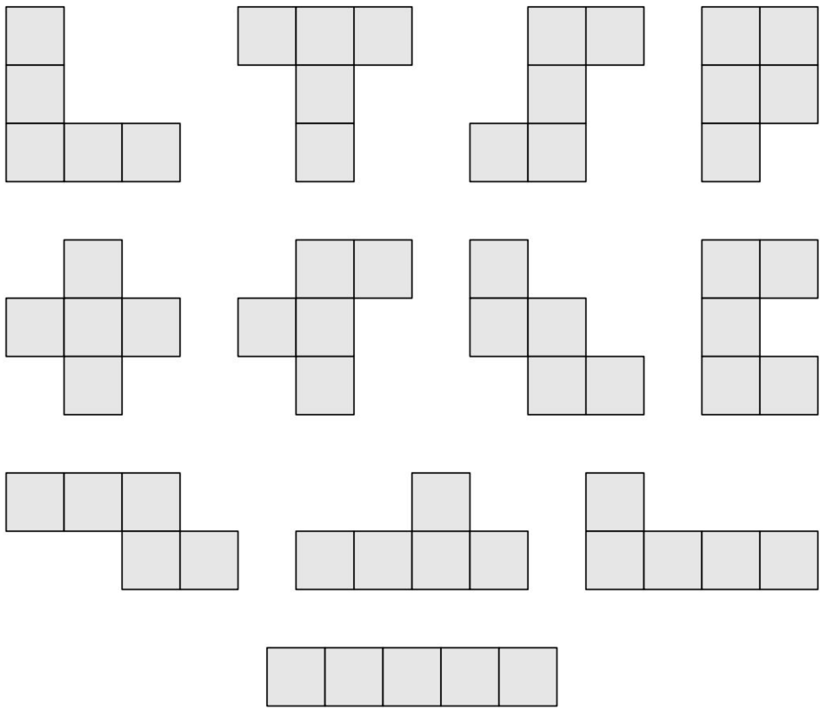
Rotation-Equivalent / Chiral Polyomino

- left/right/up/down translation allowed
- 90°/180°/270° rotation allowed

- *How many chiral 5-ominoes are there?*
- *Which of these shapes are unique when reflected?*



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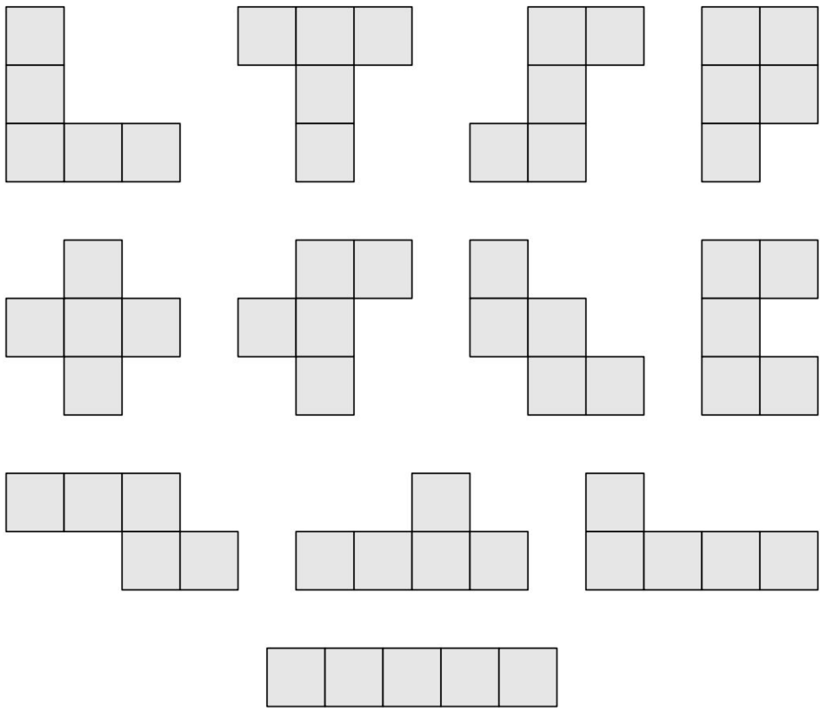
Translation-Equivalent / Fixed Polyomino

- Only left/right/up/down translation is allowed

- *How many fixed 5-ominoes are there?*
- *Which of these shapes are unique when rotated and/or reflected?*



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Counting Fixed, Chiral, and Free Polyominoes

	fixed	**chiral**	**free**
	translation-only	translation & rotation (no reflection)	translation, rotation, & reflection

n	$t(n)$	$r(n)$	$s(n)$
1	1	1	1
2	2	1	1
3	6	2	2
4	19	7	5
5	63	18	12

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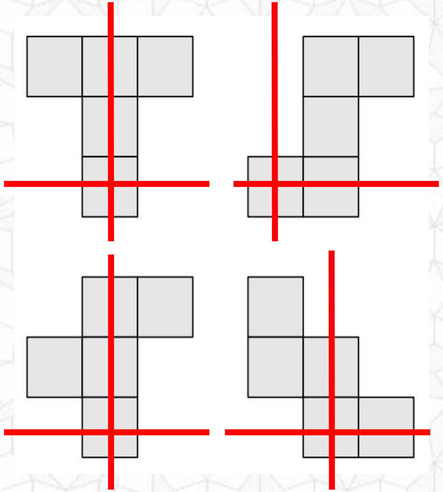
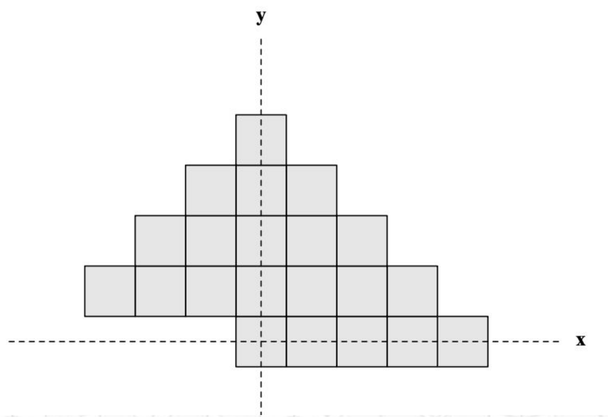
Counting Polyominoes

- *n*-omino Standard Position: Translate to place the leftmost cell in the bottom row at the origin.
- Enumerate all combinations of all possible cells
- Eliminate disconnected & duplicate ominoes
- At most

$$\binom{3(n-1)}{n-1}$$



all possible cells for 5-ominos



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Counting Polyominoes

- What is the relationship *(e.g., inequalities < > = ≤ ≥)* between *t(n)*, *r(n)*, and *s(n)*?

	fixed	chiral	free
n	$t(n)$	$r(n)$	$s(n)$
1	1	1	1
2	2	1	1
3	6	2	2
4	19	7	5
5	63	18	12
6			
7			
8			
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20			
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23			
24			

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Counting Polyominoes

- What is the relationship *(e.g., inequalities < > = ≤ ≥)* between *t(n)*, *r(n)*, and *s(n)*?

$$\frac{t(n)}{8} \leq s(n) \leq r(n) \leq t(n)$$

	fixed	chiral	free
n	$t(n)$	$r(n)$	$s(n)$
1	1	1	1
2	2	1	1
3	6	2	2
4	19	7	5
5	63	18	12
6	216	60	35
7	760	196	108
8	2725	704	369
9	9910	2500	1285
10	36446	9189	4655
11	135268	33896	17073
12	505861	126759	63600
13	1903890	476270	238591
14	7204874	1802312	901971
15	27394666	6849777	3426576
16	104592937	26152418	13079255
17	400795844	100203194	50107909
18	1540820542	385221143	192622052
19	5940738676	1485200848	742624232
20	22964779660	5741256764	2870671950
21	88983512783	22245940545	11123060678
22	345532572678	86383382827	43191857688
23	1344372335524	336093325058	168047007728
24	5239988770268	1309998125640	654999700403

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Counting Polyominoes

- The number of polyominoes, *t(n)* is exponential in *n*.

  Current unproved estimate ≈ 4.06^{n}

- The running time of the current best algorithm to count *t(n)* is also exponential (but smaller)

  $O(3^{n/2}) \approx O(1.73^{n})$

- Can t(n) be computed in poly time?
  Open problem!!

	fixed	chiral	free
n	$t(n)$	$r(n)$	$s(n)$
1	1	1	1
2	2	1	1
3	6	2	2
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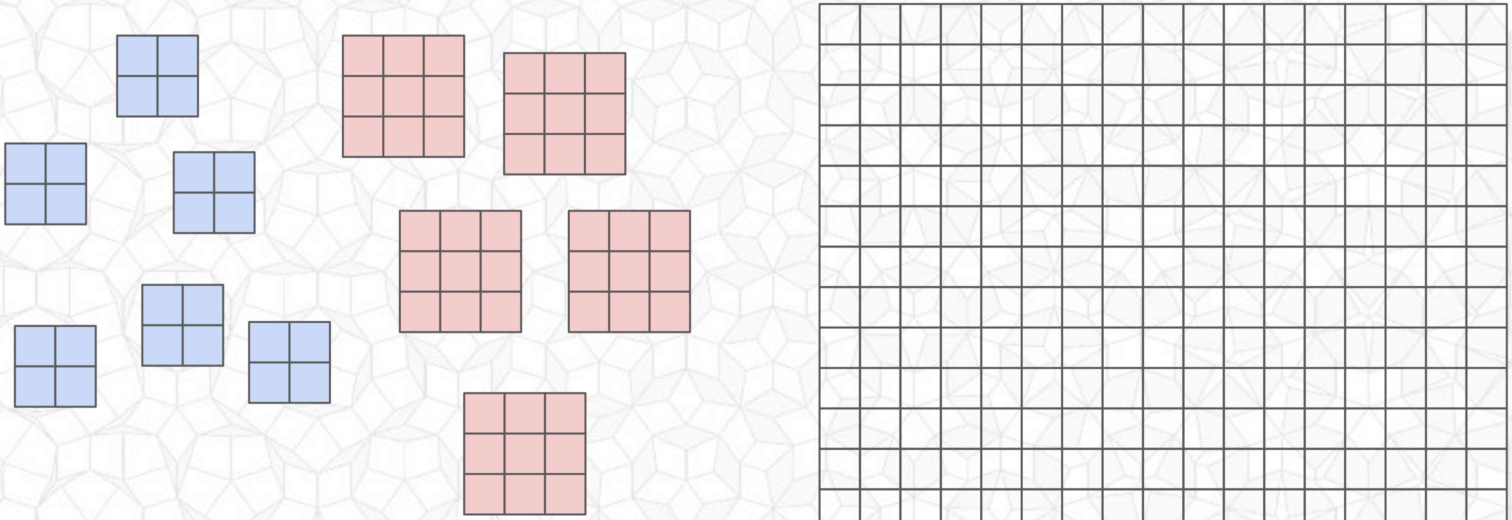
Packing Polyominoes

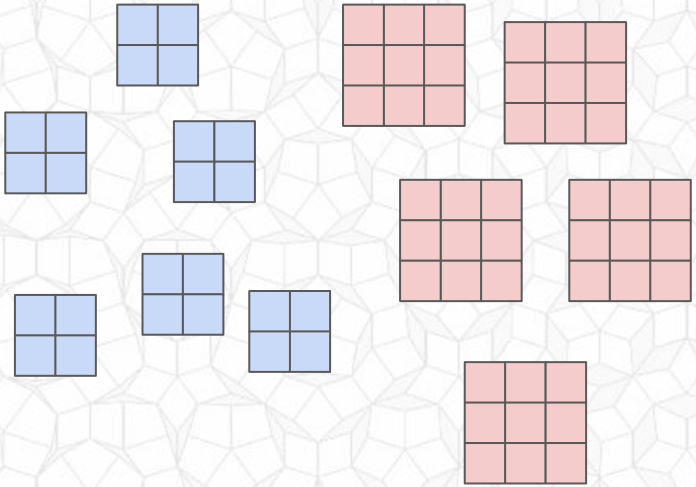
- Can we use 2x2 square 4-ominoes and 3x3 square 9-ominoes to cover (without overlaps) a 13x17 rectangle?

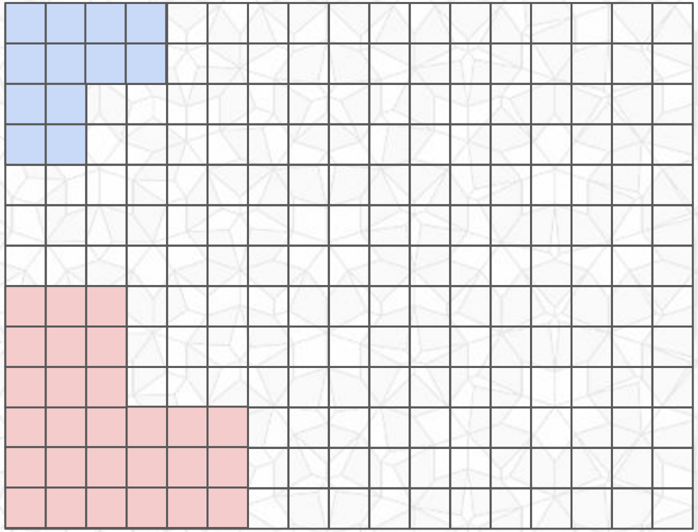


Packing Polyominoes

- Can we use 2x2 square 4-ominoes and 3x3 square 9-ominoes to cover (without overlaps) a 13x17 rectangle?

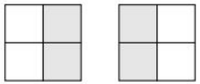
Maybe…. counting cells: $(17*4) + (17*9) = 17 * (9+4) = 17 * 13 = 221$



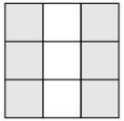


Packing Polyominoes

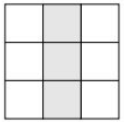
- Actually, this packing is not possible, and can be proven by contradiction using this coloring scheme



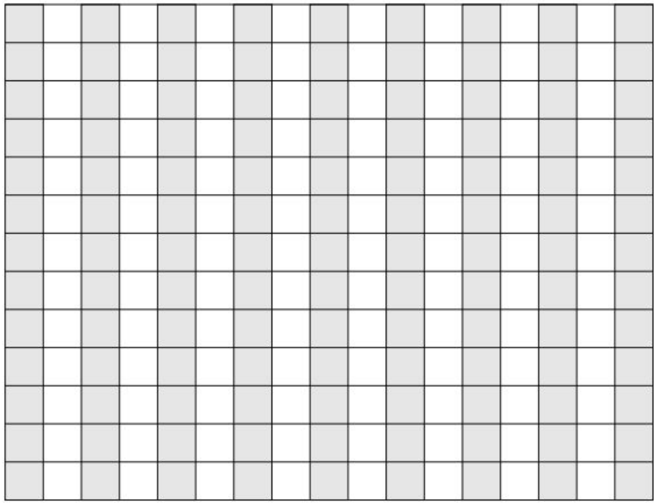
type (2,2)



type (6,3)



type (3,6)



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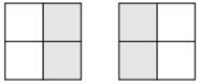

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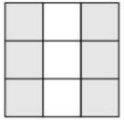

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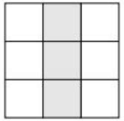


type (3,6)

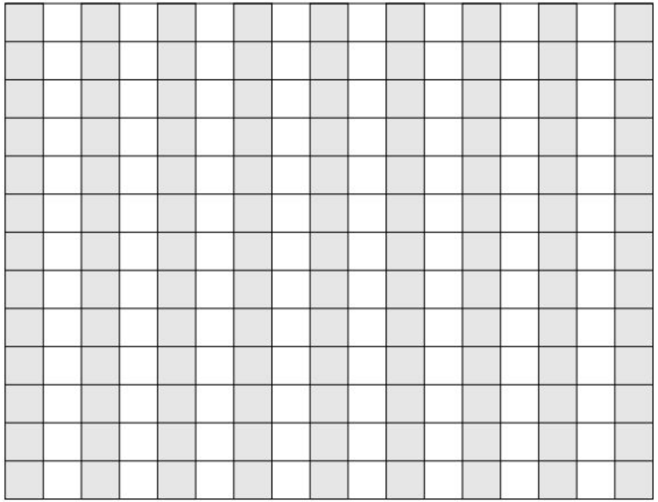
13*9=117 grey cells + 13*8=104 white cells in the rectangle

$x_a*2 + x_b*2 + y_a*6 + y_b*3 = 117$ grey cells
$x_a*2 + x_b*2 + y_a*3 + y_b*6 = 104$ white cells
in the ominoes

$117 - y_a*6 - y_b*3 = 104 - y_a*3 - y_b*6$

$13 = y_a*3 - y_b*3$

$13 = 3 * (y_a - y_b)$ *no integer solutions!*



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Packing Polyominoes

- Actually, this packing is not possible, and can be proven by contradiction using this coloring scheme

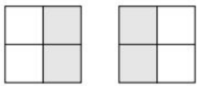
  13*9=117 grey cells + 13*8=104 white cells in the rectangle

  $x_a*2 + x_b*2 + y_a*6 + y_b*3 = 117$ grey cells
  $x_a*2 + x_b*2 + y_a*3 + y_b*6 = 104$ white cells
  in the ominoes

  $117 - y_a*6 - y_b*3 = 104 - y_a*3 - y_b*6$

  $13 = y_a*3 - y_b*3$

  $13 = 3 * (y_a - y_b)$ *no integer solutions!*



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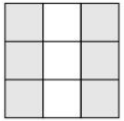

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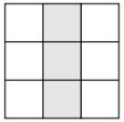

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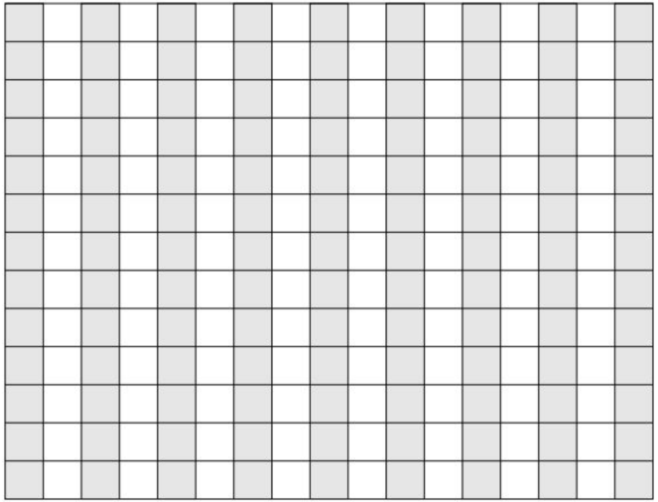
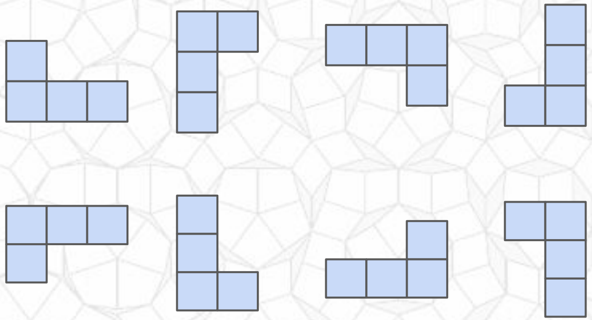
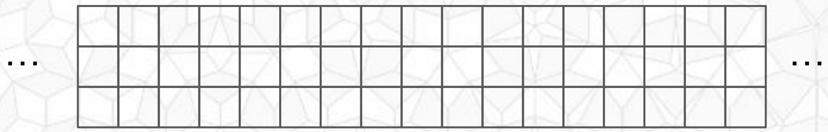


"Ch 14: Polyominoes", Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Packing Polyominoes

- Can we use the L-tetronimo, and all of its rotations and reflections to pack tile and infinite rectangle of height 3?

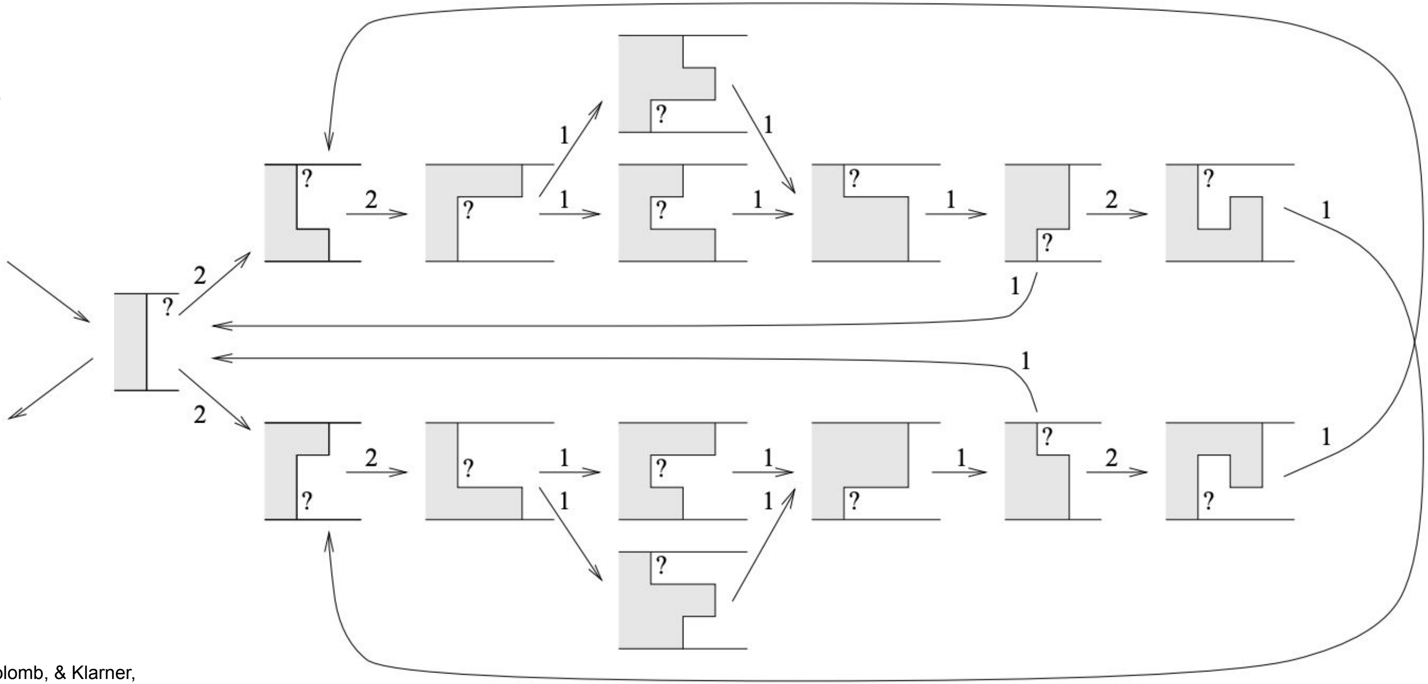




Packing Polyominoes

- Can we use the L-tetronimo, and all of its rotations and reflections to pack tile and infinite rectangle of height 3?

- *Yes, we can build the following automaton of all of states:*



“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Outline for Today

- Homework 5 Questions?
- Last Time: Signed Distance & Level Sets
- Polyominoes Terminology
- Counting Polyominoes
- Tiling / Packing Polyominoes
- Polyomino Themed Puzzles
- Next Time: More Tiling!

Puzzle from *Games Magazine* January 2022

PENTOMINO PROBLEMS

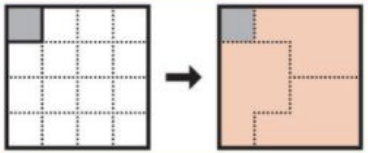
BY RODOLFO KURCHAN

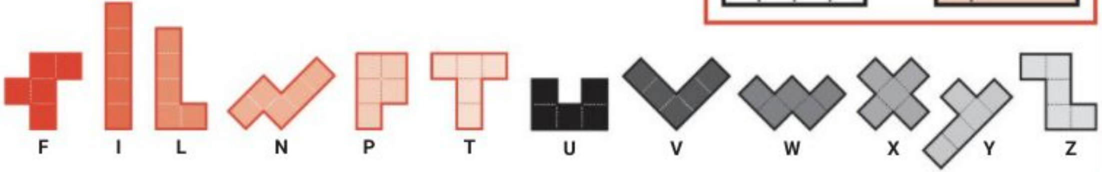
The pentominoes are the 12 different shapes that you can make with 5 unit squares. They are often identified by the letters they resemble, as shown below.

In these problems, your goal is to cover the white portion of each grid with copies of the same pentomino. Pentominoes may be rotated or reflected as needed. At right is an example of a 4×4 puzzle.

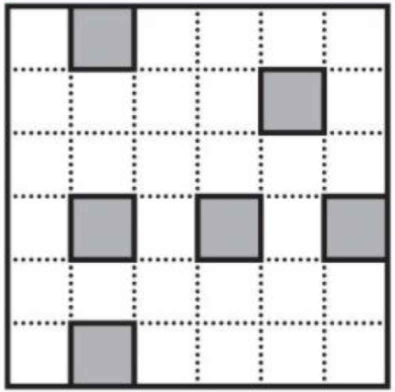
ANSWERS, PAGE 77

EXAMPLE





1



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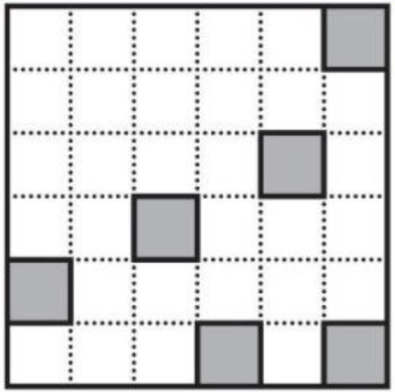
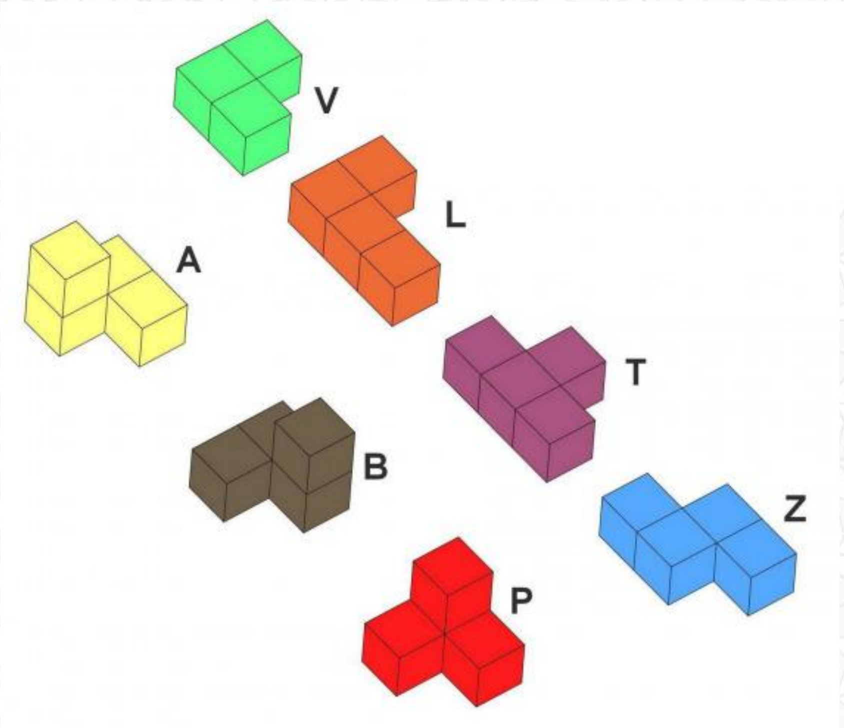


3D Packing Puzzle: Soma Cube

all possible combinations of three or four unit cubes, joined at their faces, such that at least one inside corner is formed.



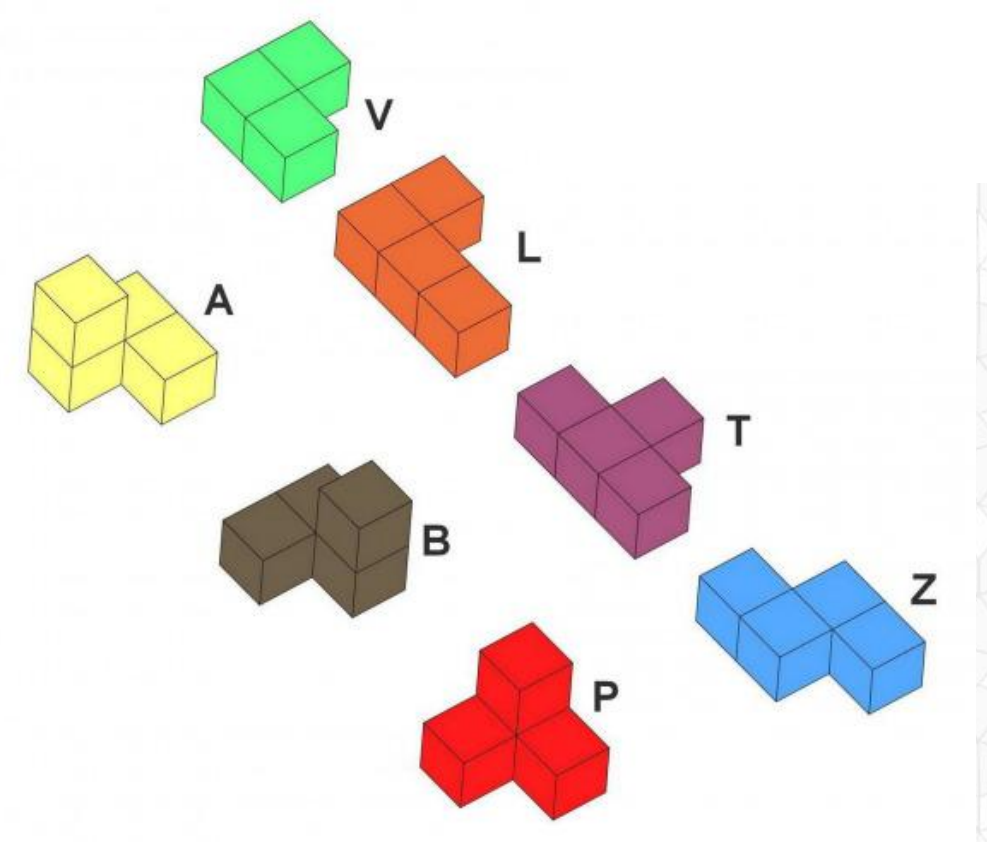
Pack into a 3x3x3 box

3D Packing Puzzle: Soma Cube

- Let's count corners…

- For each piece,
  for each possible
  placement,

  How many of the
  8 box corners
  can it cover?



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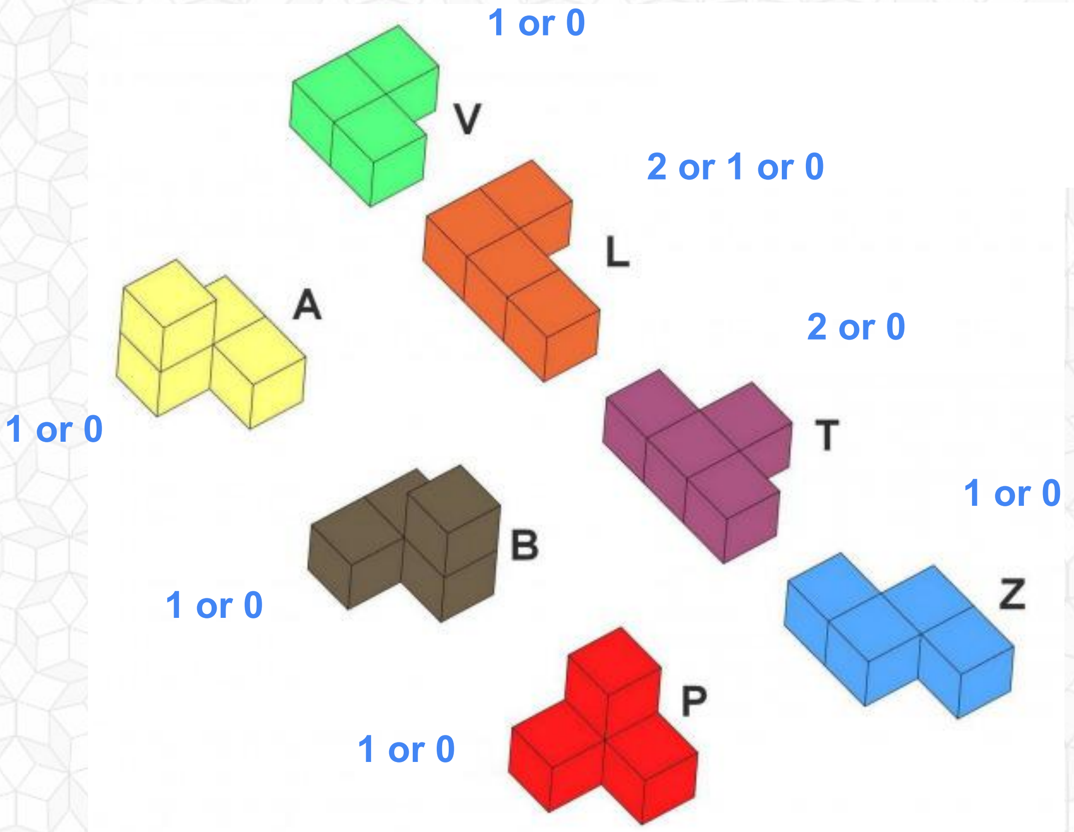

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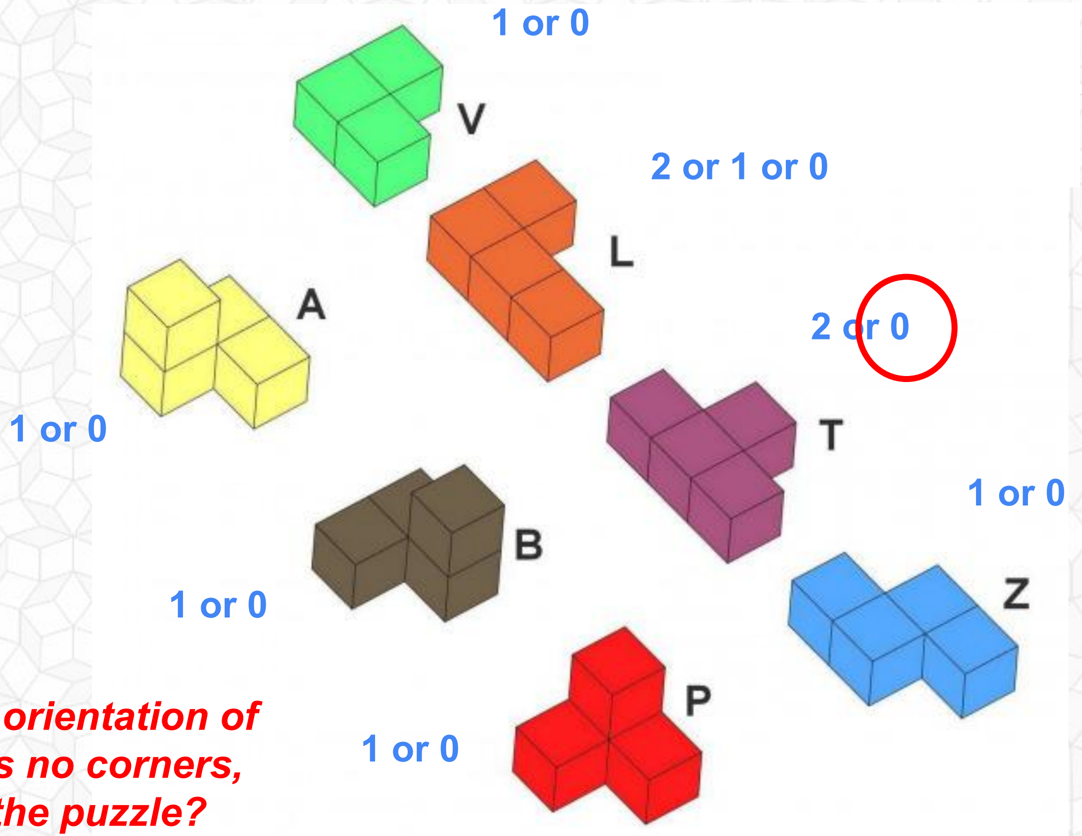
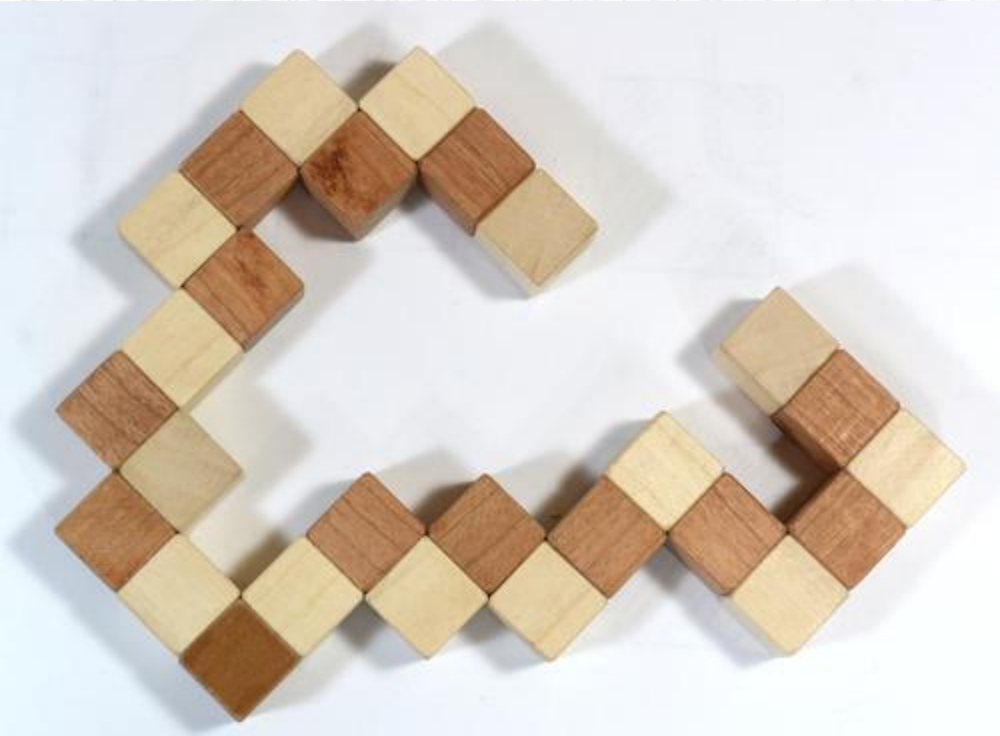


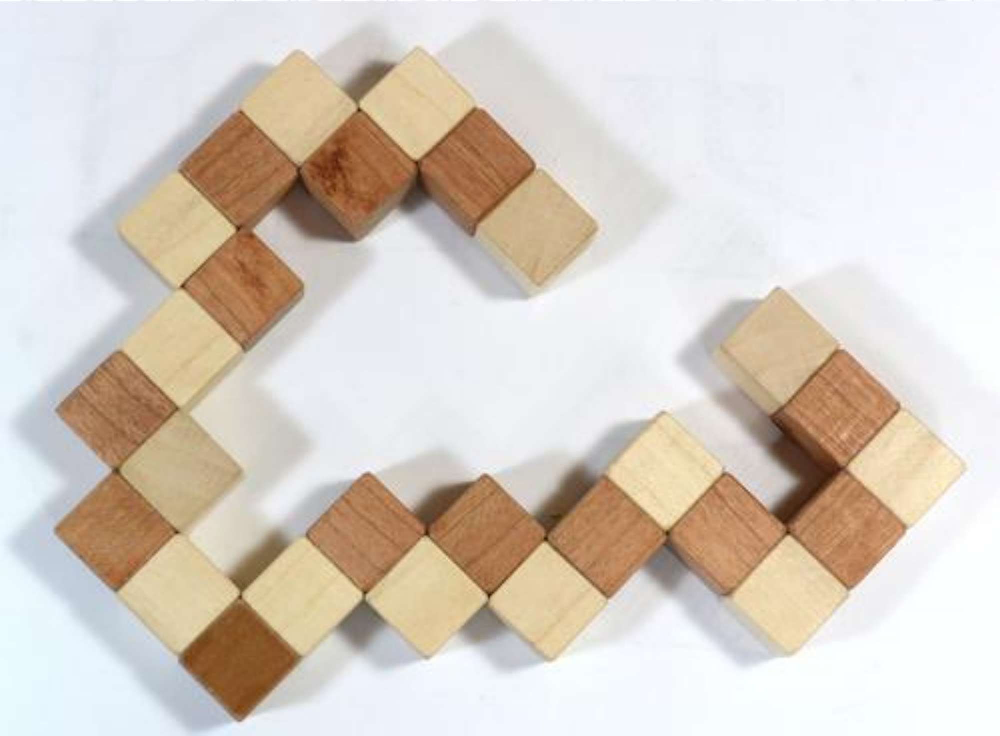
If we choose the orientation of the T that covers no corners, can we solve the puzzle?

3D Packing Puzzle: Soma Cube

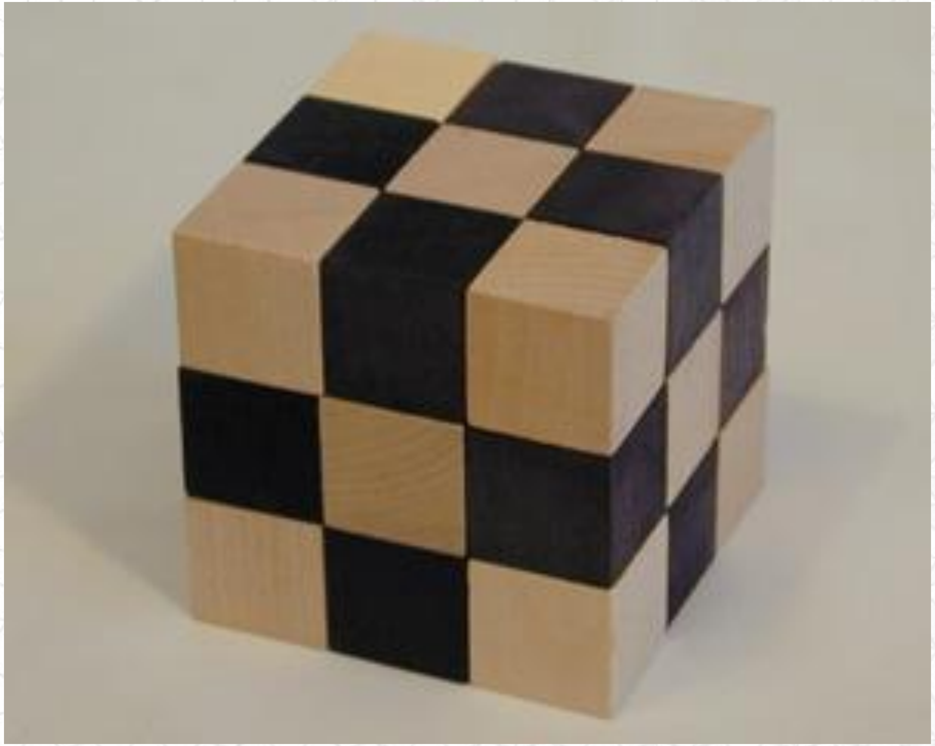
3D Packing Puzzle: Snake Cube



3D Packing Puzzle: Snake Cube



3D Packing Puzzle: Splitting Headache



http://billcutlerpuzzles.com/stock/splittingheadache.html

3D Packing Puzzle: Splitting Headache

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