CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/S22/

Lecture 20: Periodic & Non-Periodic Tiling

Outline for Today

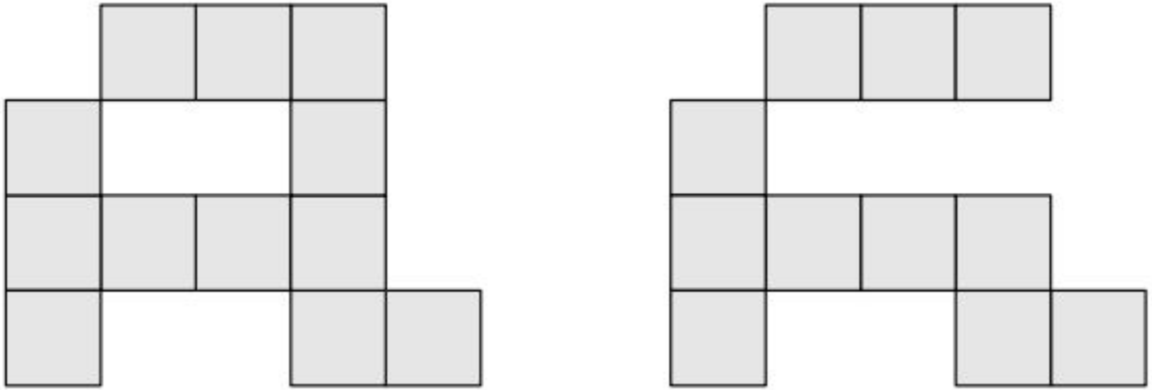
- Last Time: Polyominoes & Tiling
- Zellij - Moroccan/Islamic Mosaic Tilework
- Mashrabiya / Brise Soleil / Kinetic Architecture
- Crystals & Quasi Crystals
- Irrational Numbers
- Periodic vs. Non-Periodic Tiling
- More Tiling Terminology
- Penrose Non-Periodic Tiling
- Art: M.C. Escher, Crochet, etc.
- Next Time: ?

What is a Polyomino?

- An n-omino is a set of n cells on a square graph that is connected

is a polyomino

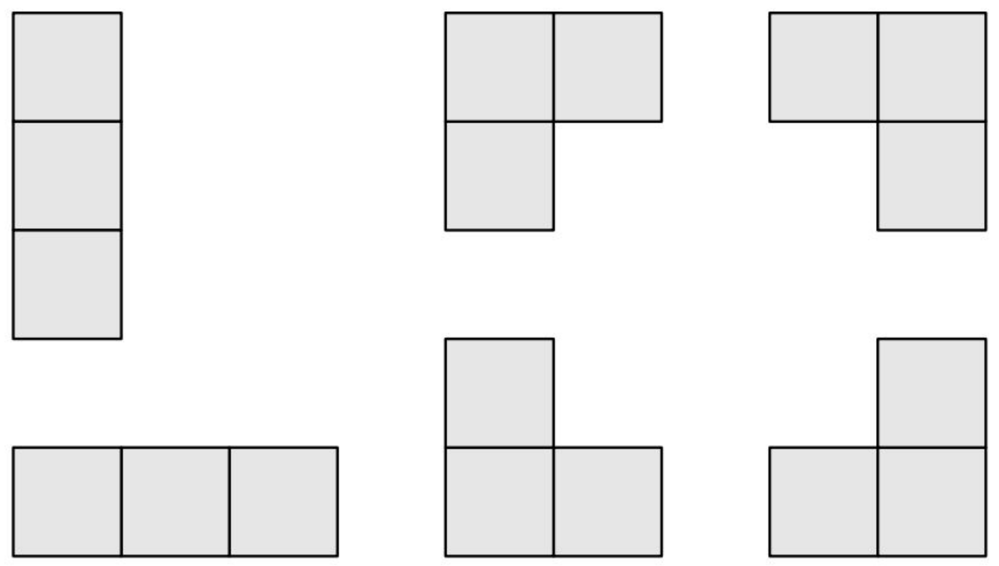
is NOT a polyomino



“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Translation-Equivalent / Fixed Polyomino

- Only left/right/up/down translation is allowed
- There are 6 unique Fixed 3-ominoes (a.k.a. trominoes):

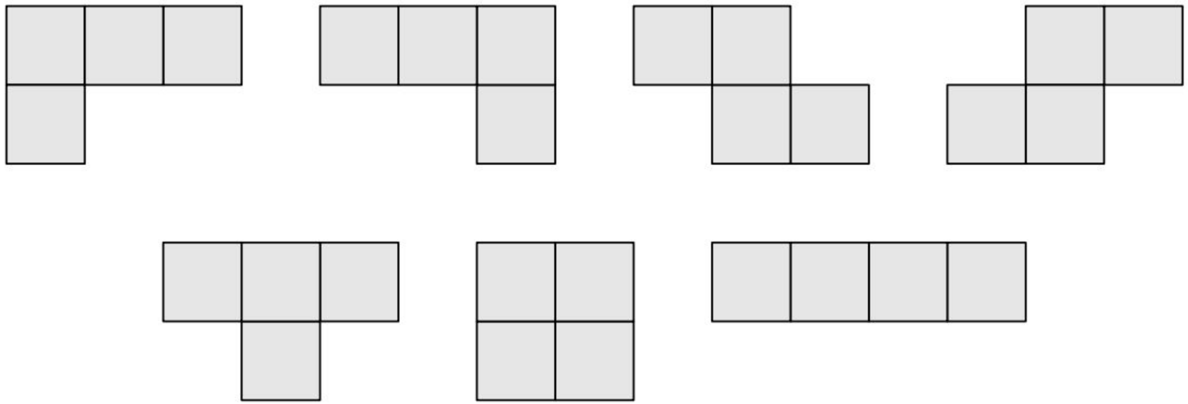


“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Rotation-Equivalent / Chiral Polyomino

- left/right/up/down translation allowed
- 90°/180°/270° rotation allowed

- There are 7 unique chiral 4-ominoes (a.k.a. tetrominoes):

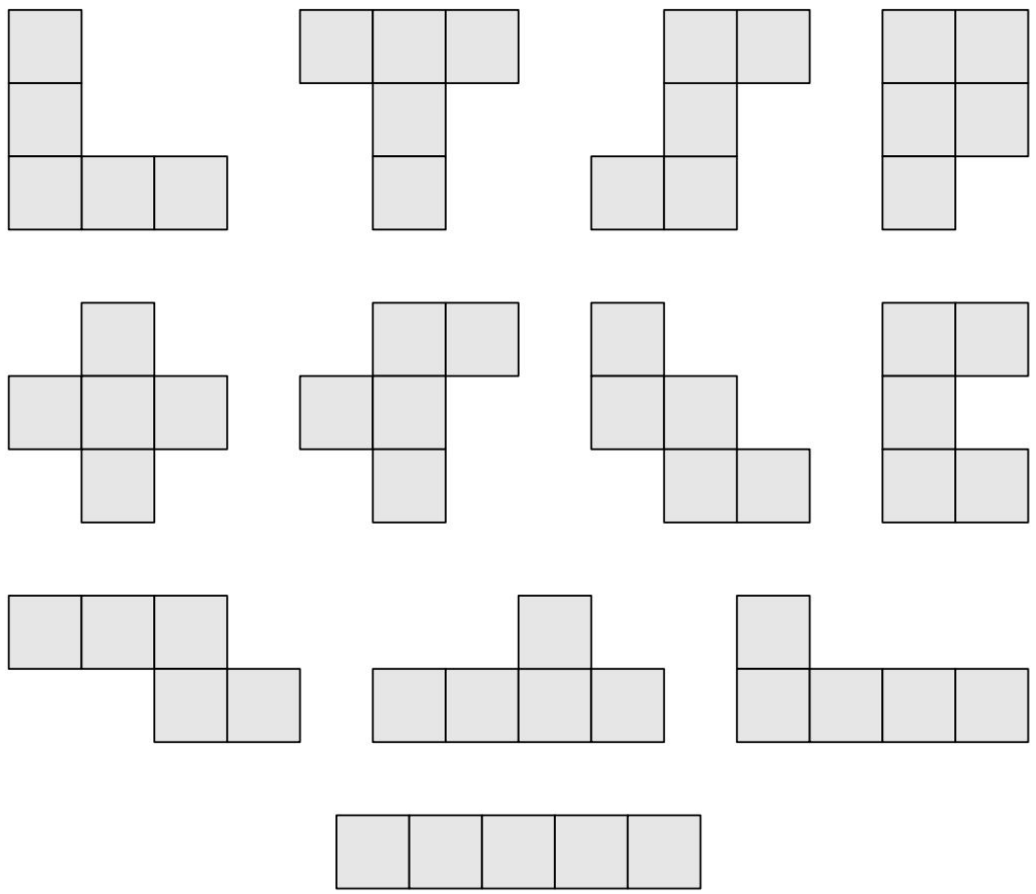


“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Free Polyomino

- Translation allowed
- Rotation allowed
- Reflection allowed

- There are 12 unique free 5-ominoes (a.k.a. pentominoes):



“Ch 14: Polyominoes”, Barequet, Golomb, & Klarner, Handbook of Discrete and Computational Geometry, 2018

Counting Fixed, Chiral, and Free Polyominoes

	fixed	chiral	free
	translation-only	translation & rotation (no reflection)	translation, rotation, & reflection

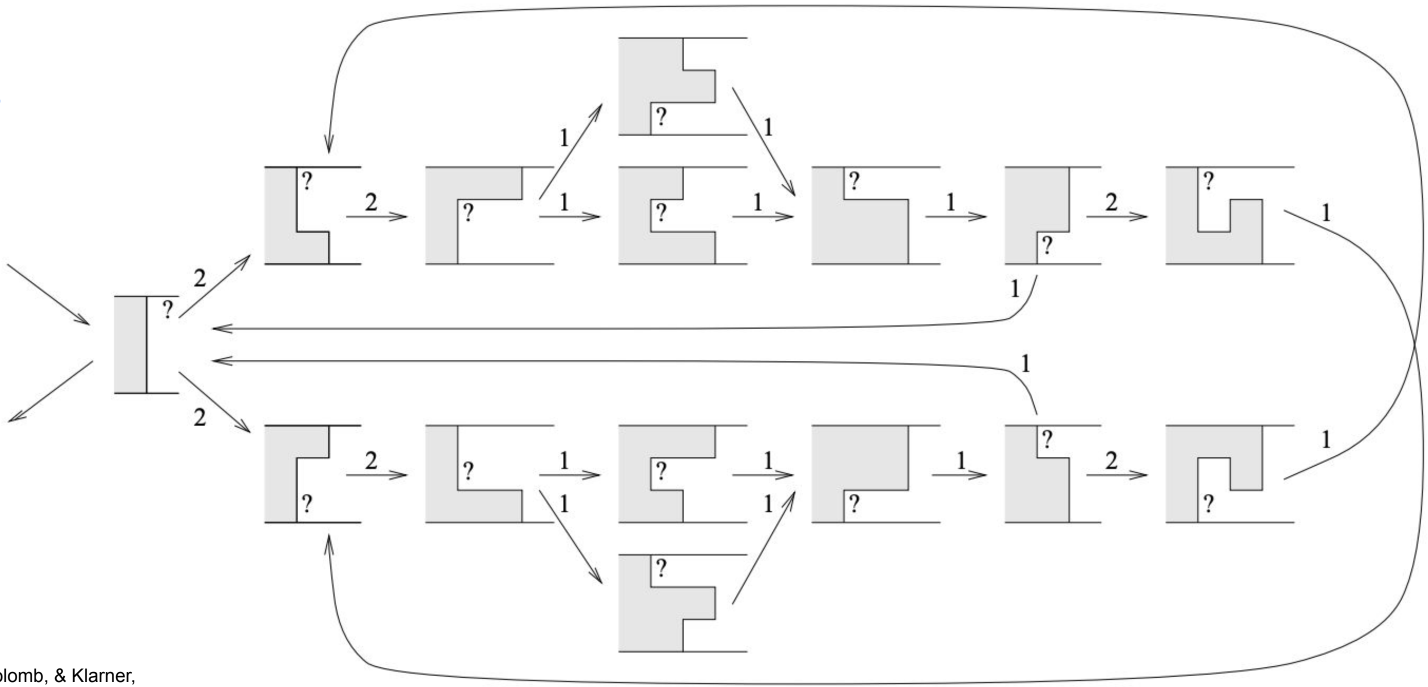
n	$t(n)$	$r(n)$	$s(n)$
1	1	1	1
2	2	1	1
3	6	2	2
4	19	7	5
5	63	18	12

"Ch 14: Polyominoes", Barequet, Golomb, & Klarner,
Handbook of Discrete and Computational Geometry, 2018

Packing Polyominoes

- Can we use the L-tetronimo, and all of its rotations and reflections to pack tile and infinite rectangle of height 3?

- *Yes, we can build the following automaton of all of states:*



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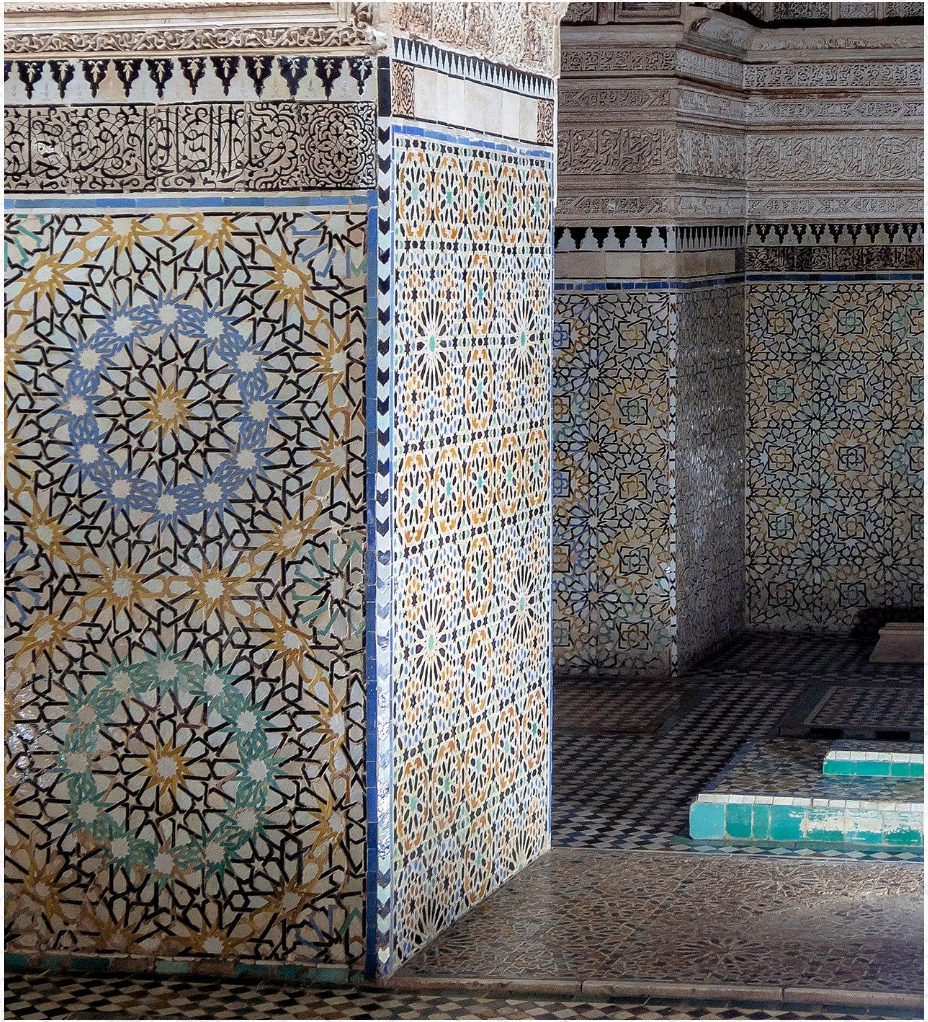
Zellij - Mosaic Tilework

- Traditional Islamic Art, Moroccan architecture, Moorish architecture
- Smooth, colorful, glazed/enamel tiles in a plaster base
- Colors:
    - Initially: white, green
    - then: yellow, blue, brown,
    - later: red
- Geometric motifs
- Avoid depictions of living things

https://en.wikipedia.org/wiki/Zellij



Zellij - Mosaic Tilework

Moroccan Zellij - Tiles - Marrakesh Tour Guide
https://www.youtube.com/watch?v=wrQsc5c-w98

https://en.wikipedia.org/wiki/Zellij

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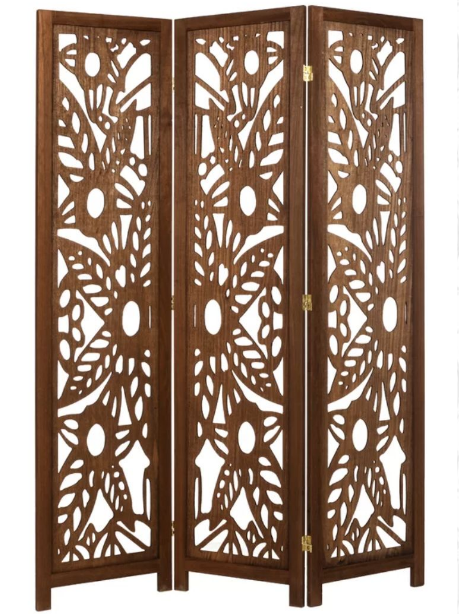
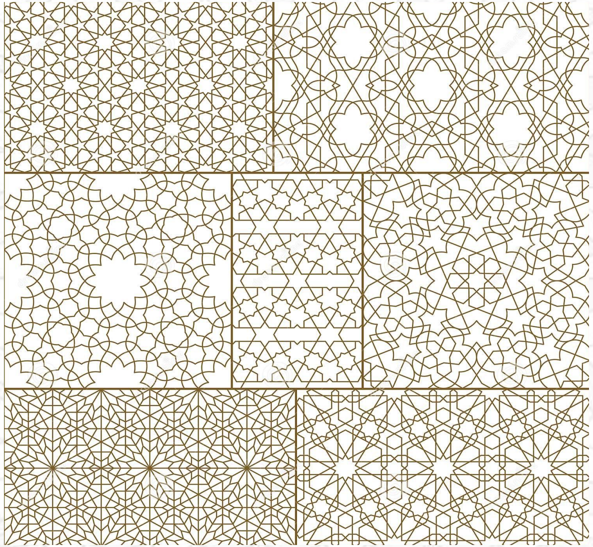
Mashrabiya

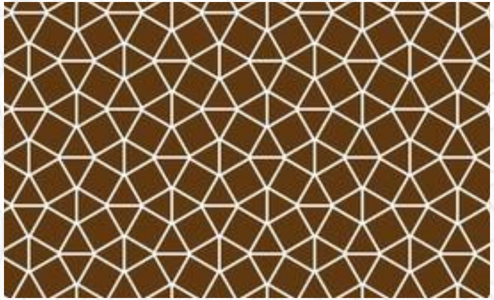
“Modern Mashrabiyas with High-tech Daylight Responsive Systems”, El Semary, Attalla, Gawad, 2017

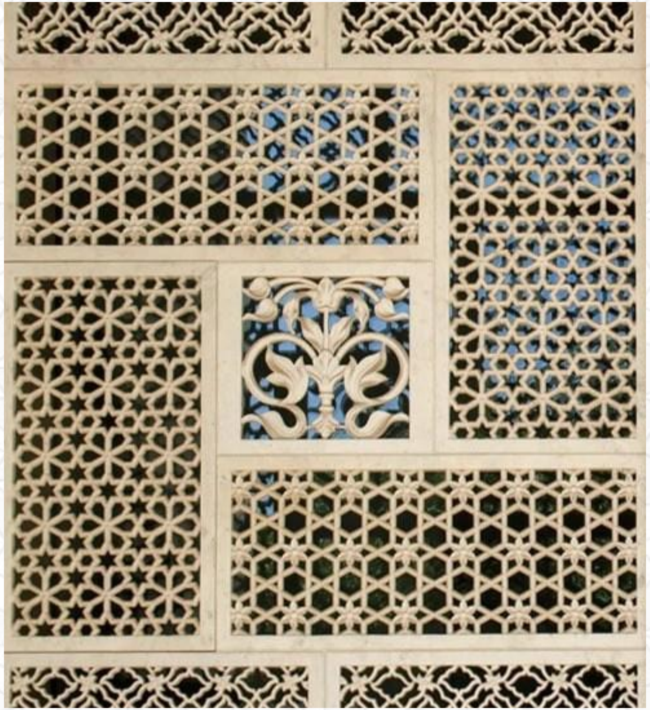
- Similar to a bay window, but enclosed with wooden latticework
- For hot & dry climates - Blocks direct sun, provides privacy
- Allows ventilation, and basins of water facilitate evaporative cooling

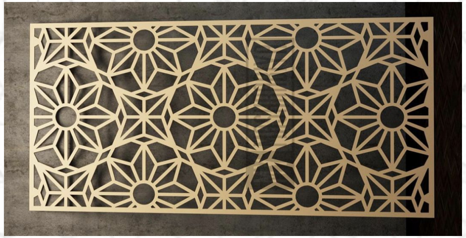
Modern Commercial Mashrabiya

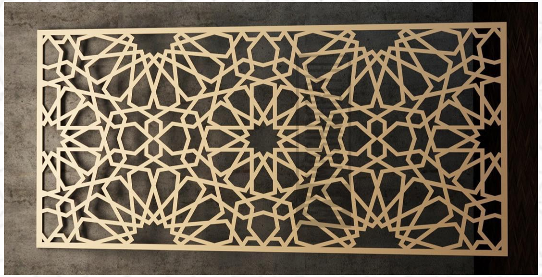












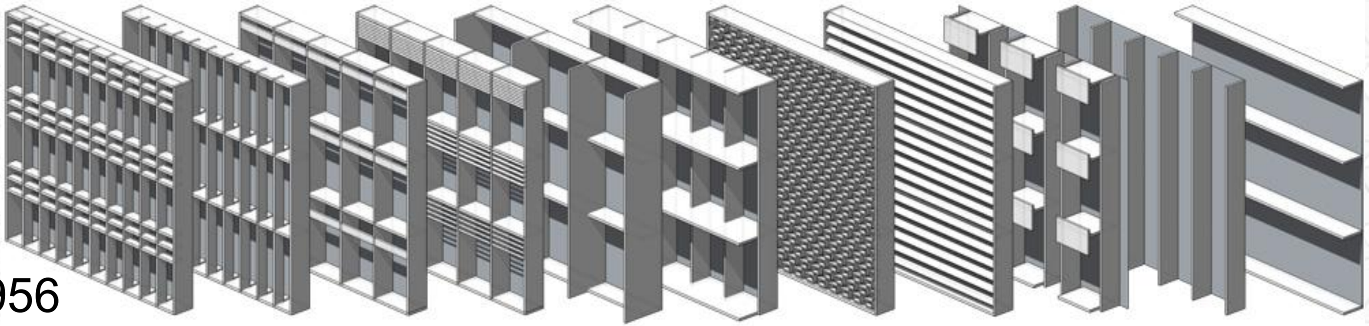
https://urbanalyse.com/research/brise-soleil-study-2/

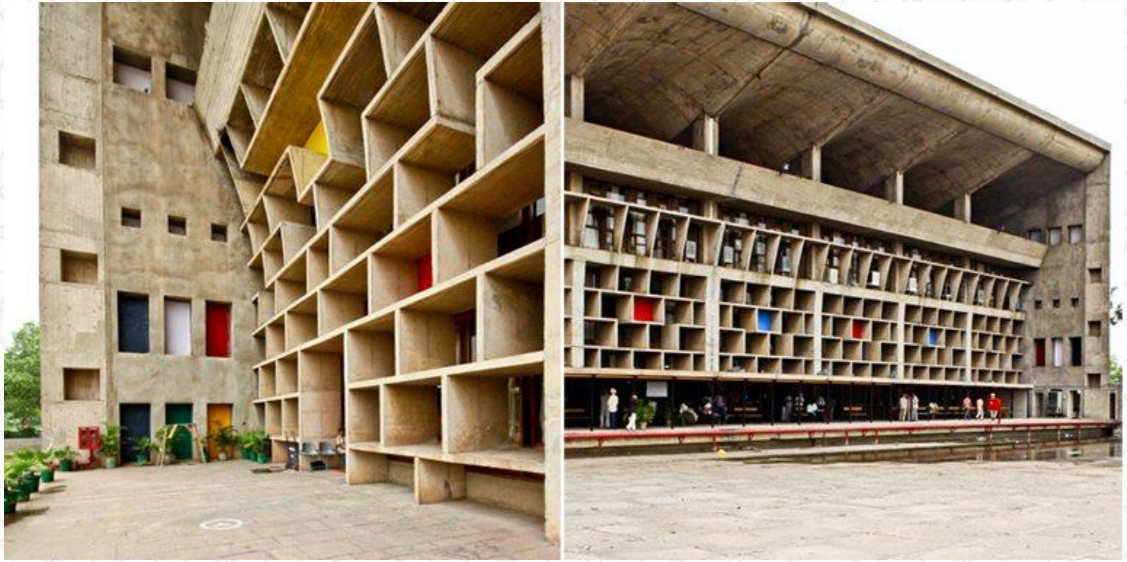
Brise Soleil

reduce heat gain by deflecting sunlight

Le Corbusier, 1951-1956

Court Chandigarh, India





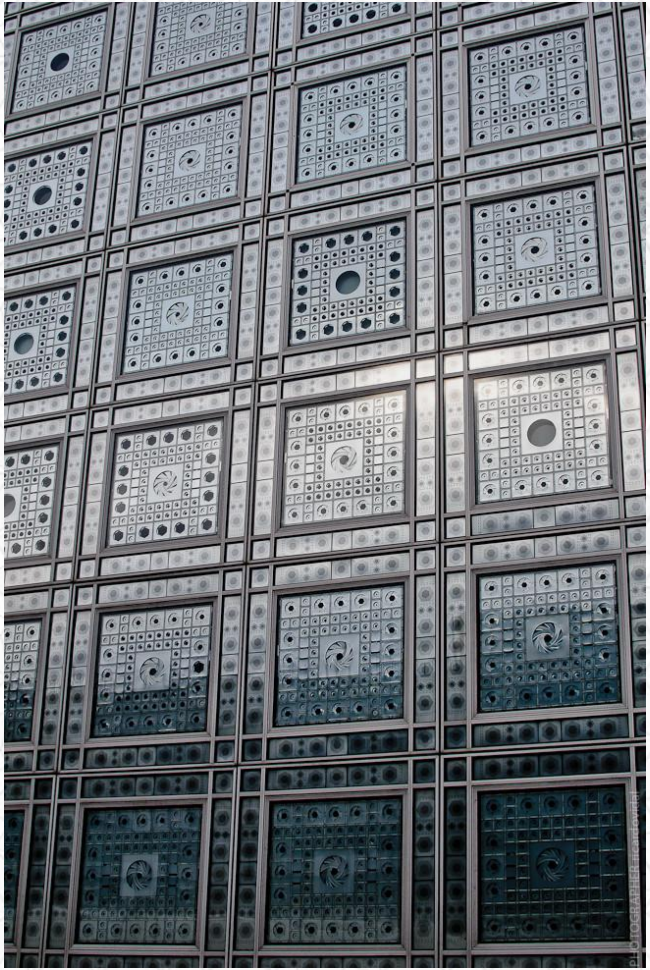
Institut du Monde Arabe

Architecture-Studio

& Jean Nouvel

Paris, France,

1987



Louvre Abu Dhabi, UAE

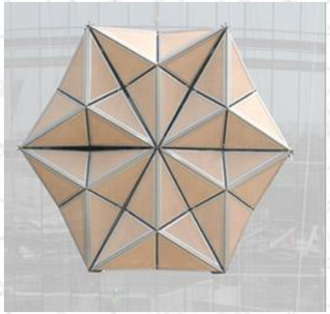
Jean Nouvel

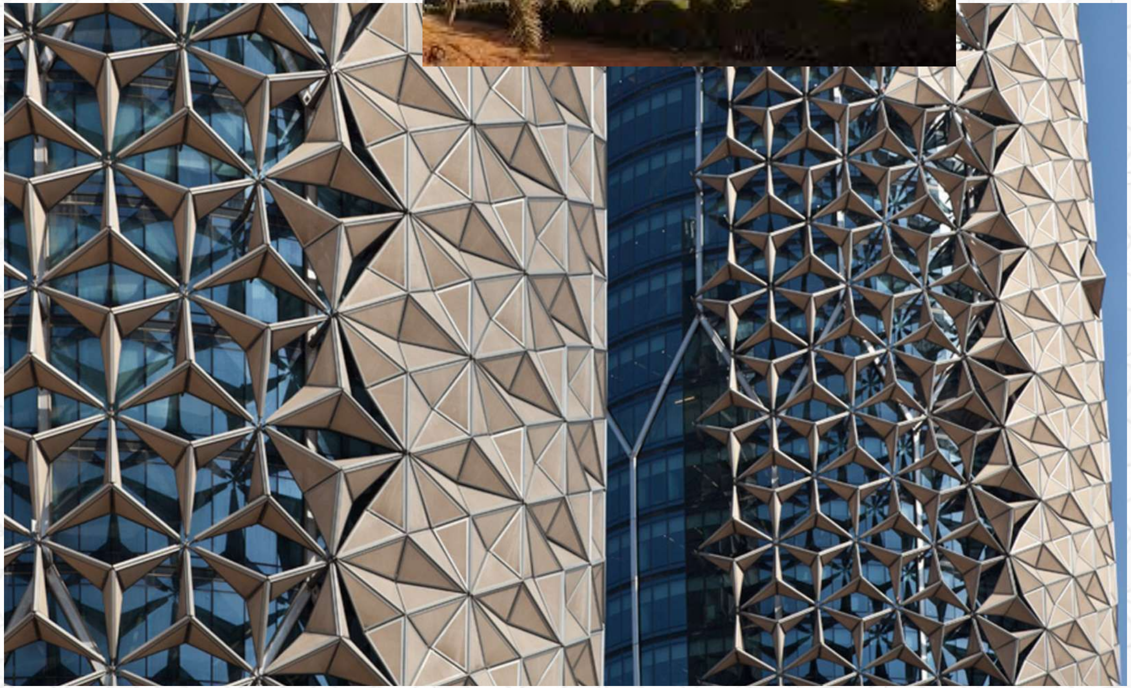
2017

Kinetic Architecture

Al Bahar Towers, Abu Dhabi, UAE

Aedas UK, Diar Consult, Arup, 2012





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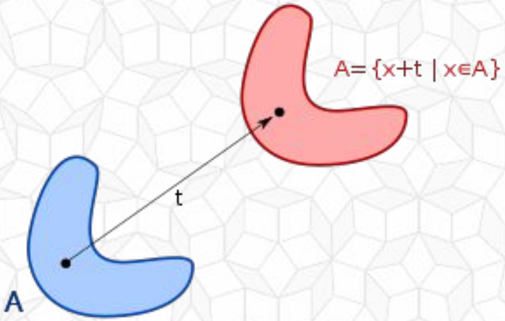
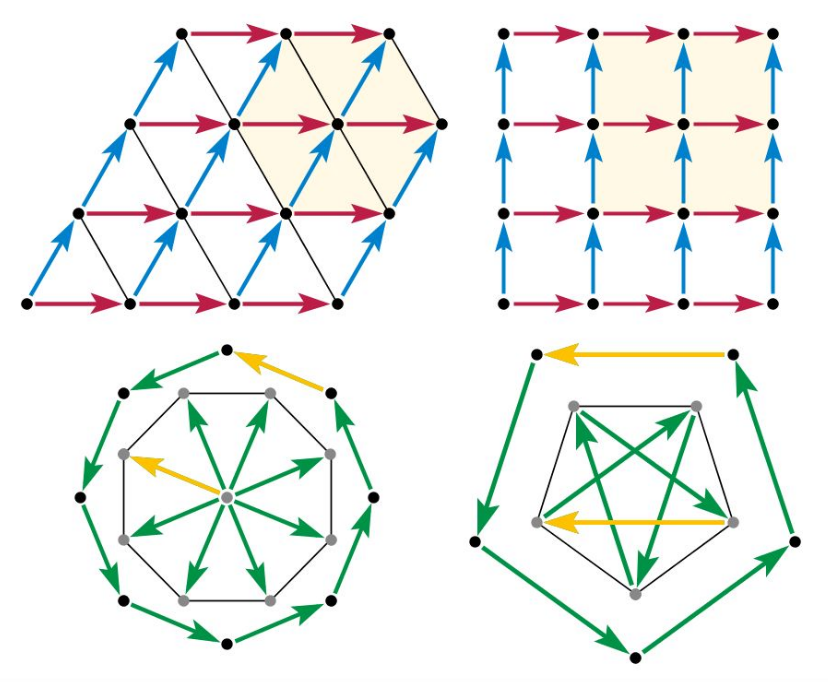
Crystal Structure

Originally assumed:

- Must have periodic, translational symmetry





- Specifically, 2-fold, 3-fold, 4-fold, or 6-fold symmetric
- And that 5-fold, 8-fold symmetry was not allowed

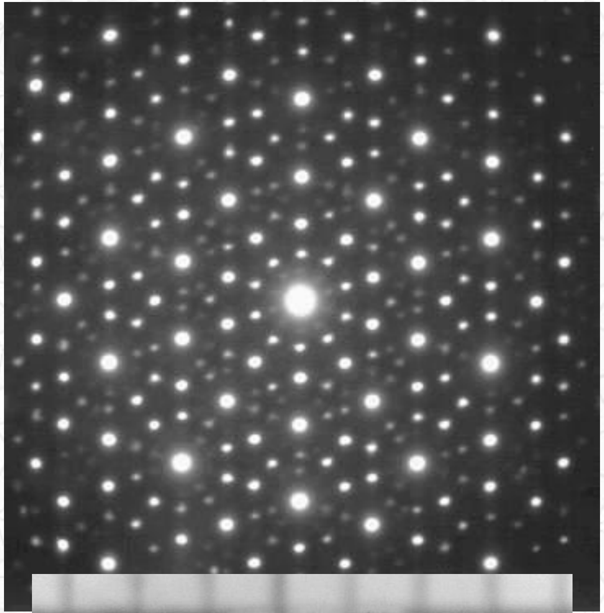
https://en.wikipedia.org/wiki/Translational_symmetry

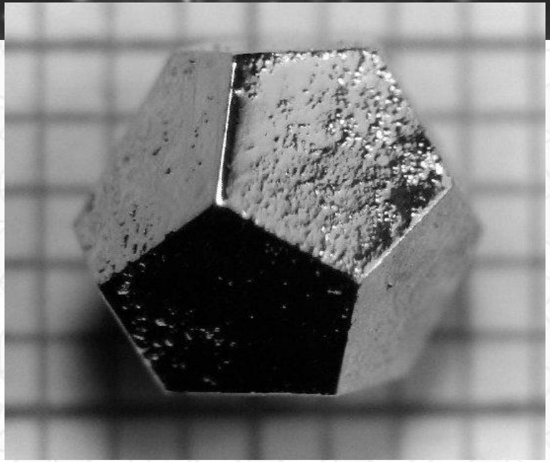
https://en.wikipedia.org/wiki/Crystallographic_restriction_theorem

Quasi-Crystal

- A nuclear bomb test in 1945 made quasi-crystal, but this was not noticed and confirmed until 2021.
- Unexpected (8-fold & 10-fold) diffraction patterns
- First investigated & published in 1980's by Dan Shechtman - *eventually won Nobel prize*
- Structure is ordered but not periodic
- Fills space (without gaps or overlaps), but lacks translational symmetry
- Properties: non-stick, heat insulating, strong
- Possible Applications: cookware, razor blades, gears, medical prosthesis, solar absorbers, …

https://en.wikipedia.org/wiki/Quasicrystal





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Irrational Numbers

- All real numbers that are not rational
- Rational numbers can be expressed as a ratio of 2 integers, e.g. “a/b”
- Examples: pi, sqrt(2), etc.
- Decimal representation does not terminate, *and* does not end with a repeating sequence

fraction	decimal expansion	ℓ_{10}	binary expansion	ℓ_2
$\frac{1}{2}$	0.5	0	0.1	0
$\frac{1}{3}$	$0.\overline{3}$	1	$0.\overline{01}$	2
$\frac{1}{4}$	0.25	0	0.01	0
$\frac{1}{5}$	0.2	0	$0.\overline{0011}$	4
$\frac{1}{6}$	$0.1\overline{6}$	1	$0.0\overline{01}$	2
$\frac{1}{7}$	$0.\overline{142857}$	6	$0.\overline{001}$	3
$\frac{1}{8}$	0.125	0	0.001	0
$\frac{1}{9}$	$0.\overline{1}$	1	$0.\overline{000111}$	6
$\frac{1}{10}$	0.1	0	$0.0\overline{0011}$	4
$\frac{1}{11}$	$0.\overline{09}$	2	$0.\overline{0001011101}$	10
$\frac{1}{12}$	$0.08\overline{3}$	1	$0.00\overline{01}$	2
$\frac{1}{13}$	$0.\overline{076923}$	6	$0.\overline{000100111011}$	12
$\frac{1}{14}$	$0.0\overline{714285}$	6	$0.0\overline{001}$	3
$\frac{1}{15}$	$0.0\overline{6}$	1	$0.\overline{0001}$	4
$\frac{1}{16}$	0.0625	0	0.0001	0

fraction	decimal expansion	ℓ_{10}
$\frac{1}{17}$	$0.\overline{0588235294117647}$	16
$\frac{1}{18}$	$0.0\overline{5}$	1
$\frac{1}{19}$	$0.\overline{052631578947368421}$	18
$\frac{1}{20}$	0.05	0
$\frac{1}{21}$	$0.\overline{047619}$	6
$\frac{1}{22}$	$0.0\overline{45}$	2
$\frac{1}{23}$	$0.\overline{0434782608695652173913}$	22
$\frac{1}{24}$	$0.041\overline{6}$	1
$\frac{1}{25}$	0.04	0
$\frac{1}{26}$	$0.0\overline{384615}$	6
$\frac{1}{27}$	$0.\overline{037}$	3
$\frac{1}{28}$	$0.03\overline{571428}$	6
$\frac{1}{29}$	$0.\overline{0344827586206896551724137931}$	28
$\frac{1}{30}$	$0.0\overline{3}$	1
$\frac{1}{31}$	$0.\overline{032258064516129}$	15

fraction	decimal expansion	ℓ_{10}
$\frac{1}{32}$	0.03125	0
$\frac{1}{33}$	$0.\overline{03}$	2
$\frac{1}{34}$	$0.0\overline{2941176470588235}$	16
$\frac{1}{35}$	$0.0\overline{285714}$	6
$\frac{1}{36}$	$0.02\overline{7}$	1
$\frac{1}{37}$	$0.\overline{027}$	3
$\frac{1}{38}$	$0.0\overline{263157894736842105}$	18
$\frac{1}{39}$	$0.\overline{025641}$	6
$\frac{1}{40}$	0.025	0
$\frac{1}{41}$	$0.\overline{02439}$	5
$\frac{1}{42}$	$0.0\overline{238095}$	6
$\frac{1}{43}$	$0.\overline{023255813953488372093}$	21
$\frac{1}{44}$	$0.02\overline{27}$	2
$\frac{1}{45}$	$0.0\overline{2}$	1
$\frac{1}{46}$	$0.0\overline{2173913043478260869565}$	22

https://en.wikipedia.org/wiki/Repeating_decimal

Digits of Pi

Let’s look for “translational symmetry”…

```
3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 5923078164 0628620899
  8628034825 3421170679 8214808651 3282306647 0938446095 5058223172 5359408128 4811174502
  8410270193 8521105559 6446229489 5493038196 4428810975 6659334461 2847564823 3786783165
  2712019091 4564856692 3460348610 4543266482 1339360726 0249141273 7245870066 0631558817
  4881520920 9628292540 9171536436 7892590360 0113305305 4882046652 1384146951 9415116094
  3305727036 5759591953 0921861173 8193261179 3105118548 0744623799 6274956735 1885752724
  8912279381 8301194912 9833673362 4406566430 8602139494 6395224737 1907021798 6094370277
  0539217176 2931767523 8467481846 7669405132 0005681271 4526356082 7785771342 7577896091
  7363717872 1468440901 2249534301 4654958537 1050792279 6892589235 4201995611 2129021960
  8640344181 5981362977 4771309960 5187072113 4999999837 2978049951 0597317328 1609631859
  5024459455 3469083026 4252230825 3344685035 2619311881 7101000313 7838752886 5875332083
  8142061717 7669147303 5982534904 2875546873 1159562863 8823537875 9375195778 1857780532
  ...
```

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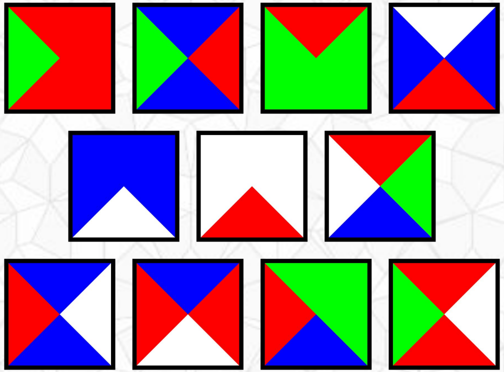
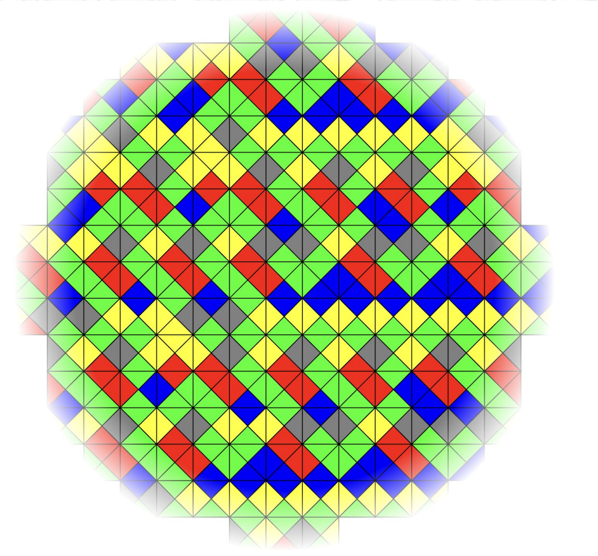
Wang Tiles / Wang Dominoes

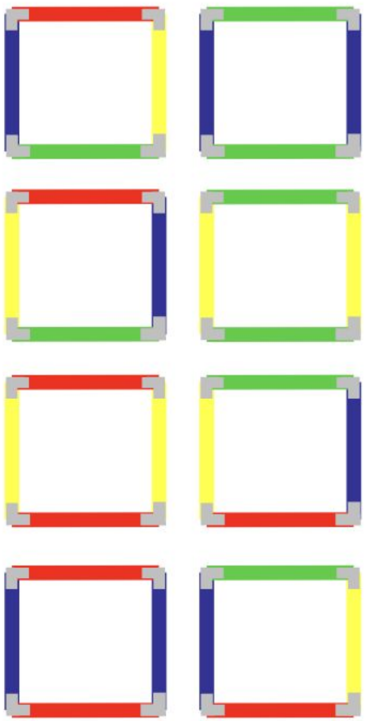
- Square tiles, edges labeled with colors, must be placed without rotation, with matching edges
- In 1961, Hao Wang conjectured that any finite set of tiles that could tile a plane infinitely, could be tiled periodically
- In 1966, Robert Berger proved that non-periodic Wang tile sets existed
- In 2015, Emmanuel Jeandel and Michael Rao proved that the smallest non-periodic Wang tile set was 11 tiles w/ 4 colors
- Applications: natural-looking, aperiodic synthesized texture, heightfields, & more

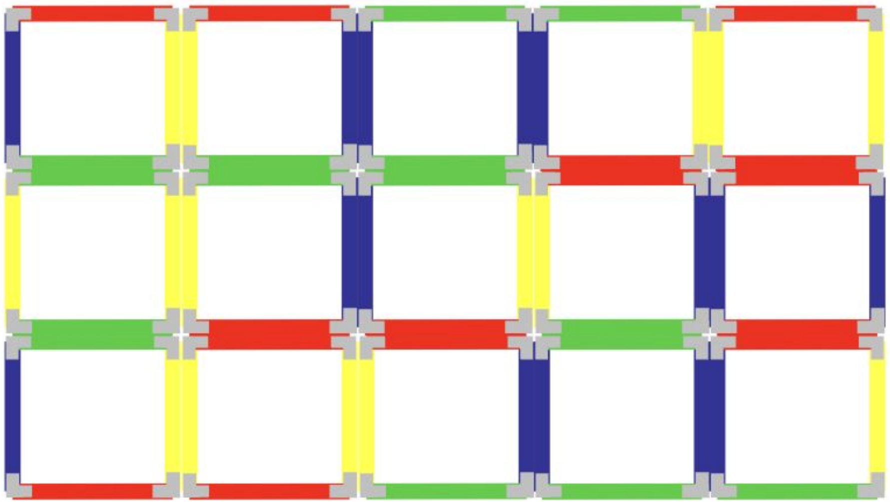




Wang Tiles

“Wang Tiles for Image and Texture Generation”,
Cohen, Shade, Hiller, Deussen, SIGGRAPH 2003





Align tiles to match edge color to create non-periodic tilings

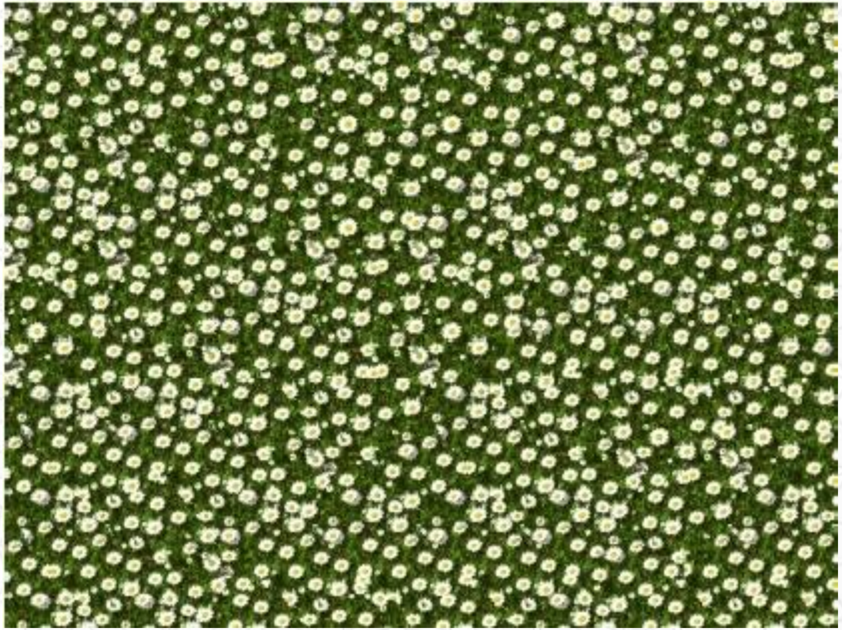
“Wang Tiles for Image and Texture Generation”, *Cohen, Shade, Hiller, Deussen,* SIGGRAPH 2003

Wang Tile Texture Synthesis

- As a precomputation, fill the tiles with texture
- Then create infinite amounts of non-periodic texture!

Input texture sample

Automatically generated set of Wang tiles



Synthesized textures using Wang tiling

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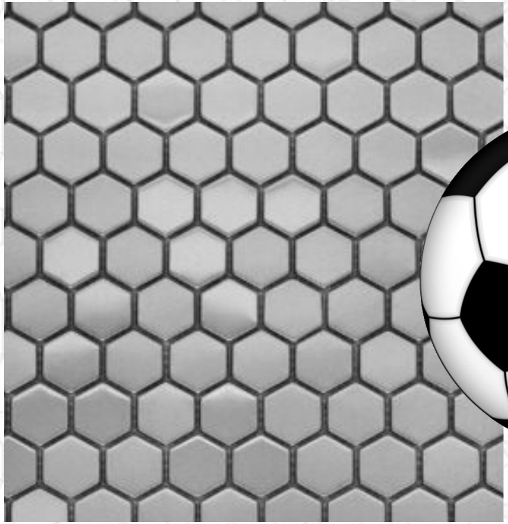
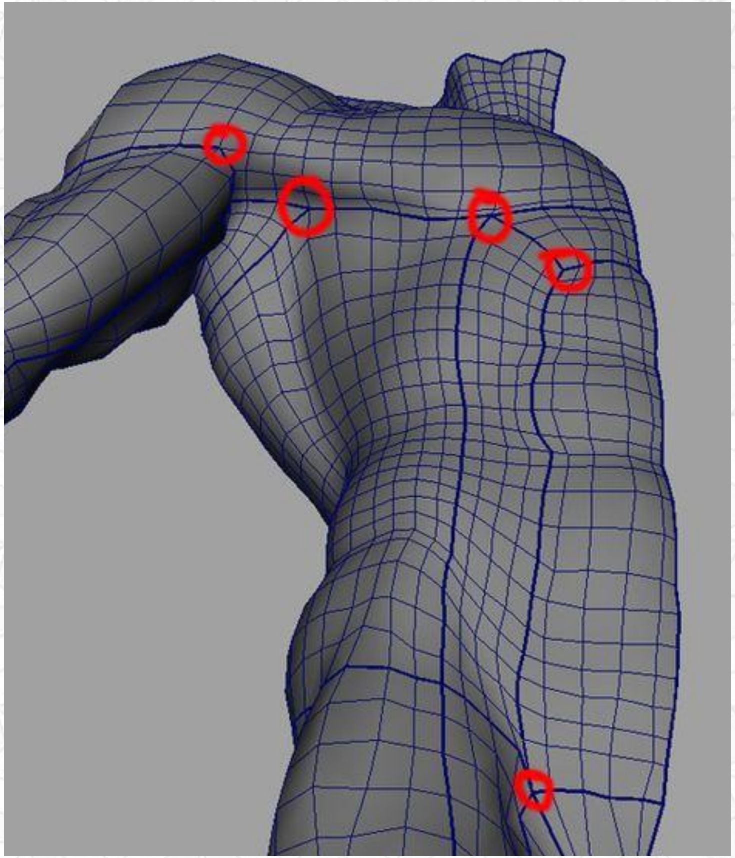
Misc. Mesh/Surface Vocabulary

- Extraordinary Vertex
  - Quad mesh: vertices w/ valence ≠ 4
  - Hex mesh: vertices w/ valence ≠ 3
  - Tri mesh: vertices w/ valence ≠ 6



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- Extraordinary Vertex
  - Quad mesh: vertices w/ valence ≠ 4
  - Hex mesh: vertices w/ valence ≠ 3
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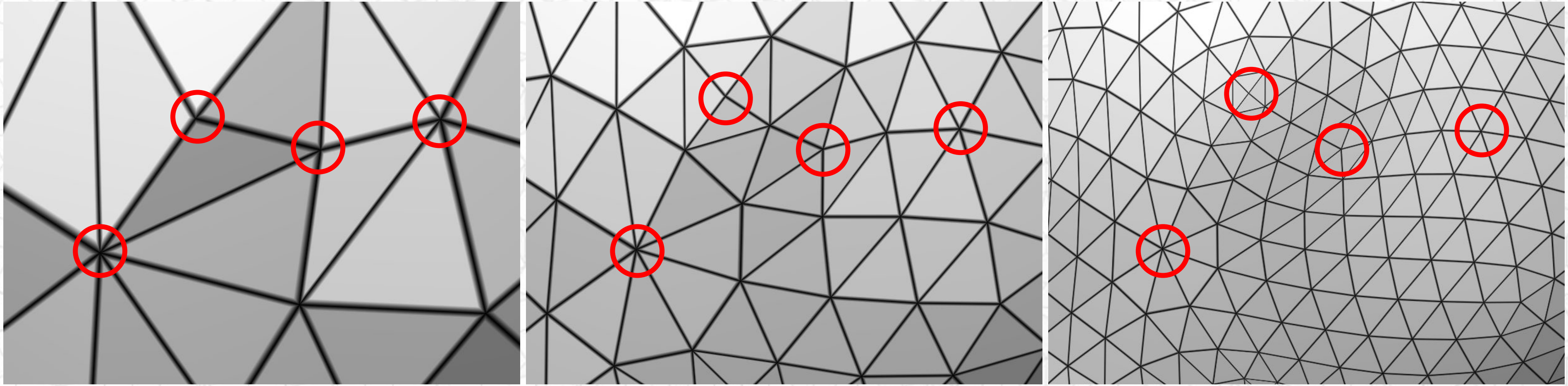
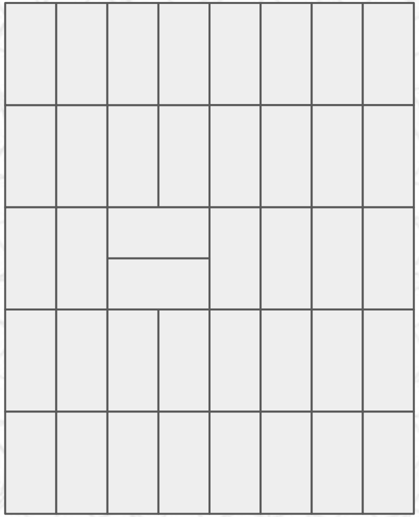
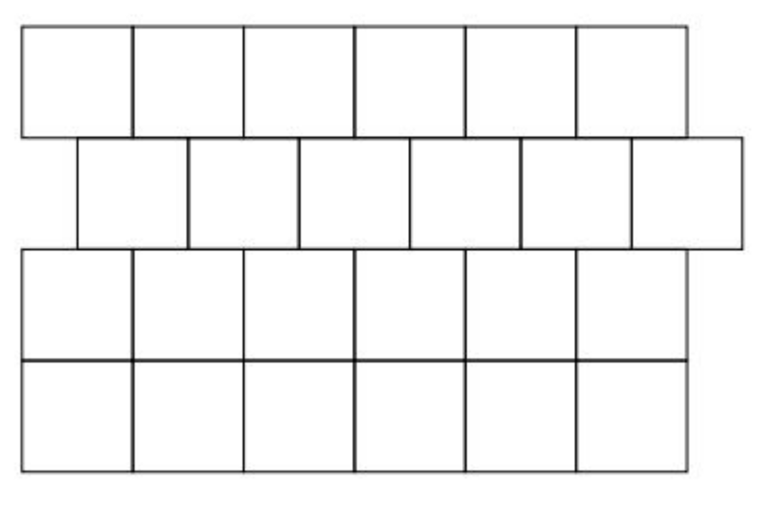
Extraordinary vertices persist through subdivision!



Non-Periodic vs. Aperiodic

- Non-Periodic: A tiling which is not translationally symmetric
- A-Periodic: A set of tiles which cannot be tiled periodically





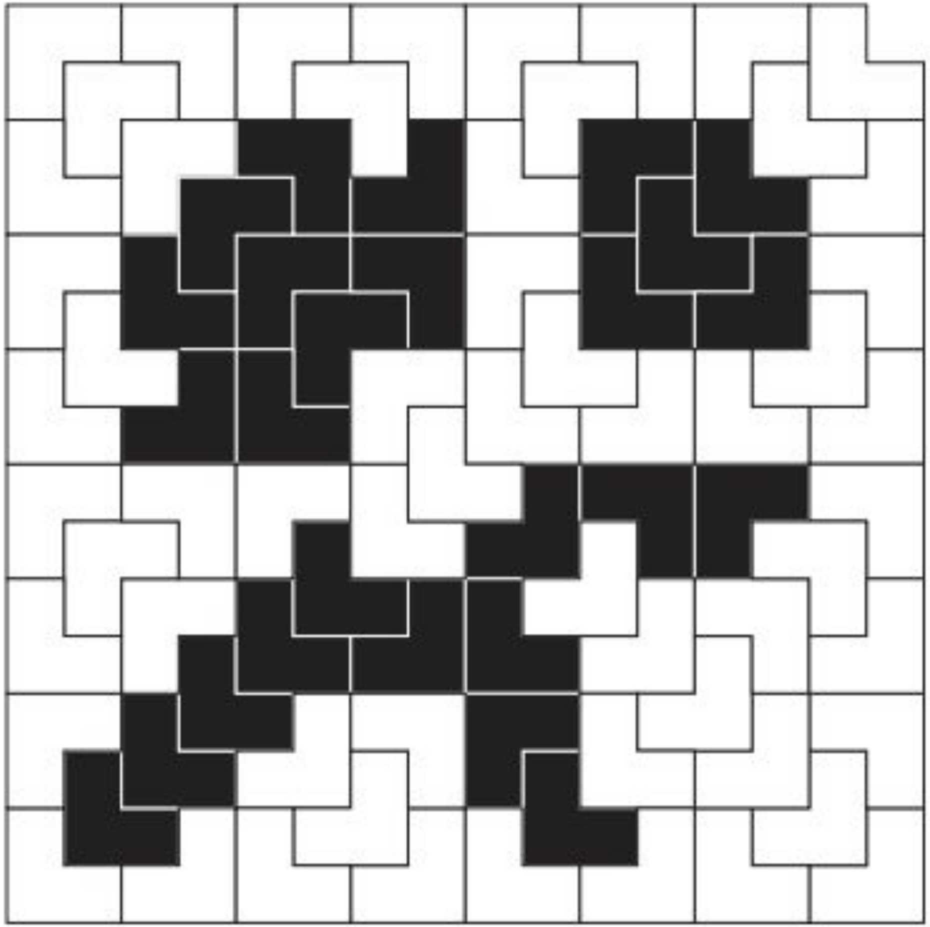
“Ch 3: Tiling”, Harriss, Schattschneider, & Senechal, Handbook of Discrete and Computational Geometry, 2018

Cluster: set of tiles that intersect a shape.

Patch: a cluster for a convex shape.

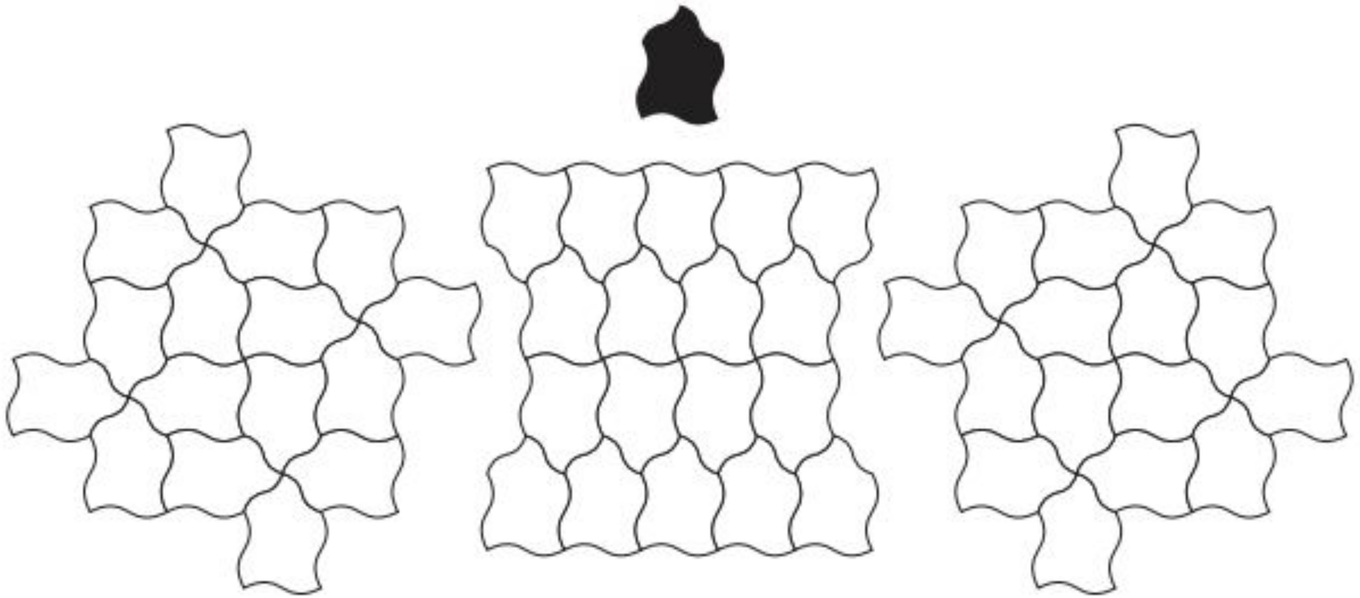
Example: Image shows 3 clusters, 2 of the clusters are patches.

“Ch 3: Tiling”, Harriss, Schattschneider, & Senechal, Handbook of Discrete and Computational Geometry, 2018



- Monohedral Tiling: Using a single shape to tile the plane
- *r*-Morphic Tile: Can be arranged in *r* different monohedral tilings

  Example: a 3-morphic (trimorphic) tile

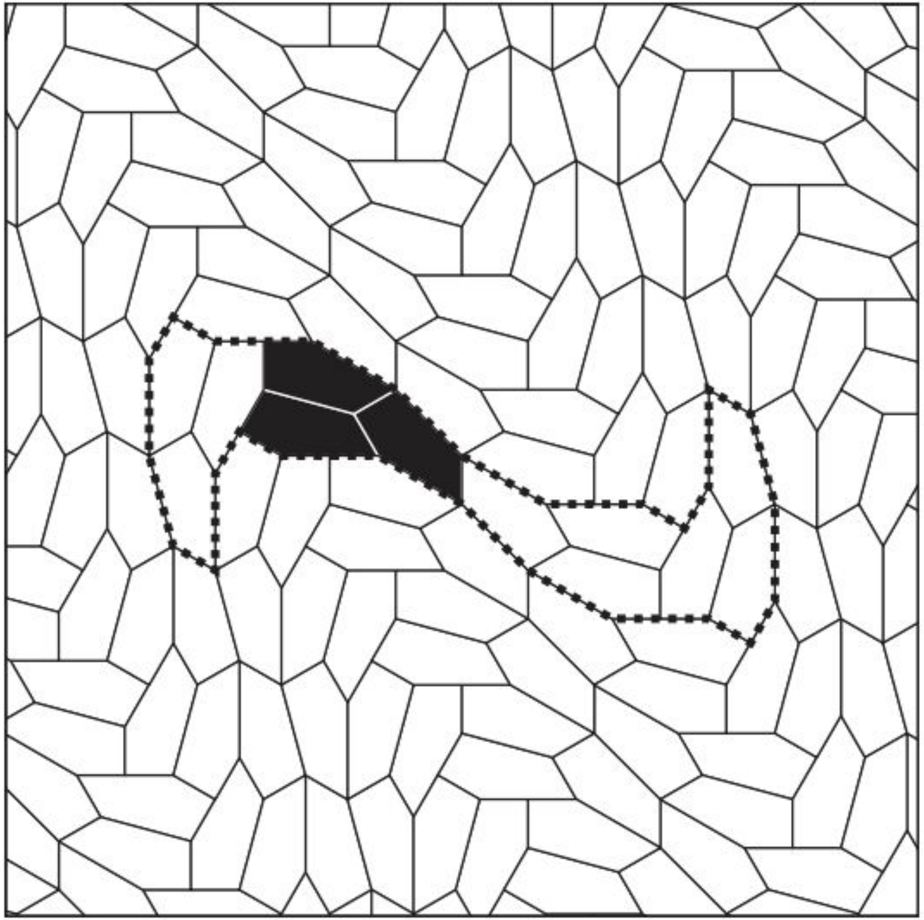


"Ch 3: Tiling", Harriss, Schattschneider, & Senechal,
Handbook of Discrete and Computational Geometry, 2018

- Isohedral (tiling): A tiling whose symmetry group acts transitively on its tiles.
- Anisohedral tile: A prototile that admits monohedral tilings but no isohedral tilings.

Example:

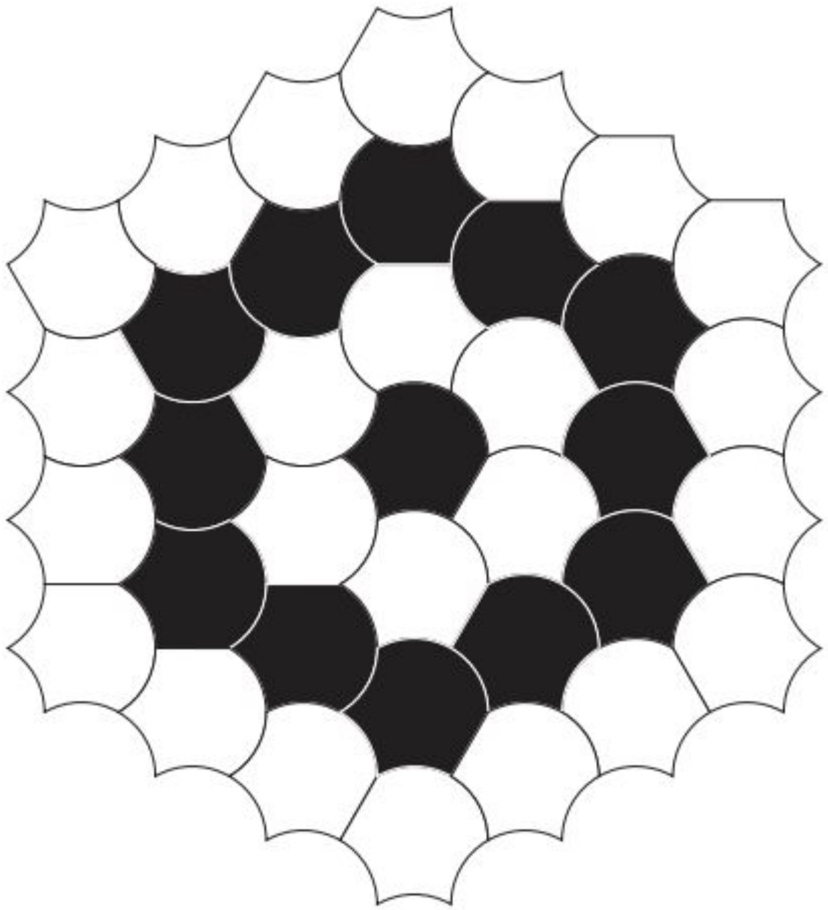
- *The prototile admits a unique non-isohedral tiling; the black tiles are each surrounded differently.*
- *This tiling is periodic.*



“Ch 3: Tiling”, Harriss, Schattschneider, & Senechal, Handbook of Discrete and Computational Geometry, 2018

k-corona of a tile:
The set of all tiles that touch the (*k-1*)-corona of the tile

Example: A 3-corona tile (It cannot be surrounded by a fourth corona.)



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Outline for Today

- Last Time: Polyominoes & Tiling
- Zellij - Moroccan/Islamic Mosaic Tilework
- Mashrabiya / Brise Soleil / Kinetic Architecture
- Crystals & Quasi Crystals
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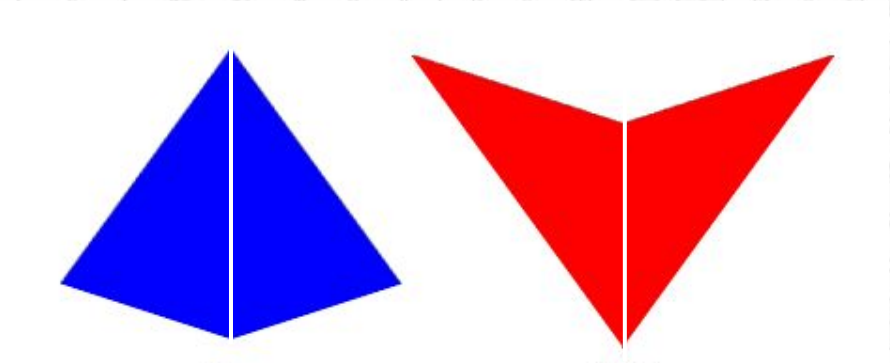
- Next Time: ?

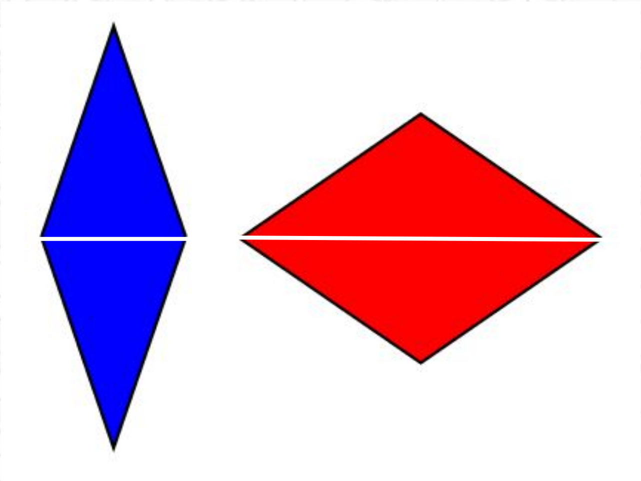
Penrose Tilings are Non-Periodic

- Discovered in 1974 by Roger Penrose
- Simple rules for which edges are allowed to match other edges
- Multiple variations of a tile sets that can fill a plane, but are non-repeating!

Kite & Dart



2 Rhomboids



https://personal.math.ubc.ca/~cass/courses/m308-02b/projects/schweber/penrose.html

- A labeling or marking of the tiles may be necessary for a specific tileset to be aperiodic.
- *E.g., Penrose Kite & Rhombus:*

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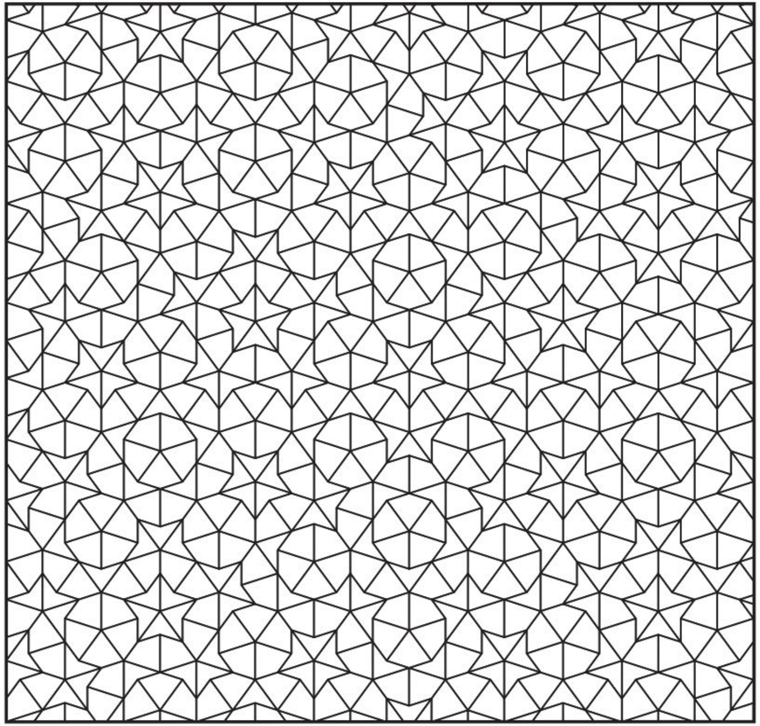
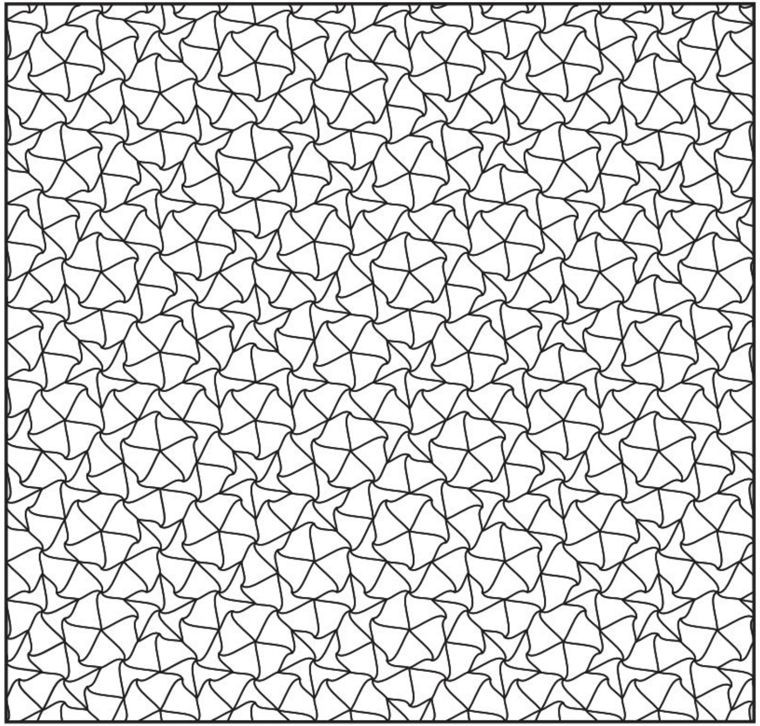
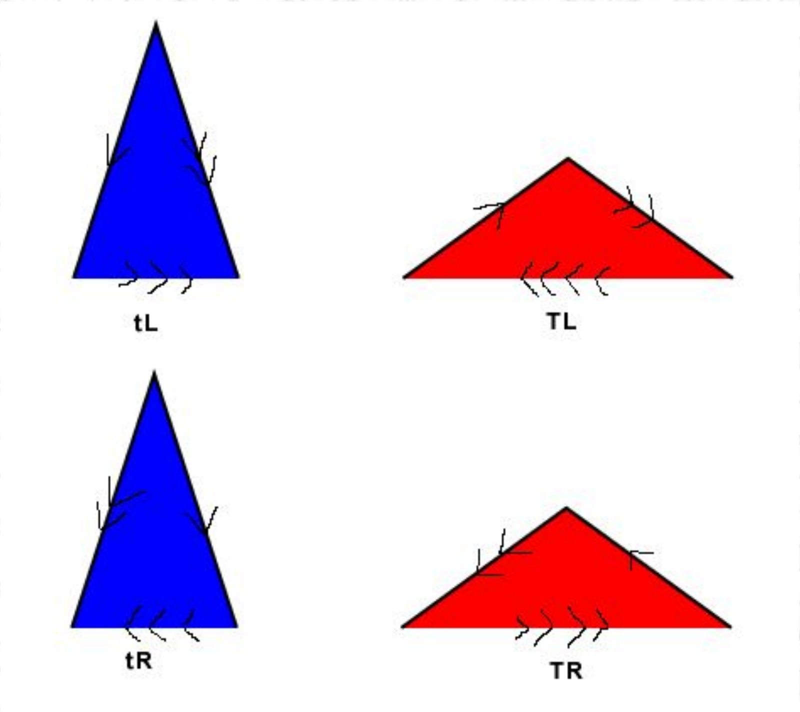
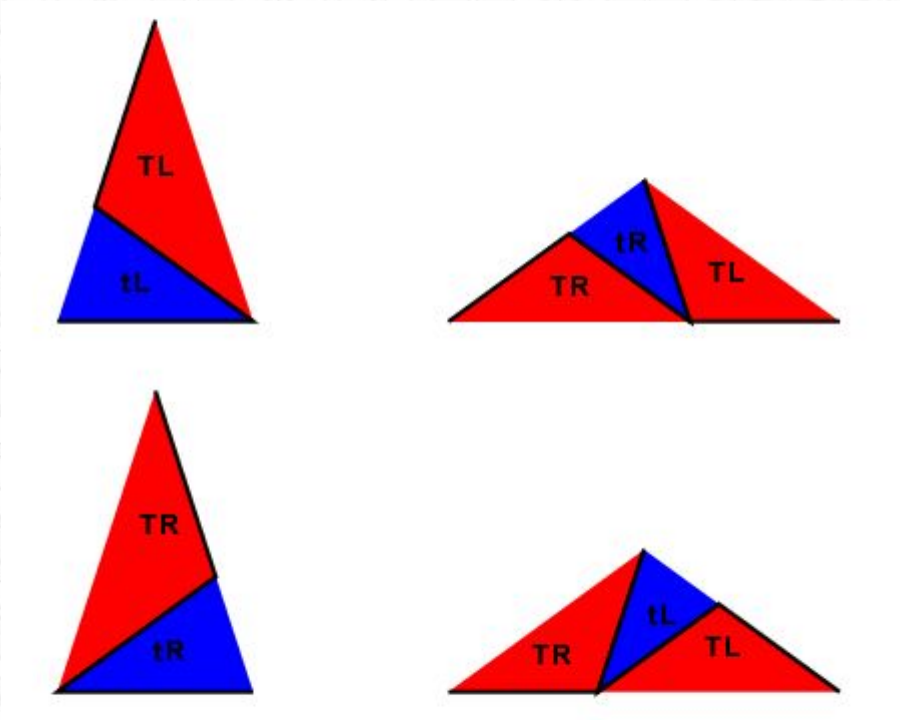




Penrose Tilings Can be Subdivided

- *And conversely, this is how they are proved to be aperiodic!*





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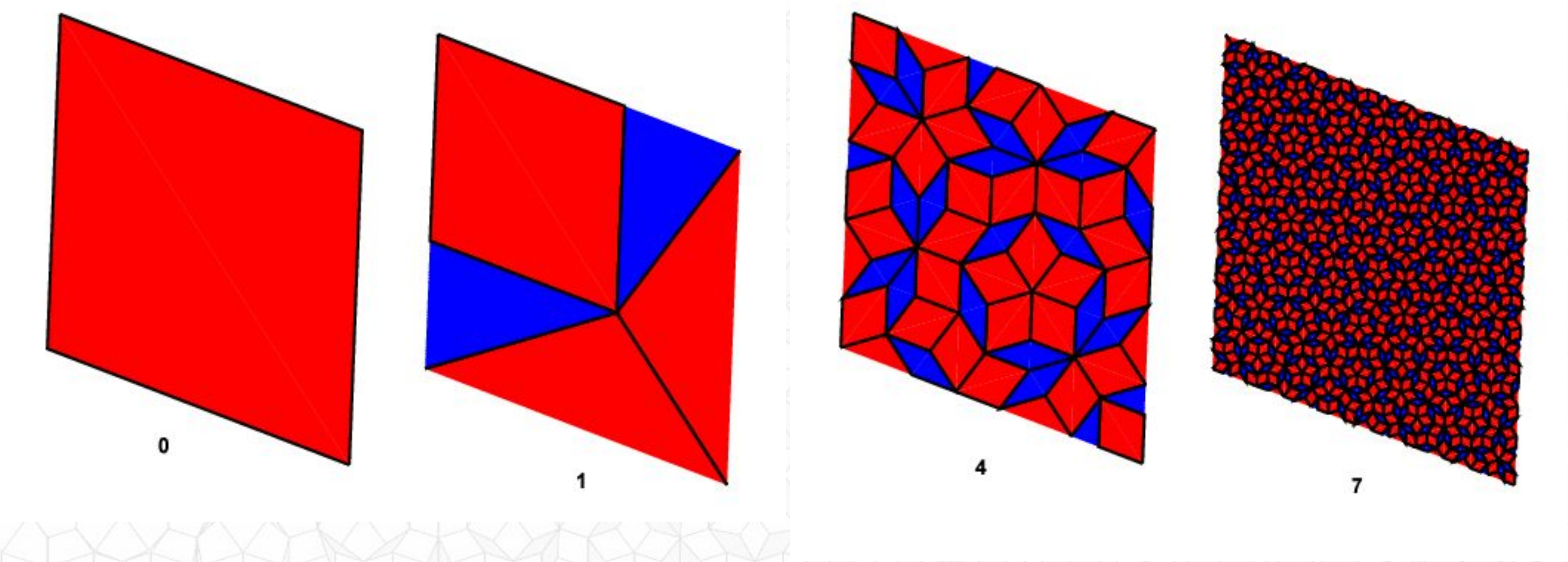

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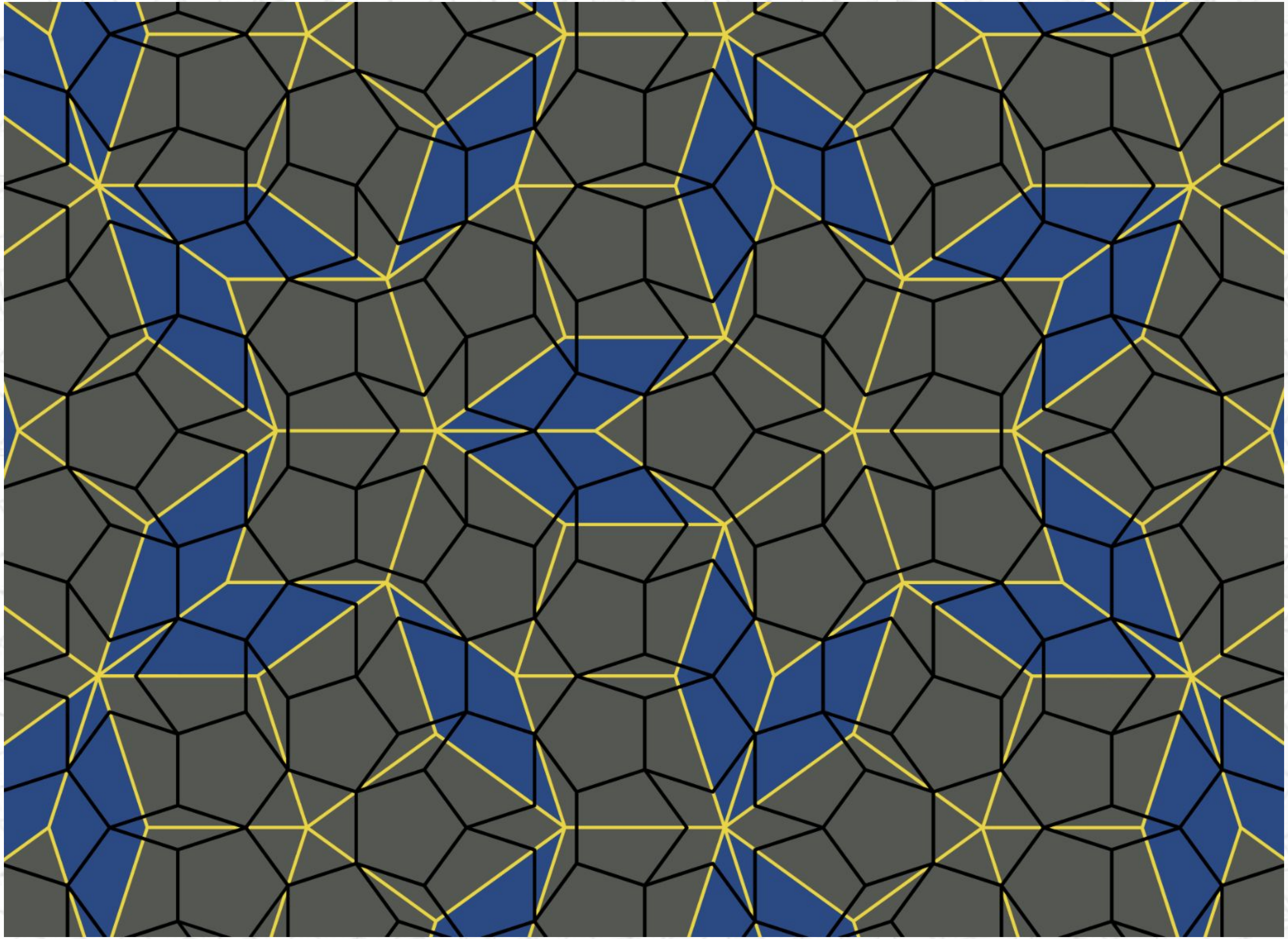
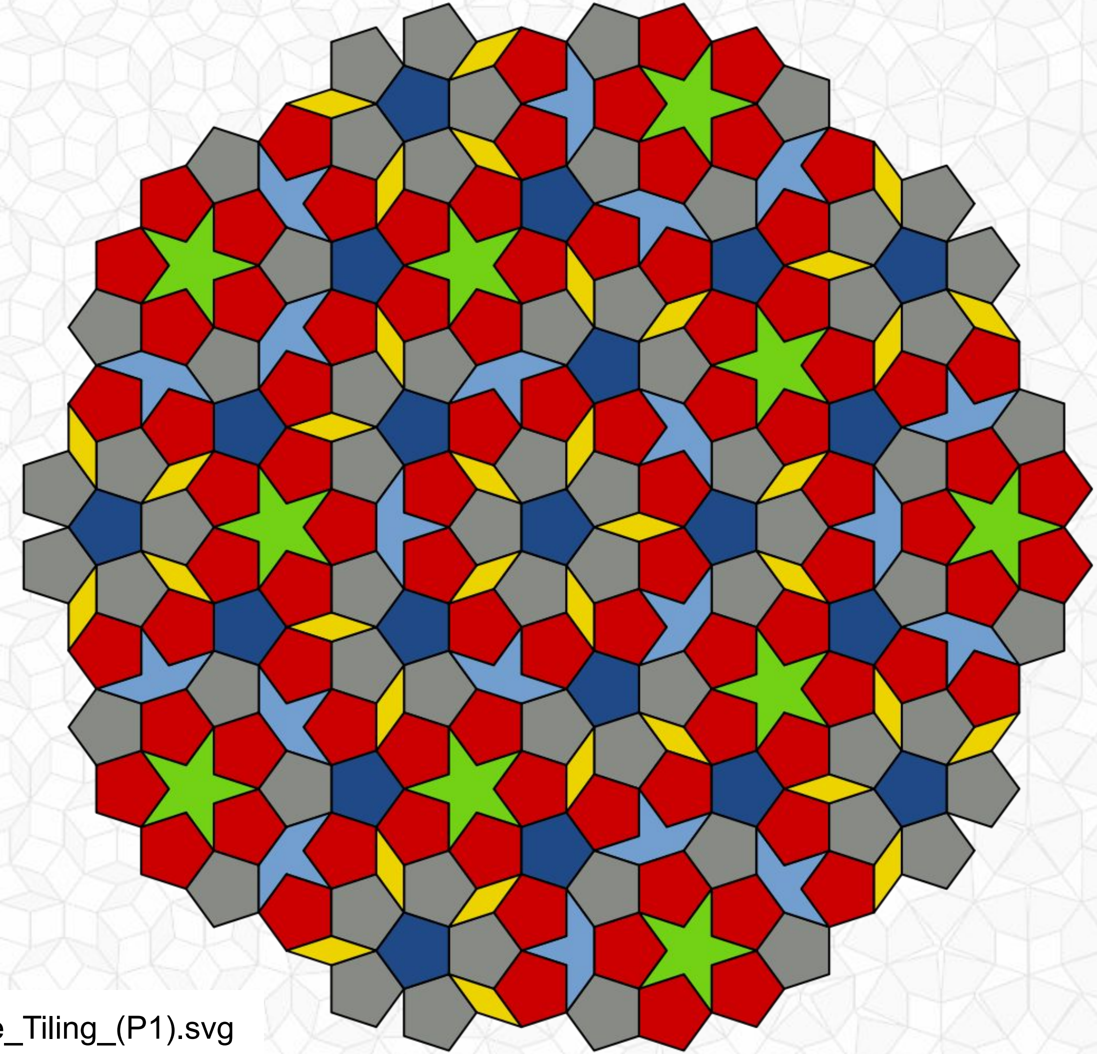


https://en.wikipedia.org/wiki/Penrose_tiling#/media/File:Penrose_Tiling_(P1_over_P3).svg

Original Penrose Tile set

Pentagons cannot tile a plane on their own!



https://commons.wikimedia.org/wiki/File:Penrose_Tiling_(P1).svg

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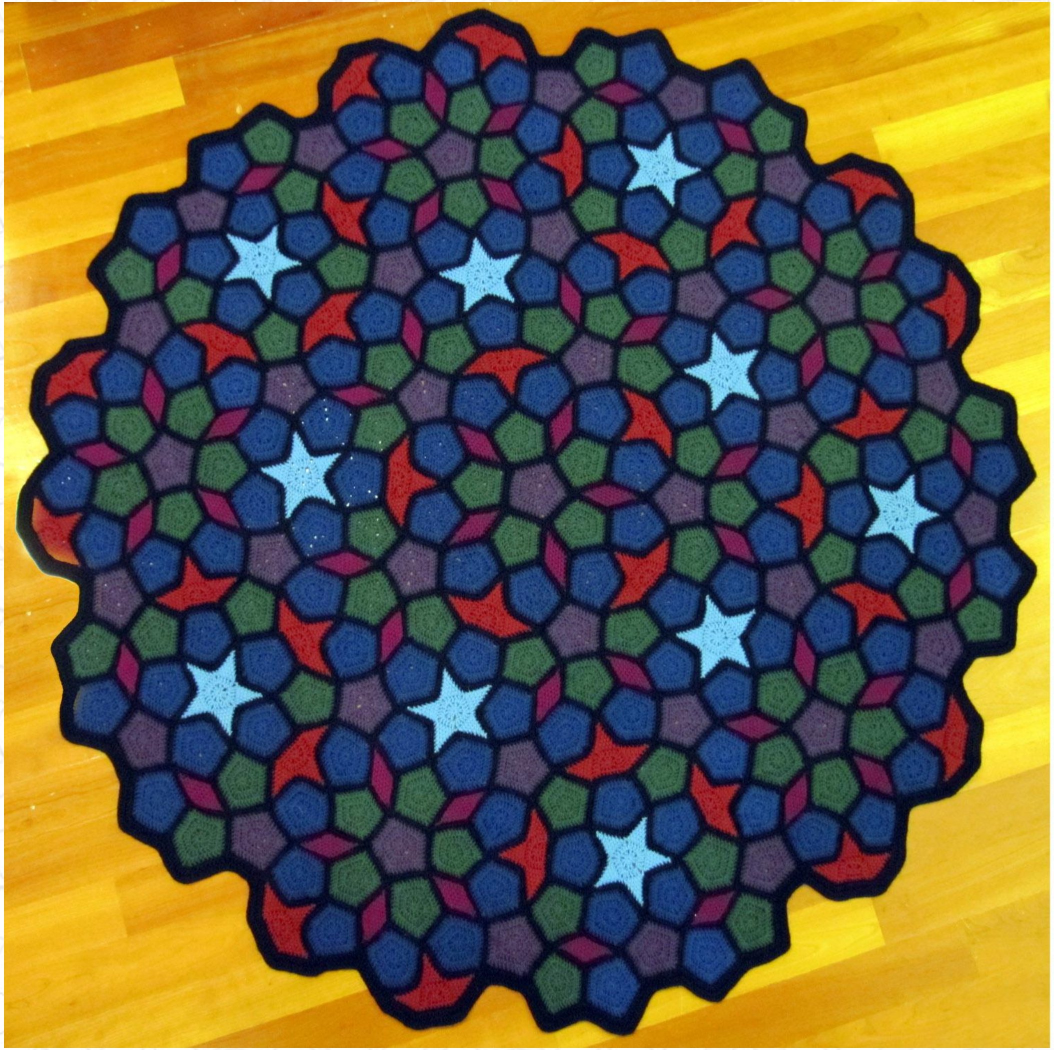


Pentagonal Penrose Crochet Project

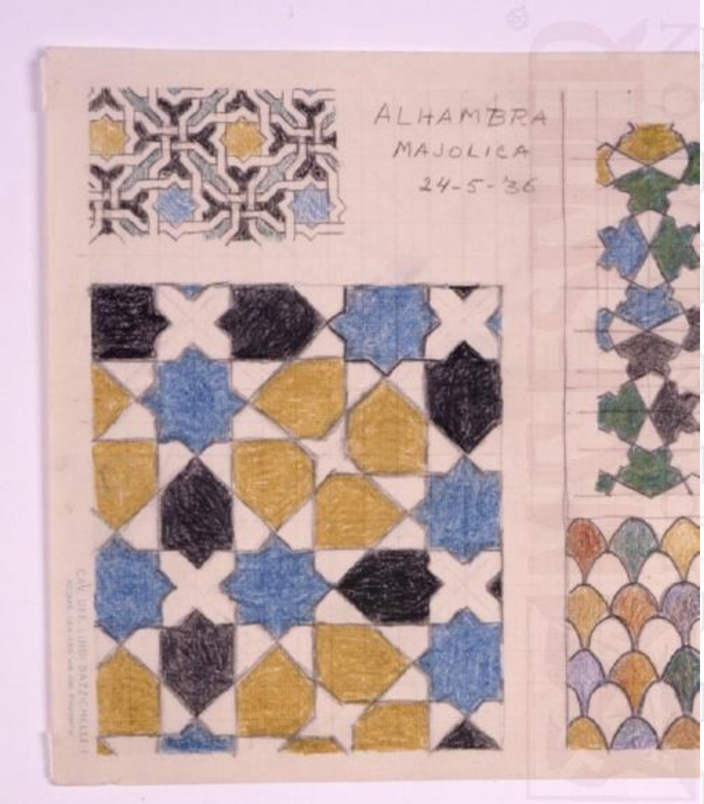
- Develop patterns for the 4 different shapes
- Crochet is not normally 5-fold symmetric!
- Crochet does not normally use 108° / 72° angles!

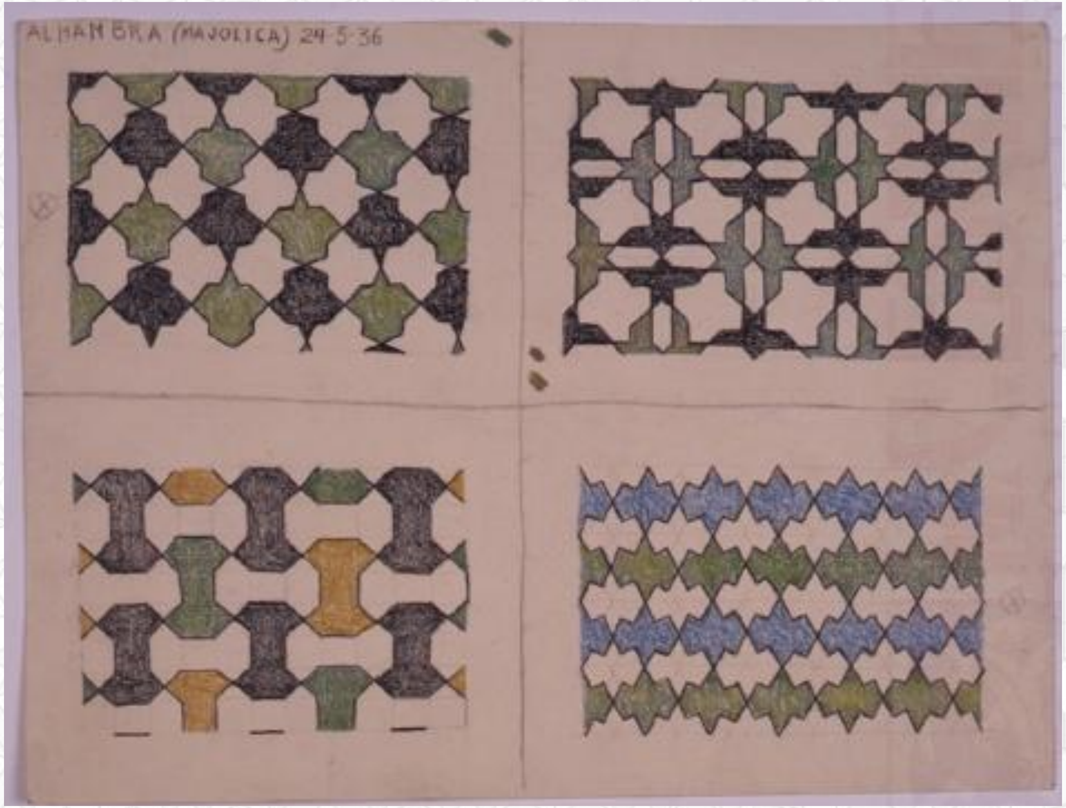
https://www.ravelry.com/patterns/library/pentagonal-penrose-throw-blanket

120	pentagons of color A
75	pentagons of color B
36	pentagons of color C
30	3 pointed stars / boats
50	diamonds / rhombuses
10	5 pointed stars

321 shapes
1284 ends to tuck

M.C.Escher https://mcescher.com/





M.C.Escher https://mcescher.com/

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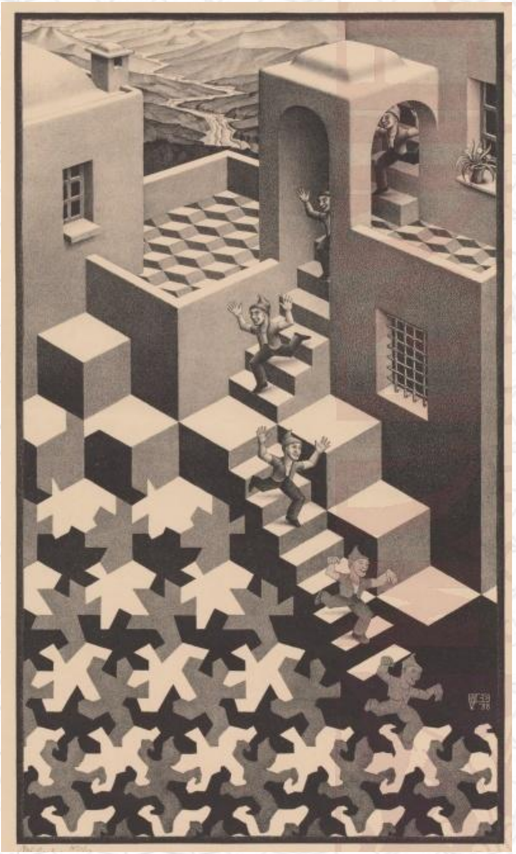

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