CSCI 4560/6560 Computational Geometry

https://www.cs.rpi.edu/~cutler/classes/computationalgeometry/S22/

Lecture 24: Curves & Sketching

Outline for Today

- Homework 6 or Homework 7 Questions?
- Last Time: Robot Motion Planning, Minkowski Sums, etc.
- Curve/Surface Continuity & Bezier Curves
- Polyline Simplification
- A Fun COVID Lockdown Project: Long Tiny Loops
- Clothoid or Cornu/Euler Spiral
- Hand-Drawn Sketch Smoothing
- Curve/Surface Reconstruction
- Next Time: ?

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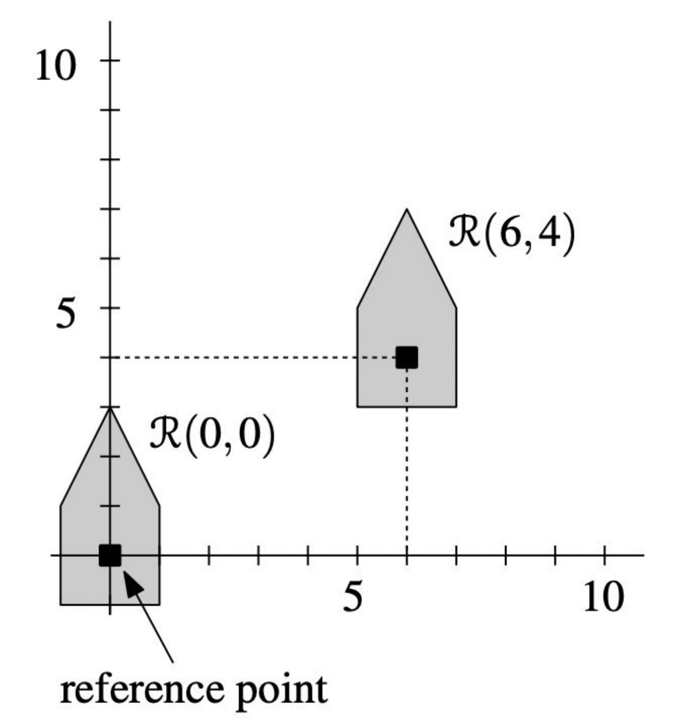
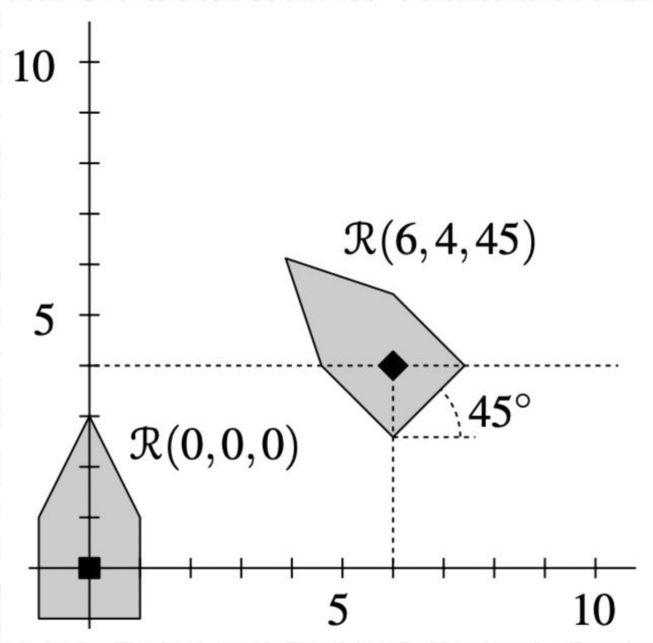
Robot Degree of Freedom (DOF)

2D w/ Translation only → 2 DOF



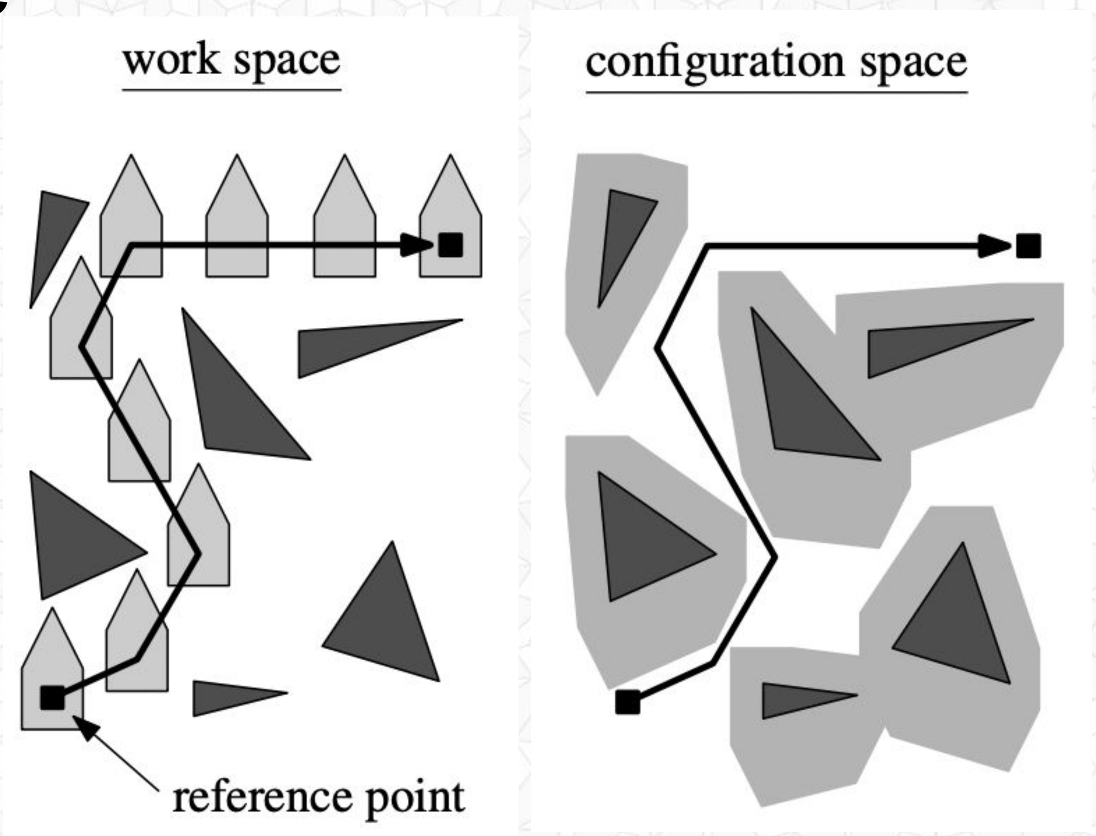
2D w/ Translation & Rotation → 3 DOF



Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 13

Configuration Space

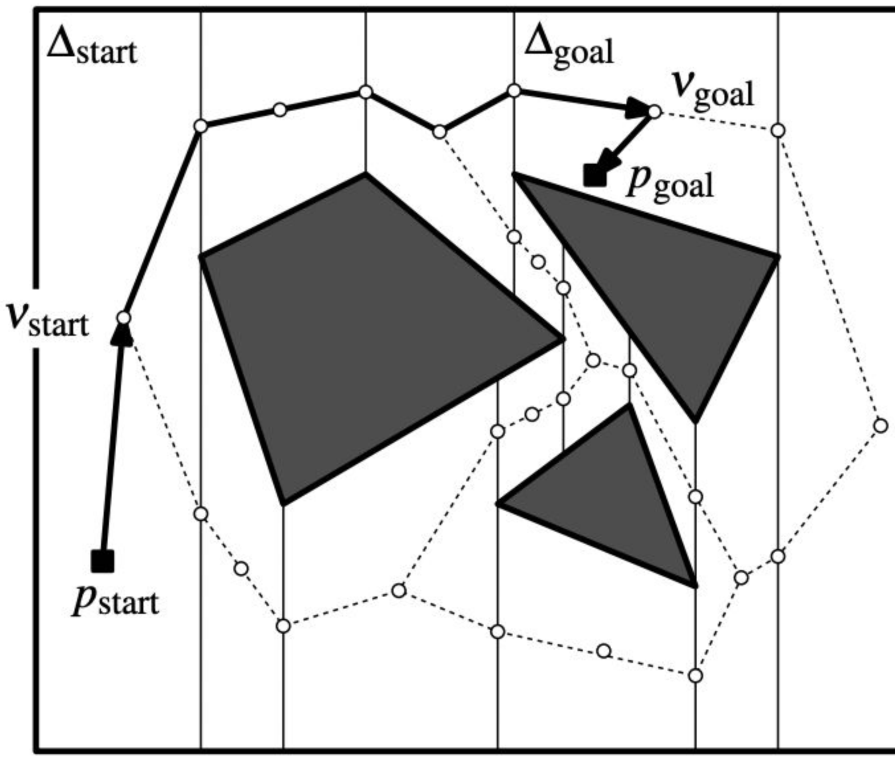
- The dimensions of configuration space match the DOF of the robot
- Usually configuration space is higher dimensional than the environment/workspace
- It is often useful to construct, visualize, and even solve the problem in “configuration space”



Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 13

Motion Planning Graph - Analysis

- Size of Trapezoid Map
  → O(n)
- Build Trapezoid Map
  → O(n log n)
- Locate start/end trapezoid
  → O(log n)
- Breadth first search
  → O(n)

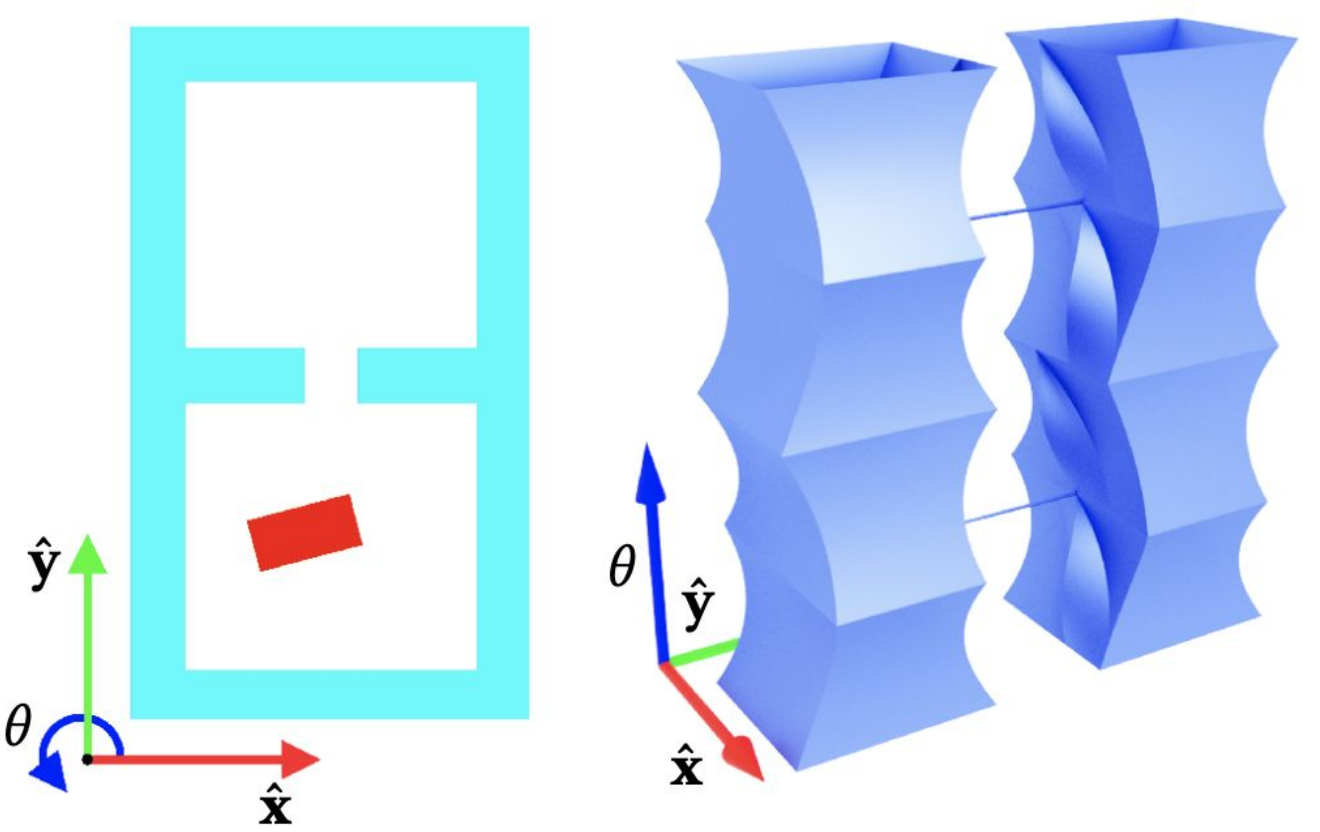


Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 13

Searching Configuration Space

"C-Space Tunnel Discovery for Puzzle Path Planning",
Zhang, Belfer, Kry, & Voucha,
SIGGRAPH 2020.

- Dimensionality becomes infeasible to construct & exhaustively search
- Randomized search is necessary



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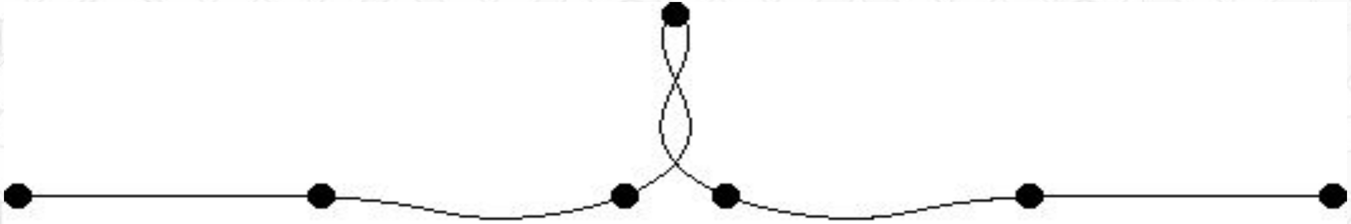
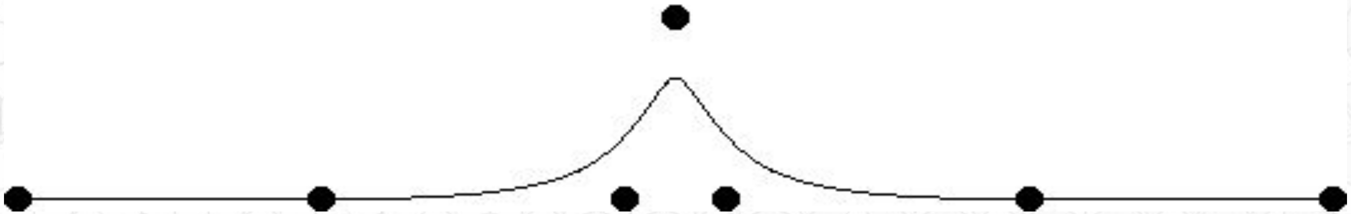
Interpolation vs. Approximation Curves

- Interpolation Curve – over constrained →
  lots of (undesirable?) oscillations

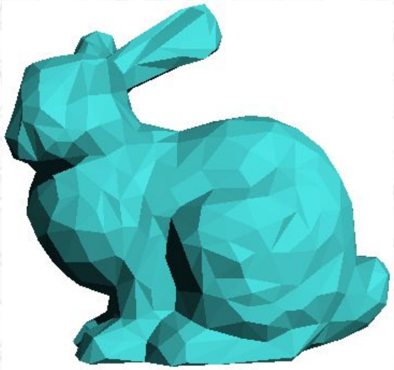


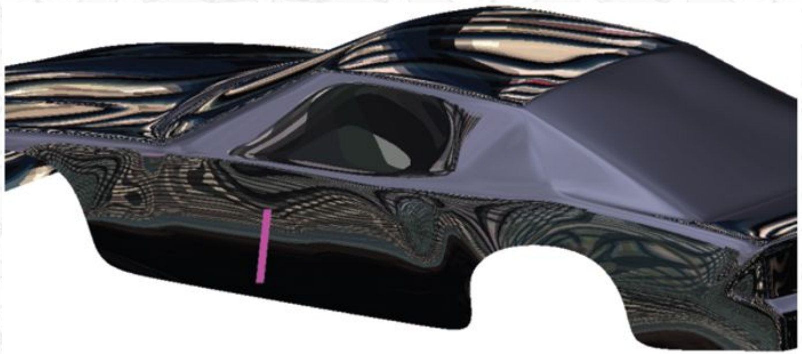
- Approximation Curve – more reasonable?



Continuity Definitions

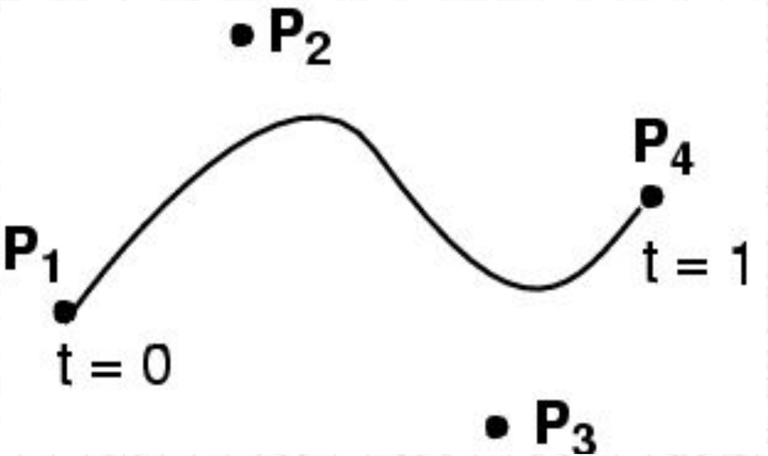
- C^0 continuous:
  - curve/surface has no breaks/gaps/holes
- G^1 continuous:
  - tangent at joint has same direction
- C^1 continuous:
  - curve/surface derivative is continuous
  - tangent at joint has same direction *and* magnitude
- C^n continuous:
  - curve/surface through n^{th} derivative is continuous
  - important for shading



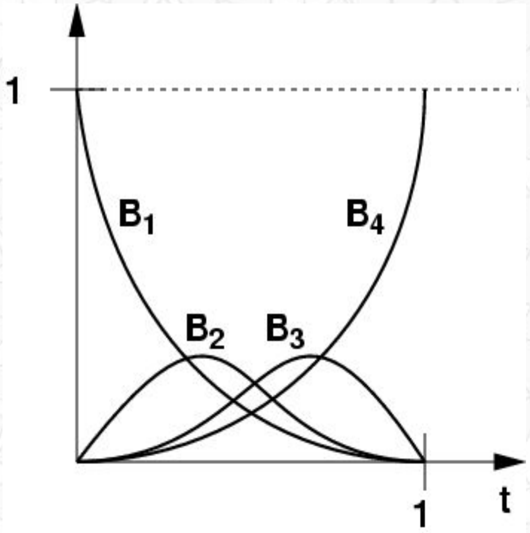


“Shape Optimization Using Reflection Lines”, Tosun et al., 2007

Cubic Bézier Curve



Asymmetric: Curve goes through some control points but misses others



weights sum to 1

Parametric equation:
Function of *t*
t* varies *0* → *1

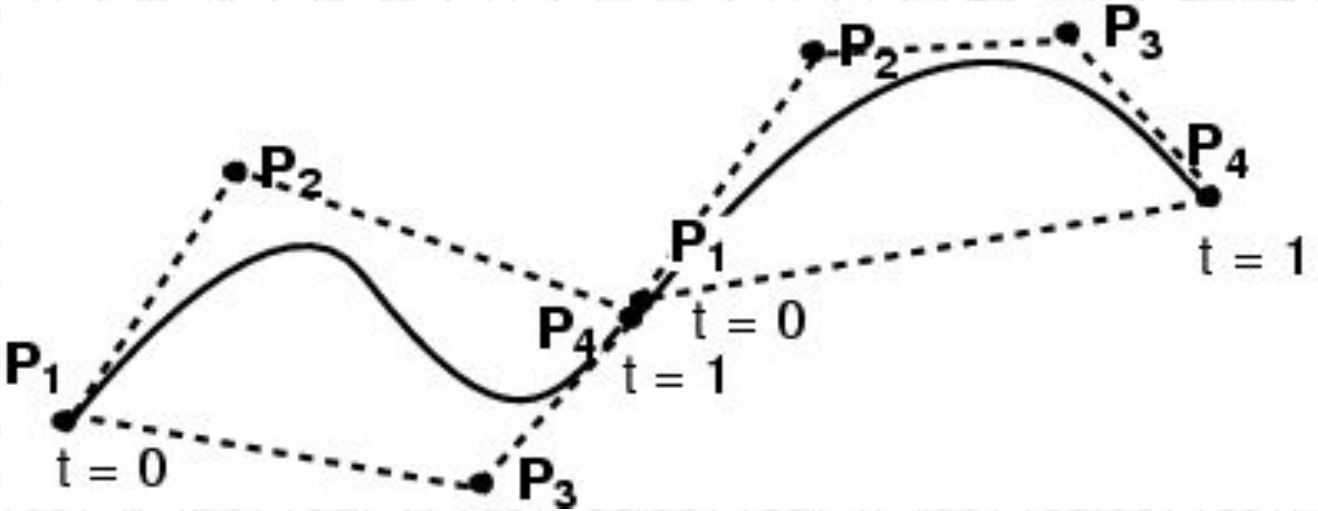
$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

control points

Connecting Cubic Bézier Curves

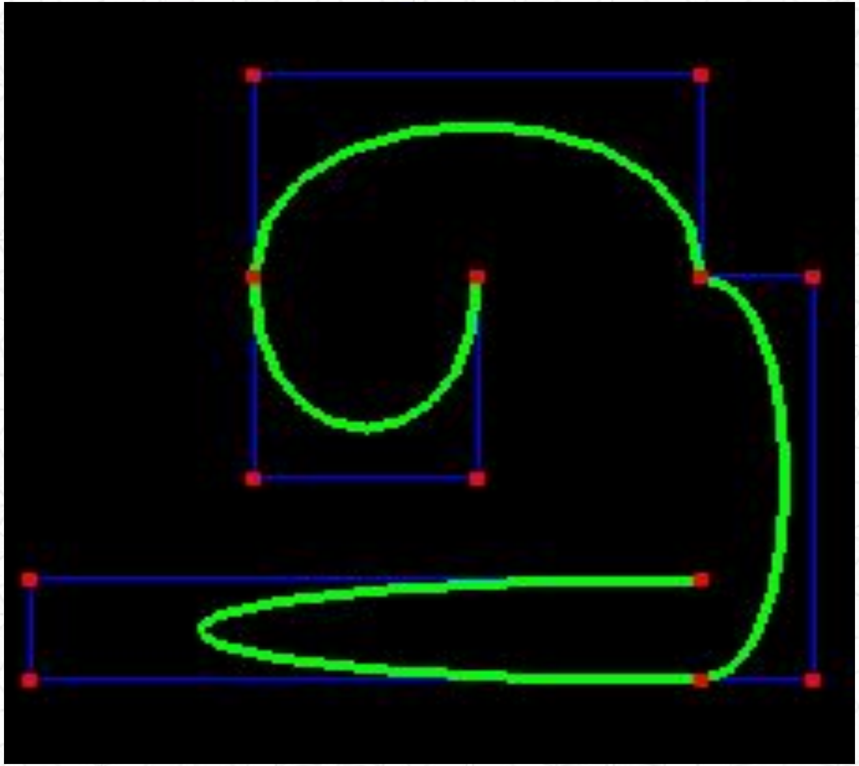
- How can we guarantee C^0 continuity?
- How can we guarantee G^1 continuity?
- How can we guarantee C^1 continuity?
- Can't guarantee higher C^2 or higher continuity

Asymmetric: Curve goes through some control points but misses others



Connecting Cubic Bézier Curves

- Where is this curve
  - C^0 continuous?
  - G^1 continuous?
  - C^1 continuous?
- What's the relationship between:
  - the # of control points, and
  - the # of
    cubic Bézier subcurves?



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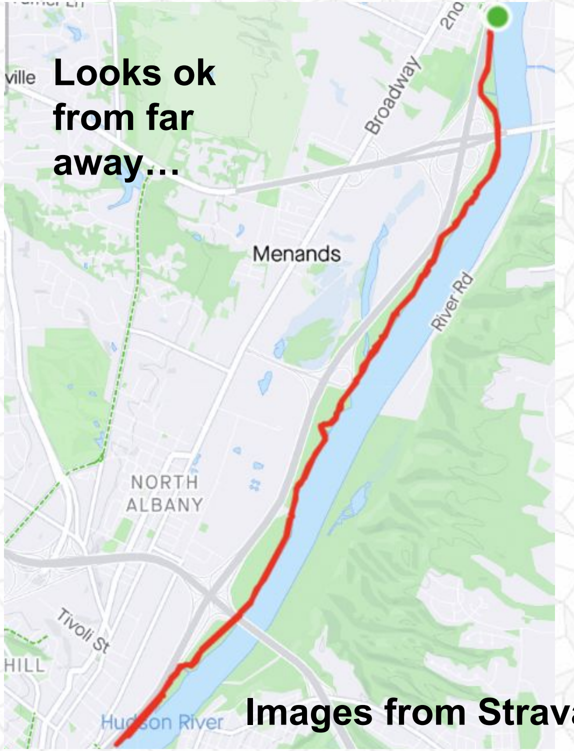
Noisy GPS Running Data

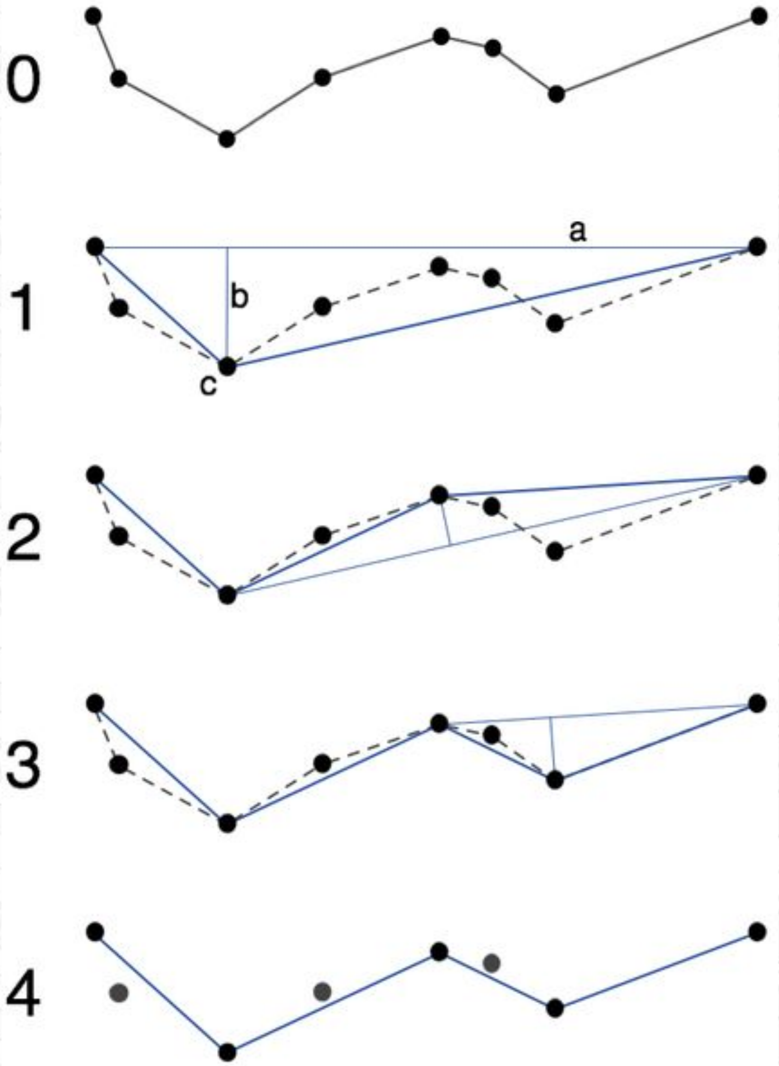
- Can overestimate distance by ~10% !!



Images from Strava

Polyline Simplification: Ramer–Douglas–Peucker

- Originally developed for cartography
- Reduce number of points necessary to represent a polyline
- Identify most important points
- Discards points that are < ε from the simplified shape



https://commons.wikimedia.org/wiki/File:Douglas_Peucker.png

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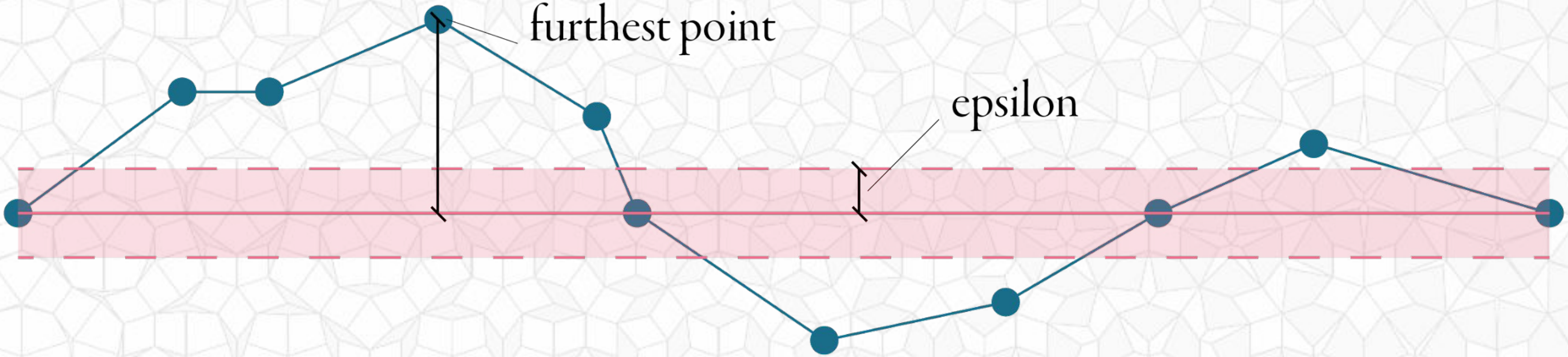
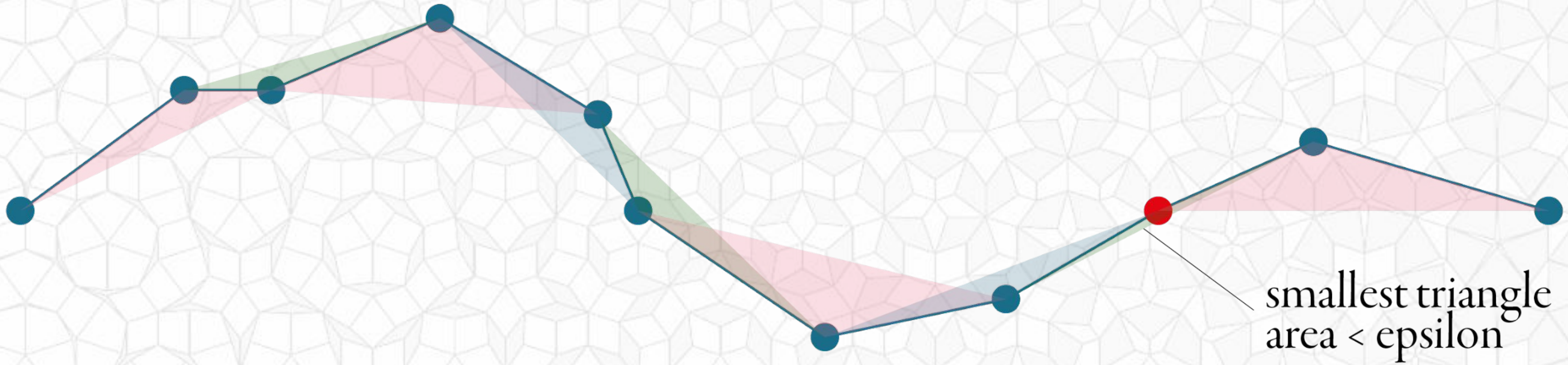


https://martinfleischmann.net/line-simplification-algorithms/

Polyline Simplification: Visvalingam-Whyatt

- Similar algorithm to Ramer-Douglas-Peucker (sometimes but not always the same result)
- Remove a point if the triangle formed by that point & two immediate neighbors has area < ε



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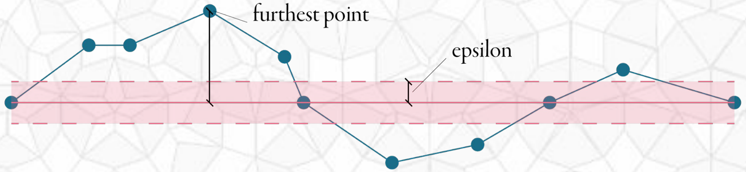
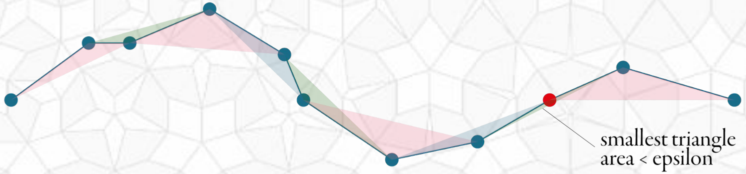
Polyline Simplification Analysis

- Ramer–Douglas–Peucker



- Visvalingam-Whyatt



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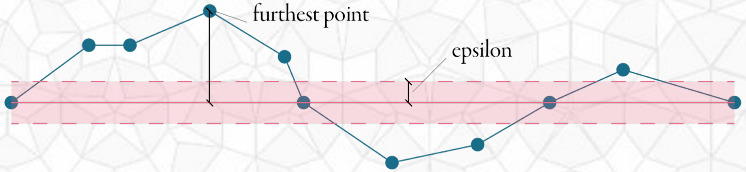
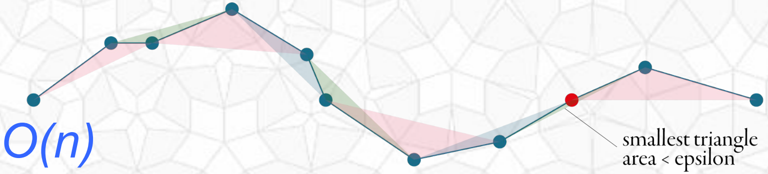
Polyline Simplification Analysis



- Ramer–Douglas–Peucker
  - Connect endpoints, find split point furthest from current segment: → $O(n)$
  - Recurse on each side of split
  - Average case (even split): → $O(n \log n)$
  - Worst case (uneven split): → $O(n^2)$
- Visvalingam-Whyatt
  - Compute all high resolution triangles: → $O(n)$
    - Store in priority queue
  - Remove a point requires 2 new triangle computes
    - Priority queue update: → $O(\log n)$
  - Overall: → $O(n \log n)$



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Long Tiny Loops by Dan Aminzade

- Inspired by 2020 COVID lockdown
- How far can you run without repeating roads or intersections while staying close to home?

- GPS tracked run or bike
- Closed loop
- Streets or bike paths only
- Non-intersecting
- No repeated streets (even in opposite direction)
- Score = distance / max diameter

https://longtinyloop.com/faq

Long Tiny Loops by Dan Aminzade

- Extract GPS data from Strava API
- Ramer-Douglas-Peucker:
  Simplify input (remove false positive intersections due to noise)
- Verify closed loop
- Check for segment intersections
- Compute convex hull
- Rotating calipers maximum diameter

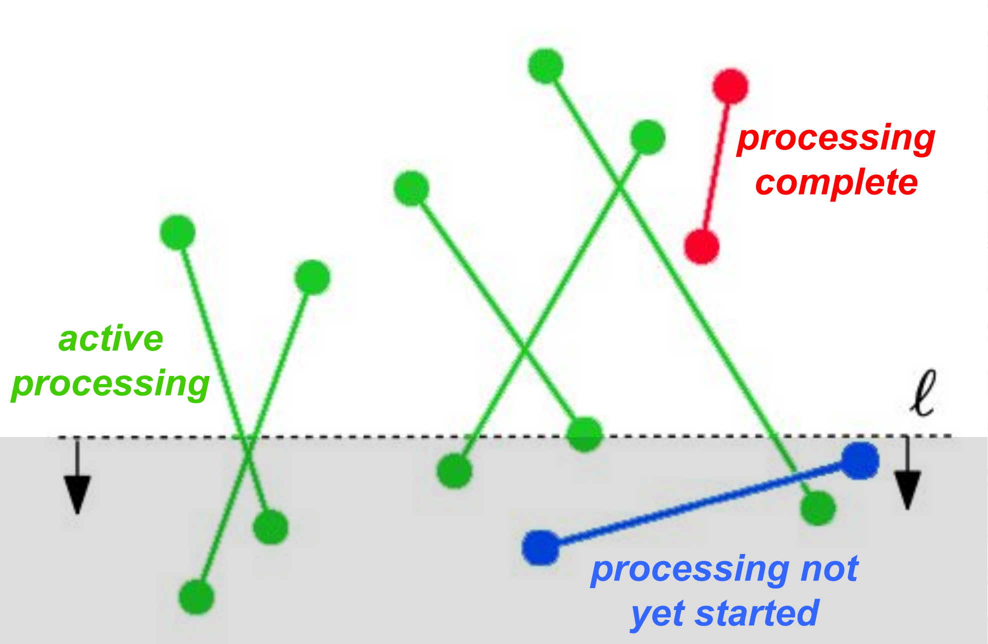
→ Compute final score
= distance / max diameter

https://longtinyloop.com/faq

Intersection Detection: Line-Sweep Algorithm

- (Review from Lecture 2)
- Sort all endpoints vertically
- Maintain horizontally sorted list of segments intersecting with current sweep line
- Check for intersections with adjacent segments only

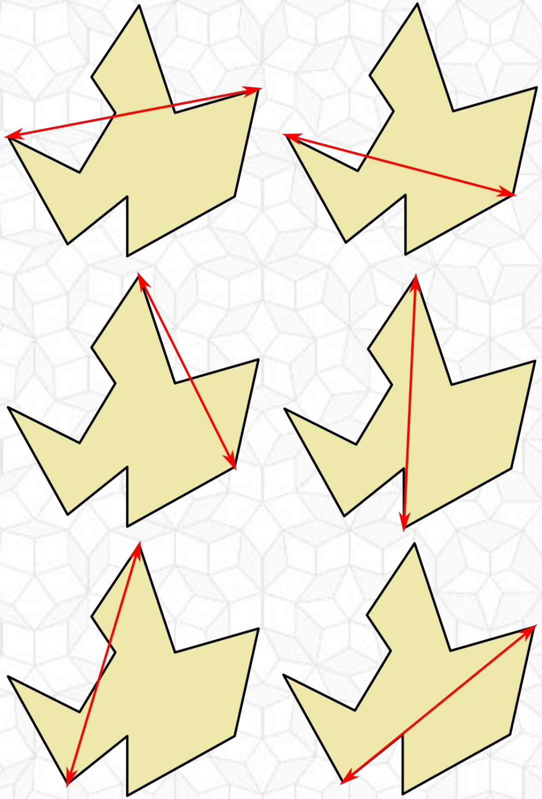
- Overall: → $O((k+n) \log n)$
  - n segments,
  - k intersections

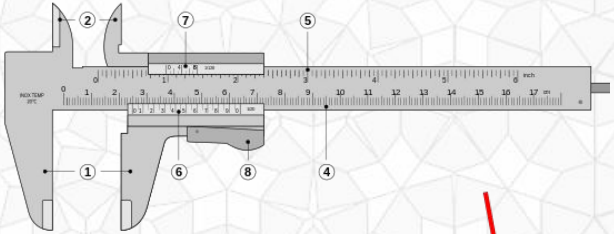


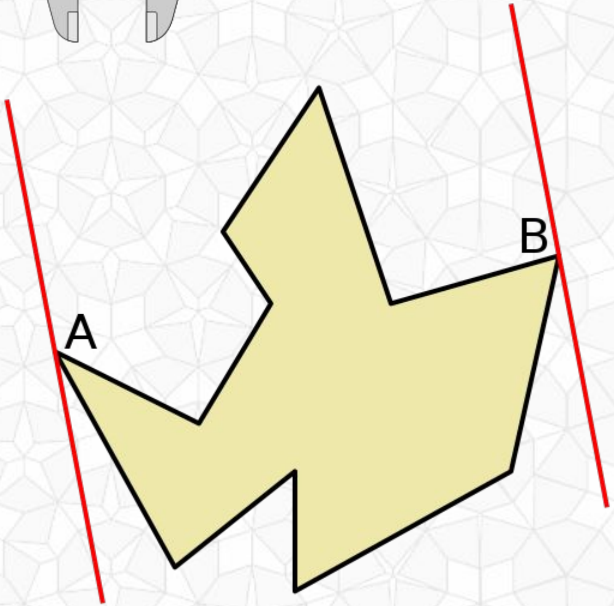
Computational Geometry Algorithms and Applications,
de Berg, Cheong, van Kreveld and Overmars, Chapter 2

Maximum Diameter: Rotating Calipers

- Efficient algorithm to consider all pairs of *antipodal points*
  - also useful for other computations
- Return the maximum distance
- Analysis:



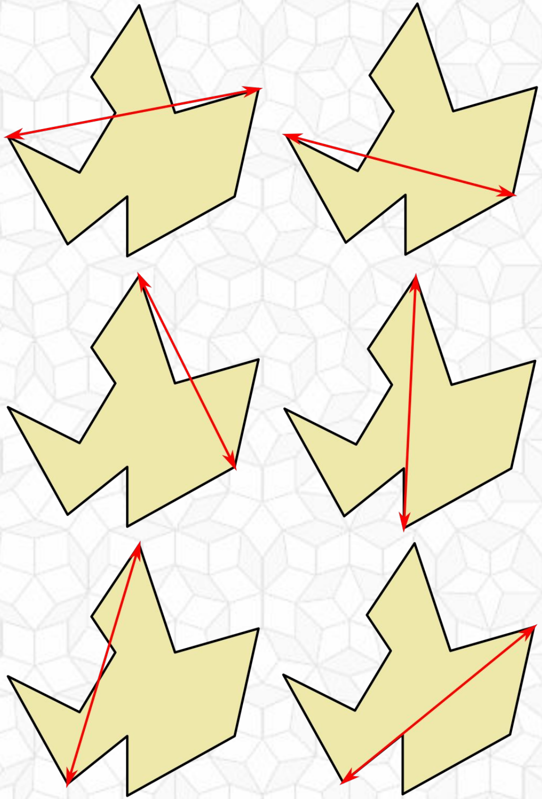


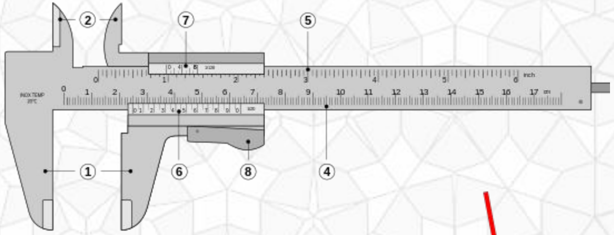


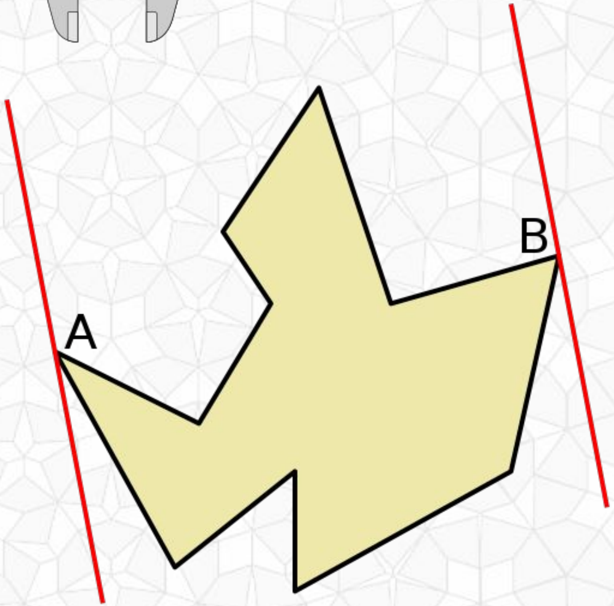
https://en.wikipedia.org/wiki/Rotating_calipers

Maximum Diameter: Rotating Calipers

- Efficient algorithm to consider all pairs of *antipodal points*
  - also useful for other computations
- Return the maximum distance
- Analysis:
  → O(n)







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Long Tiny Loops

Current high score:
Nathan Rooy

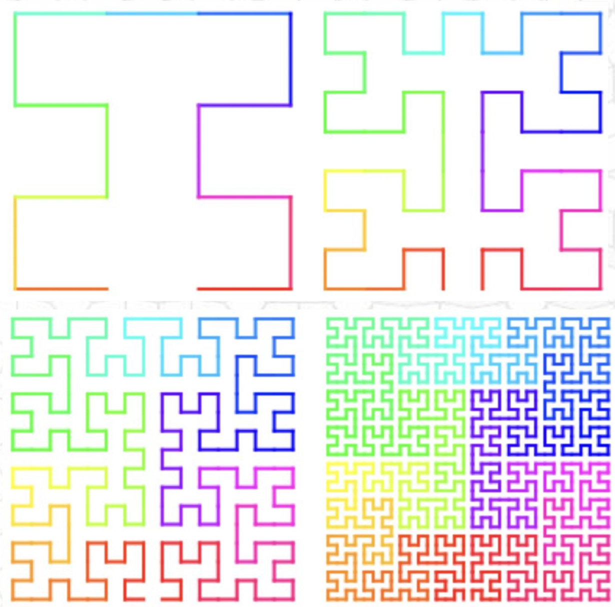
Distance:
190 km (118 miles)
Diameter:
4.7km (2.92 miles)

Score: 40.51

by Dan Aminzade
https://longtinyloop.com/

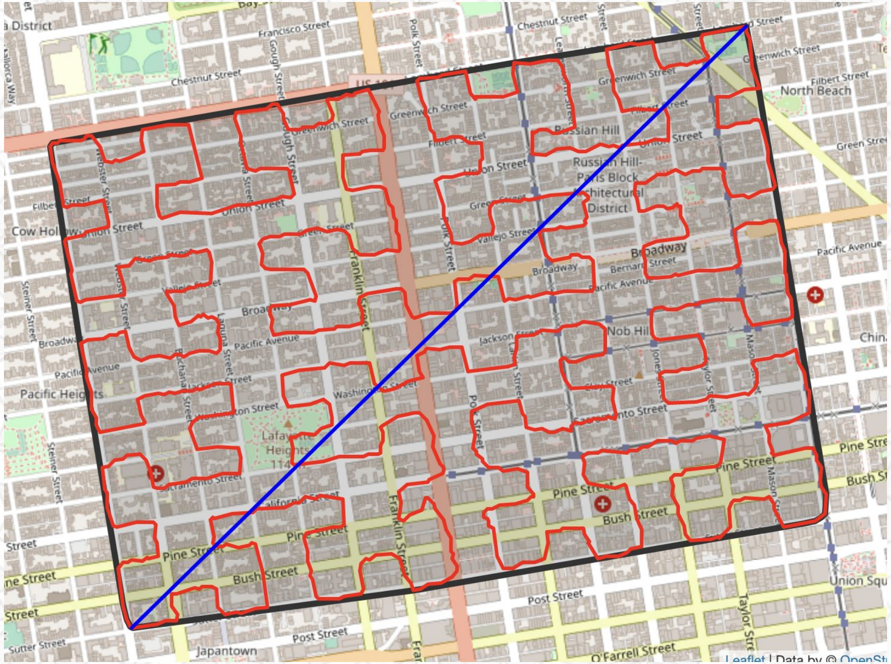
Space Filling Curve - A Fractal

- Peano curve (Guiseppe Peano, 1890)
- Hilbert Curve (David Hilbert, 1891)
- Moore Curve (E.H. Moore 1900)



https://en.wikipedia.org/wiki/Moore_curve

path by Octavian Voicu
https://longtinyloop.com/



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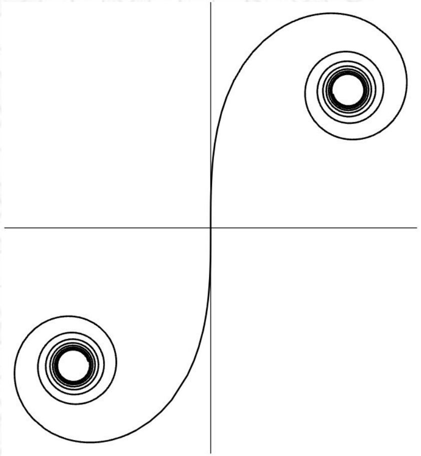
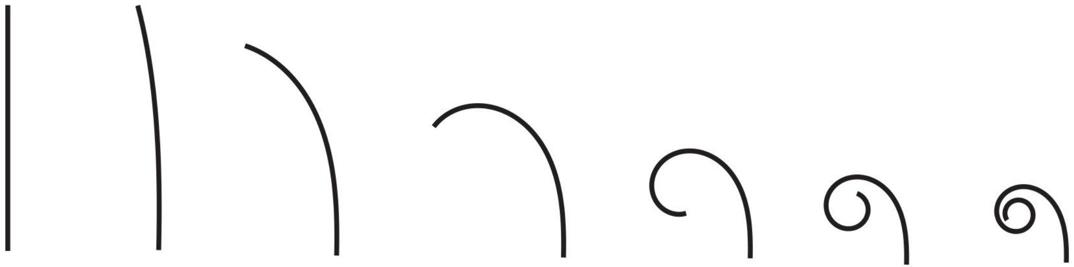
Clothoid or Cornu/Euler Spiral

"A new, simple and accurate transition curve type, for use in road and railway alignment design"
European Transport Research Review,
Eliou & Kaliabetsos, 2014

- For railroads, roads, rollercoasters, etc.
  - Avoid instantaneous curvature changes at high speed
- Linear correlation between curvature/radius & length





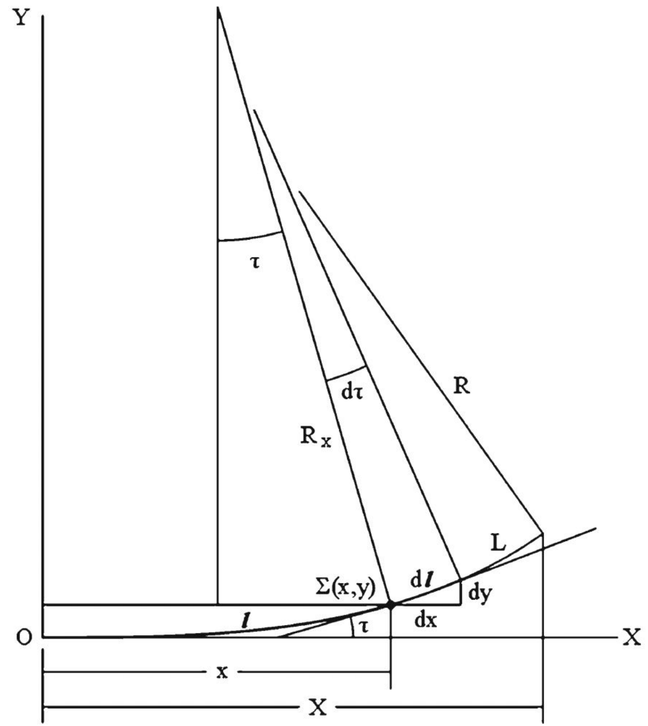


Fig. 2 Transition curve graph in detail

French Curve / Burmester Set

- Metal, wood, or plastic template
- For manual drafting/design
- Created from different segments of Clothoid or Cornu/Euler Spiral
- Invented by Ludwig Burmester

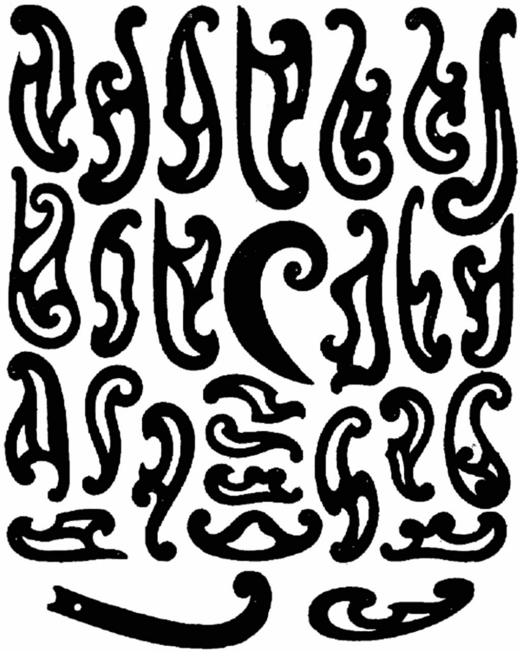


Fig. 6. Kurvenlineale von Gebrüder Wichmann, Berlin.

https://en.wikipedia.org/wiki/French_curve

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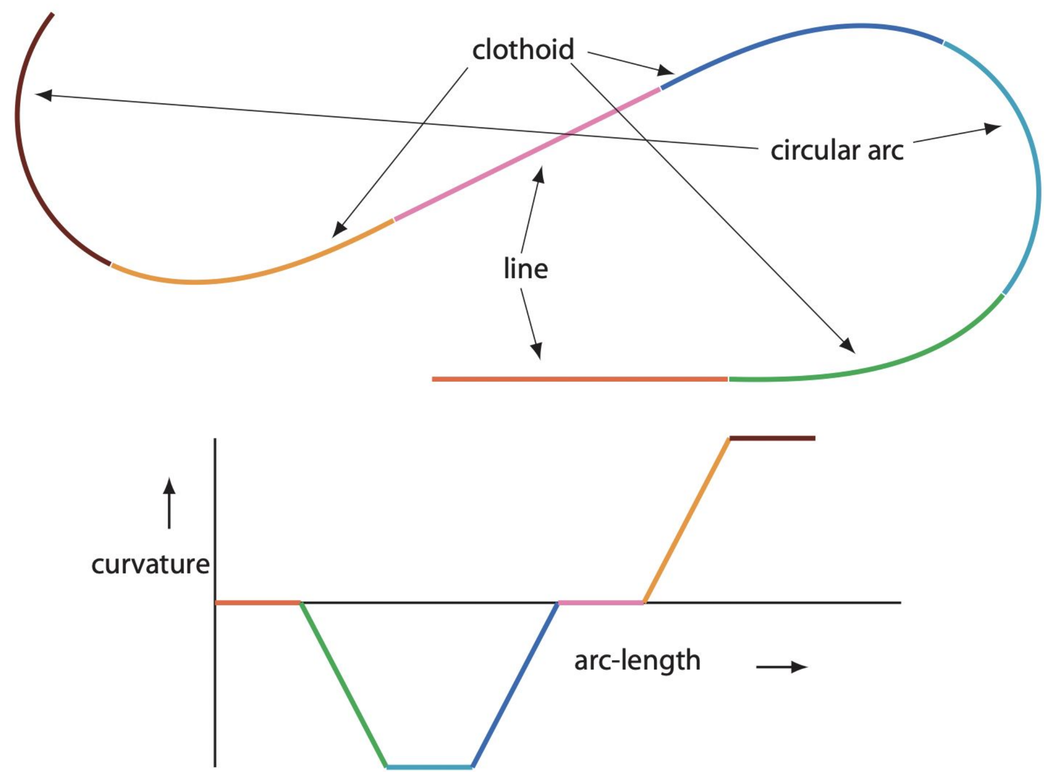
Piecewise Clothoid + Circular Arc + Line

- Aesthetically pleasing
- Fairness
- Can ensure G2 or G3 continuity
- Also model sharp discontinuities as appropriate



“Sketching Piecewise Clothoid Curves”
McCrae & Singh, 2008

Fairing (definition)

- Reduce undesirable, unaesthetic, unnecessary bumps and wiggles in a curve/surface

- Also: An additional part or structure added to an aircraft, tractor-trailer, etc. to smooth the outline and thus reduce drag

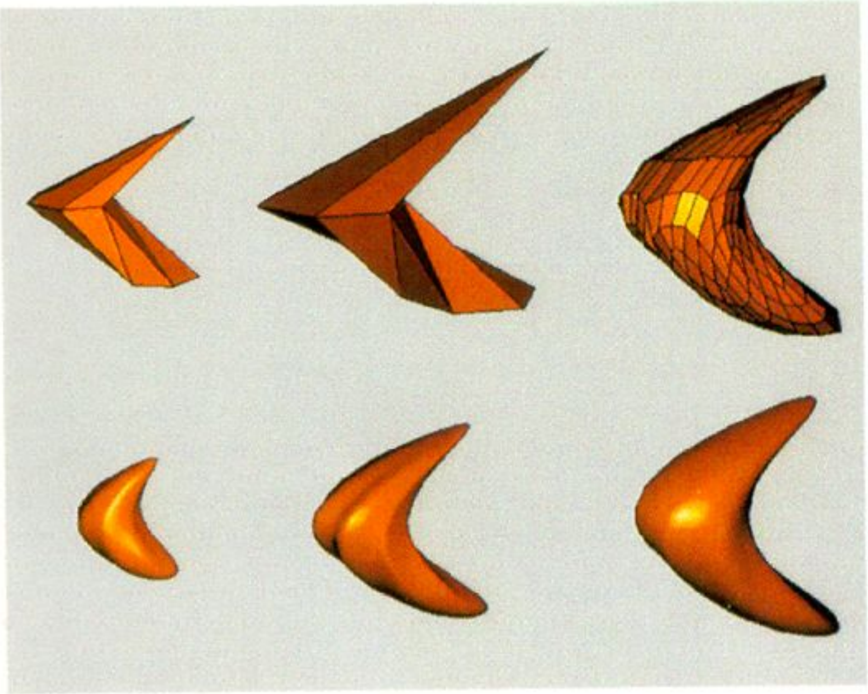


Figure 5: Top row: Original mesh, Interpolating mesh, Faired interpolating mesh. Bottom row: Corresponding Catmull-Clark surfaces. Interpolation introduces wiggles which are removed by fairing.

"Efficient, fair interpolation using Catmull-Clark surfaces", Halstead, Kass & DeRose, SIGGRAPH 1993

Advantages of Clothoids

- Interactive digital sketching data has noise and high frequency wiggles
- Clothoid fitting tends be smoothest and to minimize the *variation of curvature*

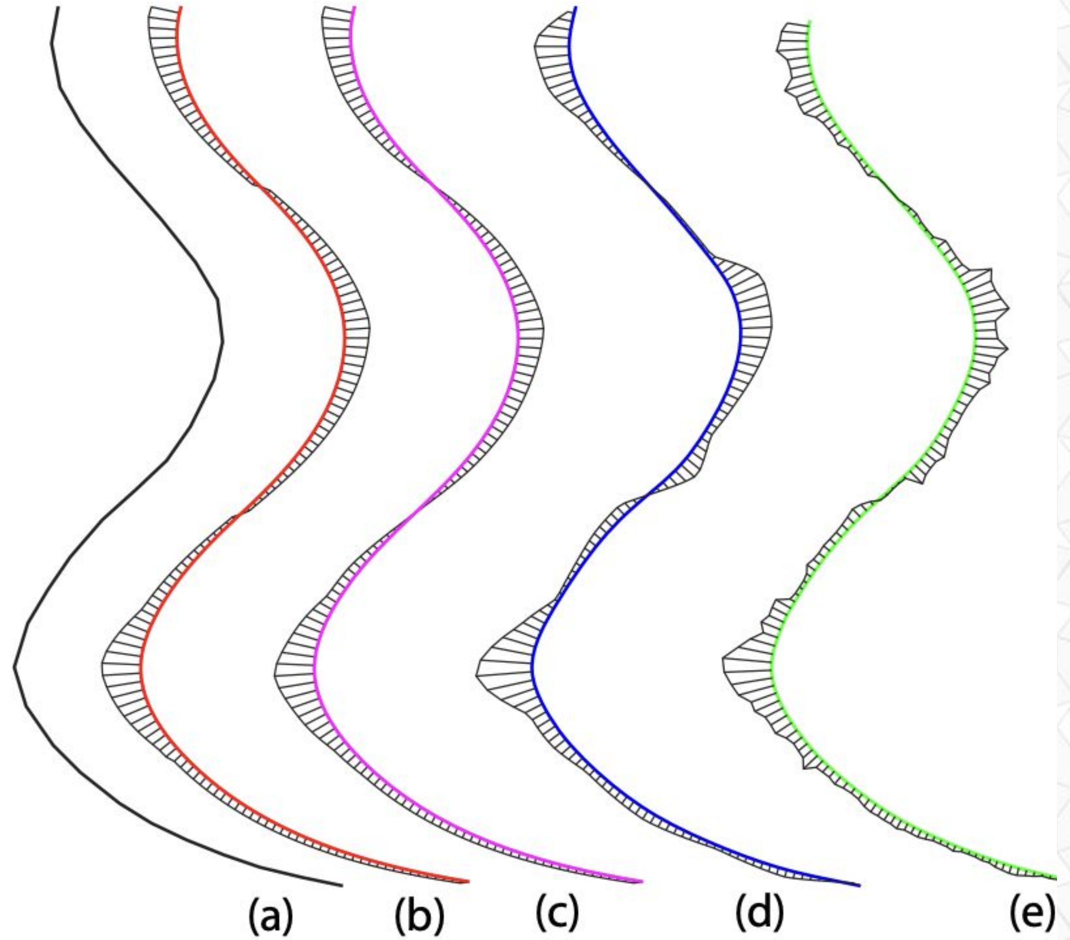
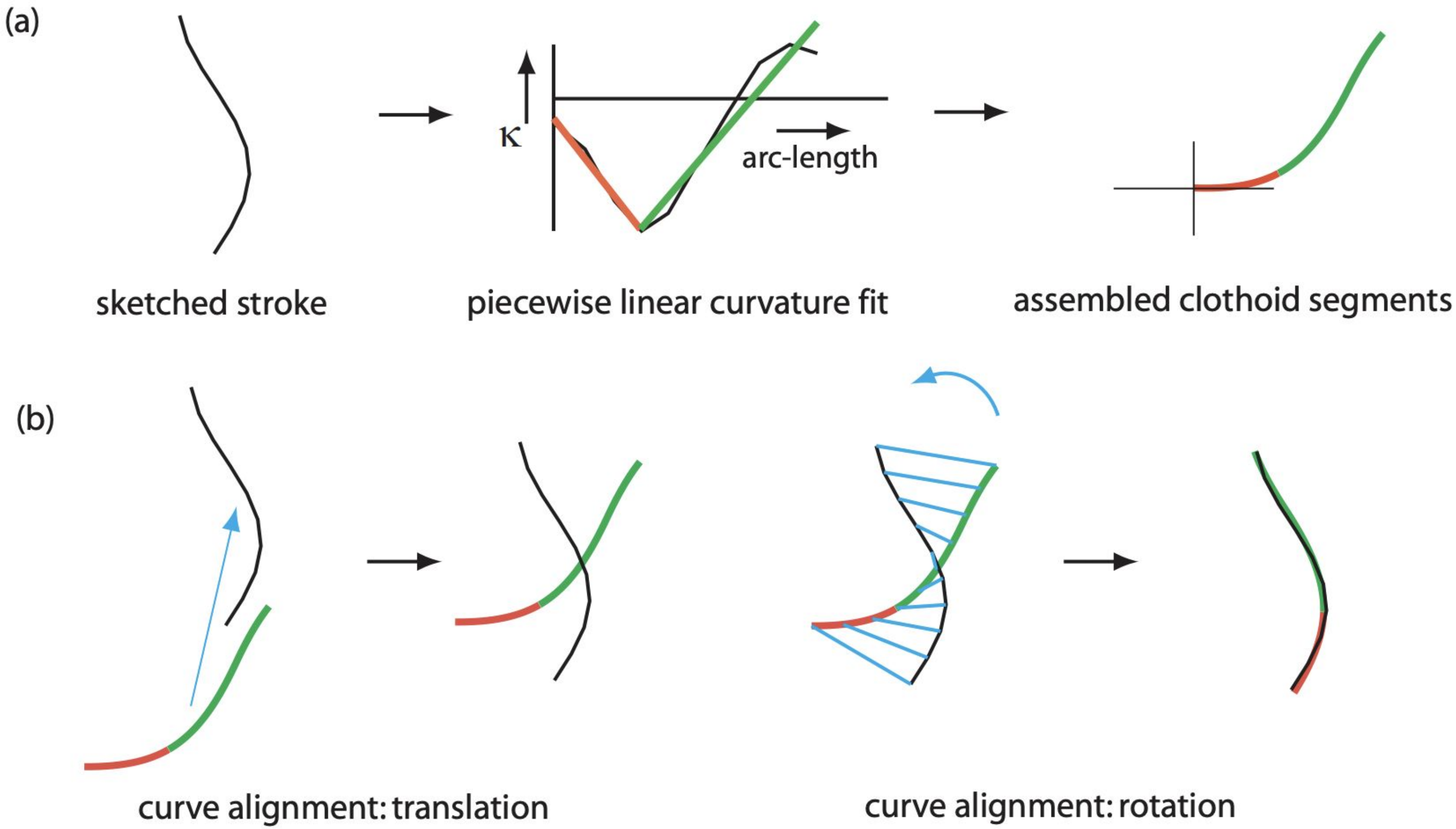
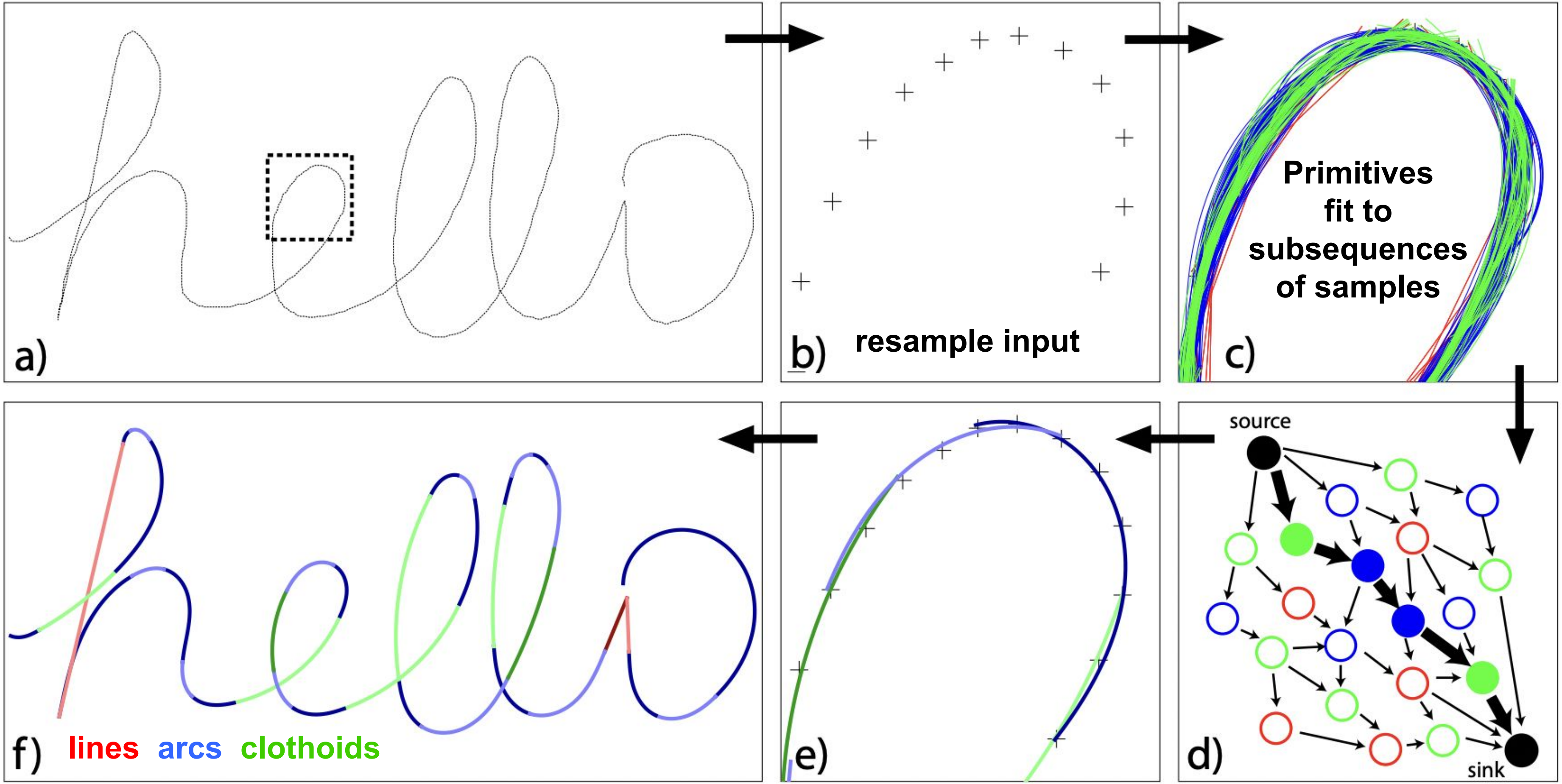


Figure 3: *Stroke fairing: (a) A sketched stroke. (b) Clothoid fitting the stroke (a). (c) Cubic spline fitting the clothoid curves in (b). (d) Cubic spline fitting the stroke (a). (e) Laplacian smoothing (4 iterations at 10%) the stroke (a). Curvatures are plotted uncolored along the length of processed strokes (b-d) to evaluate smoothness.*

“Sketching Piecewise Clothoid Curves”, McCrae & Singh, 2008



“Sketching Piecewise Clothoid Curves”, McCrae & Singh, 2008



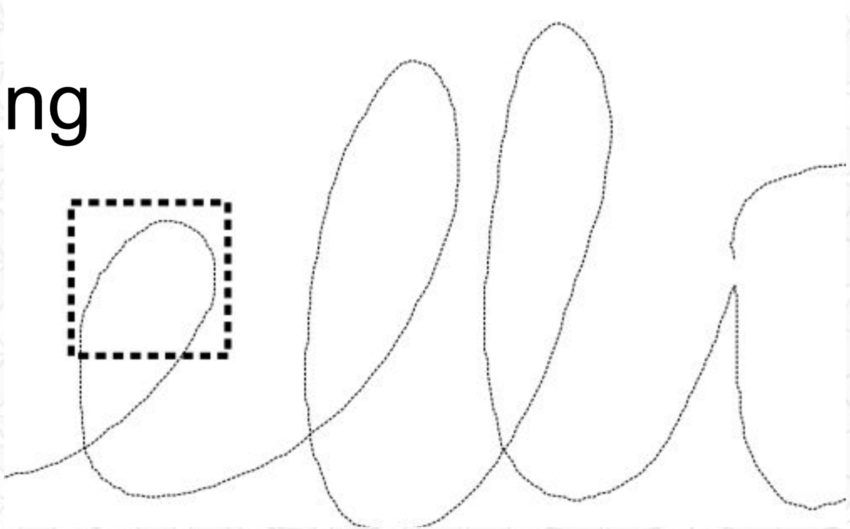
“Sketching Clothoid Splines Using Shortest Paths”, Baran, Lehtinen, & Popović, 2010

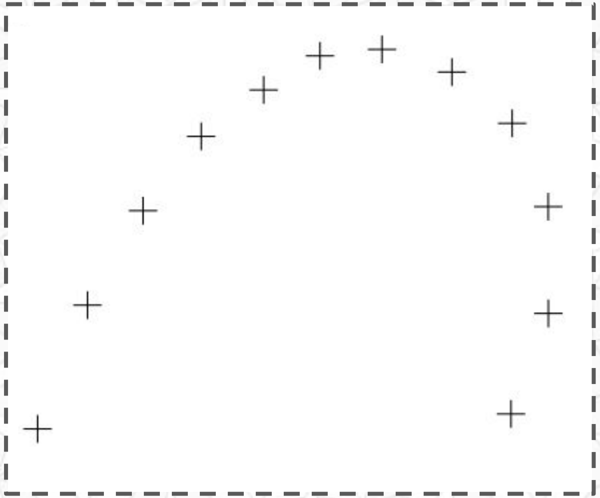
non-linear program to enforce continuity

find shortest path through all subsequences

Curvature-Based Resampling

- Raw sketch input usually has noise and overall too many samples
- Reduce total number
- Regularize the spacing of samples
- Have more samples where the curvature is higher





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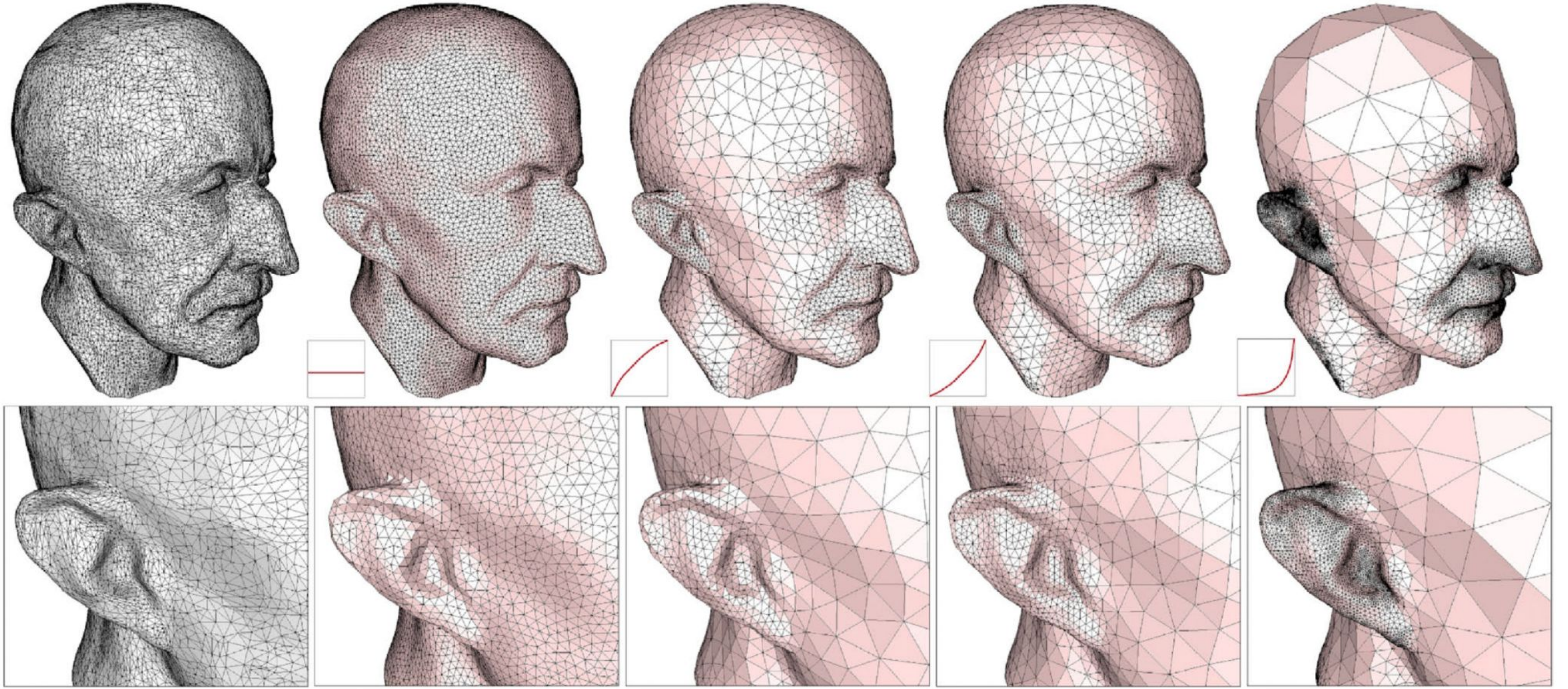

Figure 11: *Remeshing of the MaxPlanck model with various distribution of the sampling with respect to the curvature. The original model (left) is remeshed uniformly and with an increasing importance placed on highly curved areas (left to right) as the magnified area shows.*

"Interactive Geometry Remeshing"
Alliez, Meyer, & Desbrun, SIGGRAPH 2002

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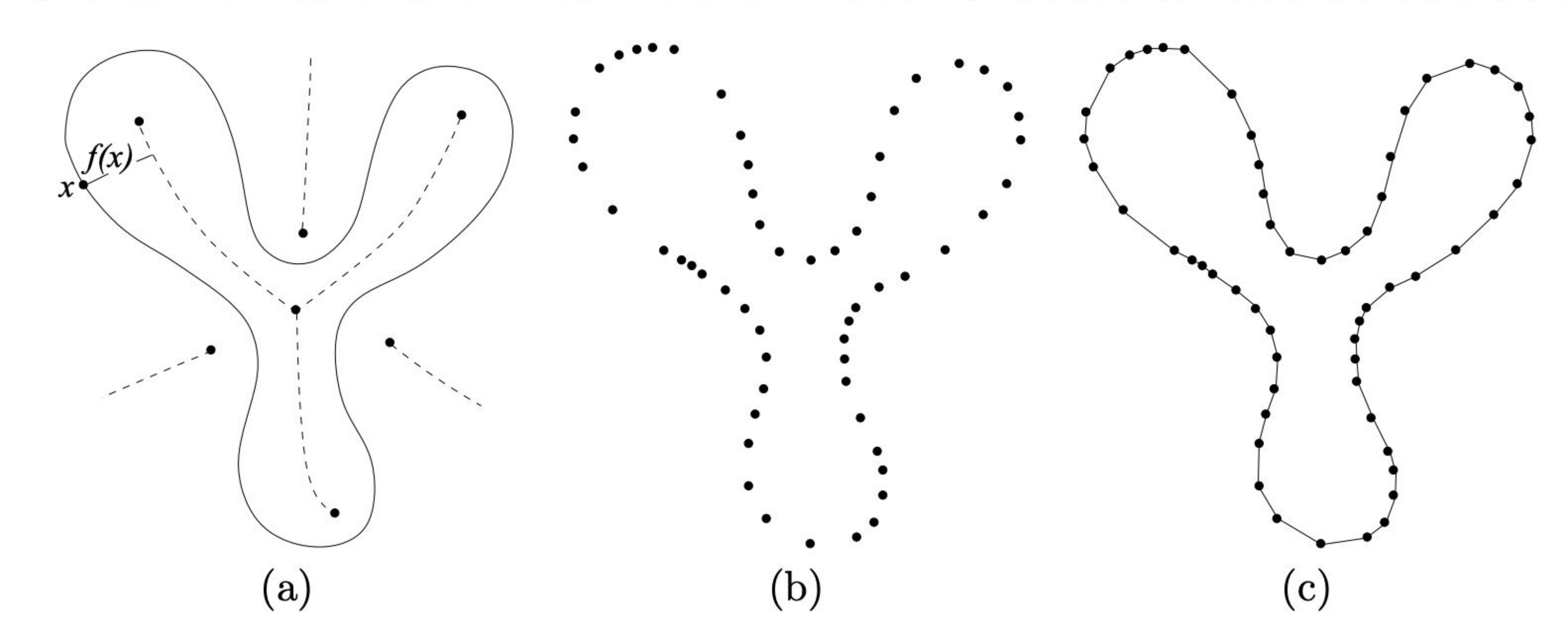
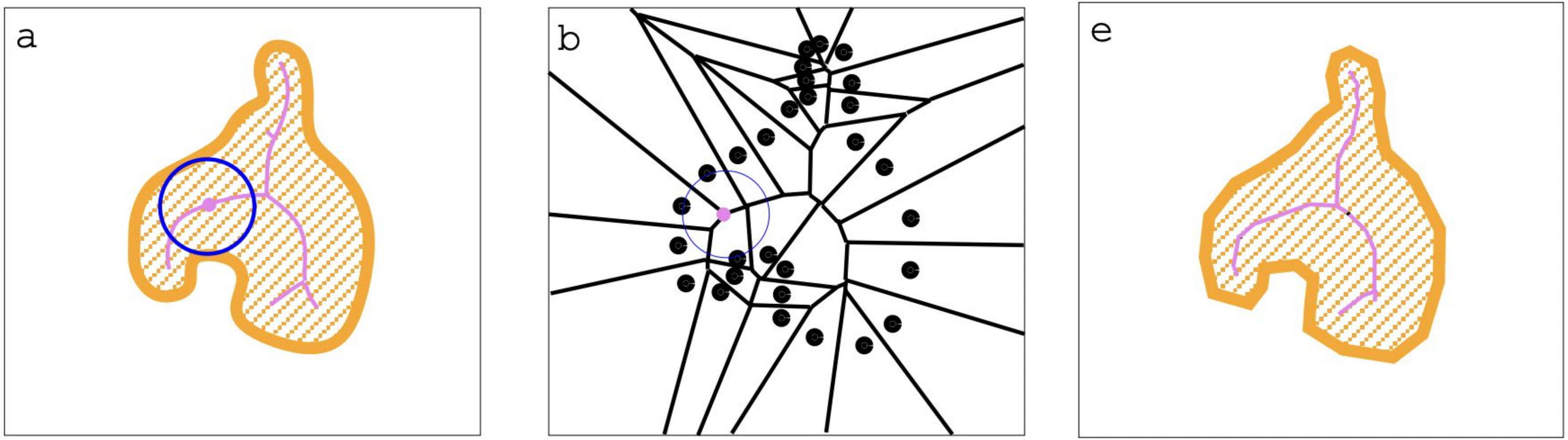
Curve Reconstruction

- Guaranteed reconstruction if sufficient sampling requirements are met.



"Chapter 35/36: Curve and Surface Reconstruction", Dey, Handbook of Discrete and Computational Geometry, 2018



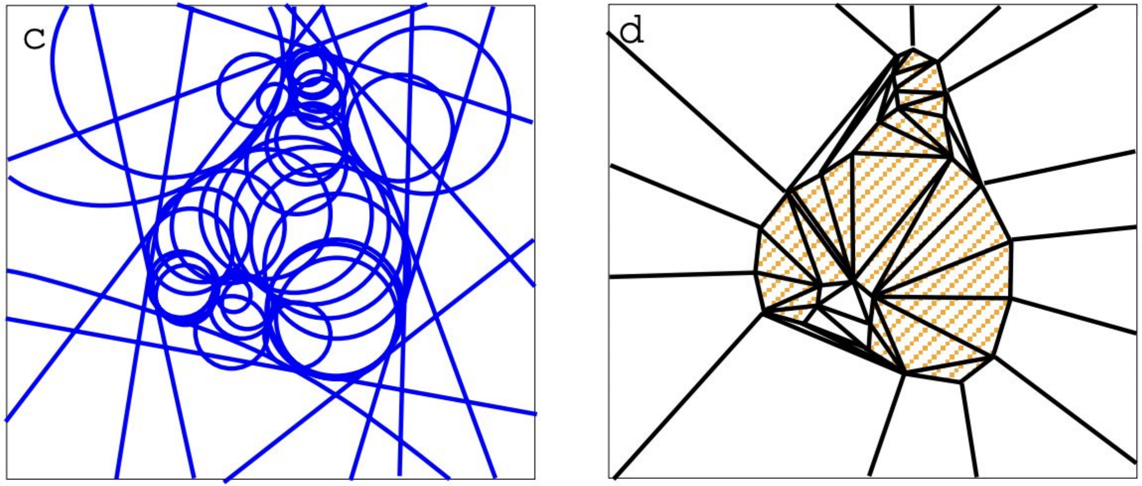
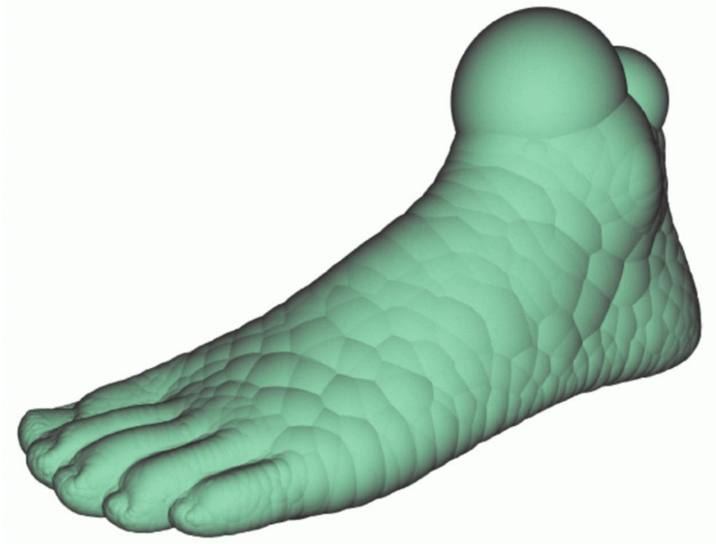
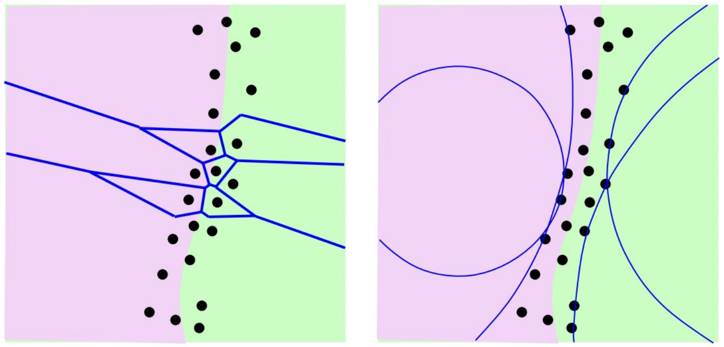


Figure 2. Two-dimensional example of power crust construction. a) An object with its medial axis; one maximal interior ball is shown. b) The Voronoi diagram of *S*, with the Voronoi ball surrounding one pole shown. In 2D, we can select all Voronoi vertices as poles, but not in 3D. c) The inner and outer polar balls. Outer polar balls with centers at infinity degenerate to halfspaces on the convex hull. d) The power diagram cells of the poles, labeled inner and outer. e) The power crust and the power shape of its interior solid.

"The Power Crust",
Amenta, Choi, Kolluri,
2001





“The Power Crust”,
Amenta, Choi, Kolluri,
2001

Outline for Today

- Homework 6 or Homework 7 Questions?
- Last Time: Robot Motion Planning, Minkowski Sums, etc.
- Curve/Surface Continuity & Bezier Curves
- Polyline Simplification
- A Fun COVID Lockdown Project: Long Tiny Loops
- Clothoid or Cornu/Euler Spiral
- Hand-Drawn Sketch Smoothing
- Curve/Surface Reconstruction
- Next Time: ?