Lecture 25:
Sprouts &
Brussel Sprouts
Outline for Today

● Homework Questions
● Quiz on Friday
● Final Project Presentation Schedule
● Last Time: Bezier Curves, Polyline Simplification, Clothoid Sketches
● Paper & Pencil Game of Sprouts
● Computer Analysis of Sprouts
● Brussel Sprouts
● Hybrid Sprouts & Brussel Sprouts
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Cubic Bézier Curve

Parametric equation:
Function of \( t \)
\( t \) varies \( 0 \rightarrow 1 \)

\[ Q(t) = (1 - t)^3 P_1 + 3t(1 - t)^2 P_2 + 3t^2(1 - t) P_3 + t^3 P_4 \]

Asymmetric: Curve goes through some control points but misses others

weights sum to 1

control points
Connecting Cubic Bézier Curves

- How can we guarantee $C^0$ continuity?
- How can we guarantee $C^1$ continuity?
- How can we guarantee $G^1$ continuity?
- Can’t guarantee higher $C^2$ or higher continuity

Asymmetric: Curve goes through some control points but misses others
Noisy GPS Running Data

- Can overestimate distance by ~10% !!

Looks ok from far away...

Close up shows a problem!

Images from Strava

 iPhones app

running watch
Polyline Simplification: Ramer–Douglas–Peucker

- Originally developed for cartography
- Reduce number of points necessary to represent a polyline
- Identify most important points
- Discards points that are $< \varepsilon$ from the simplified shape

https://commons.wikimedia.org/wiki/File:Douglas_Peucker.png
Long Tiny Loops by Dan Aminzade

- Extract GPS data from Strava API
- Ramer-Douglas-Peucker:
  Simplify input (remove false positive intersections due to noise)
- Verify closed loop
- Check for segment intersections
- Compute convex hull
- Rotating calipers maximum diameter

→ Compute final score
= distance / max diameter

https://longtinyloop.com/faq
Piecewise Clothoid + Circular Arc + Line

- Aesthetically pleasing
- Fairness
- Can ensure G2 or G3 continuity
- Also model sharp discontinuities as appropriate

“Sketching Piecewise Clothoid Curves”
McCrae & Singh, 2008
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Sprouts Game Rules

- Draw $n$ spots
- Players take turns:
  - Draw a line joining two spots, or a single spot to itself.
  - The line must not cross another line or pass through another spot.
  - Draw a spot on the new line.
  - No more than three lines can emerge from any spot.
- Normal Winning Condition: Winner is last person to make a move
- Misère Winning Condition: Winner is first person who cannot make a move
Sprouts Analysis

- Starting with $n$ dots
- What’s the maximum number of turns?
  What’s the maximum number of lines drawn?
  What’s the maximum number of new points added?
Sprouts Analysis

- Starting with \( n \) dots
- What’s the minimum number of turns?
Definition: Planar Graph

- Can be drawn in 2D without any edges crossing

planar: [Diagram](https://en.wikipedia.org/wiki/Planar_graph#/media/File:Dodecahedron_schlegel.svg)

NOT planar: [Diagram](https://en.wikipedia.org/wiki/Petersen_graph#/media/File:Petersen1_tiny.svg)
Sprout Game Tree

Even for just 2 starting spots, the full tree of moves is very large!

Sprout Game Tree
Normal Play

Maximum # moves = 5

If the game ends in 5 moves, Player 1 makes the last move and wins

Sprout Game Tree - Normal Play

If Player 2 can separate and isolate the 2 final dots, ending in 4 moves, they win!

**Sprout Game Tree - Misère Play**

*Flip the condition: Player who makes last move loses*

*Does this mean Player 1 can guarantee a win?*

Sprout Game Tree - Misère Play

Nope! Player 2 is still the winner with perfect play.
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Sprouts Computer Analysis

A “1” means the first player to move has a winning strategy, a “2” means the second player has a winning strategy, and an asterisk indicates a new result obtained by our program.

The *n*-spot Sprouts positions evaluated so far fall into a remarkably simple pattern, characterized by the following conjecture:

**Sprouts conjecture.** The first player has a winning strategy in *n*-spot Sprouts if and only if *n* is 3, 4, or 5 modulo 6. **STILL HOLDS!**

The data for misère Sprouts fit a similar pattern.

**Misère sprouts conjecture.** The first player has a winning strategy in *n*-spot misère Sprouts if and only if *n* is 0 or 1 modulo 5. **LATER DISPROVEN**
Definition: Graph Isomorphism

Two graphs are isomorphic if there is a bijection between the labels of the graphs, and an edge exists between a pair of vertices in one graph if and only if an edge exists between the corresponding vertices in the other graph.

https://en.wikipedia.org/wiki/Graph_isomorphism
Sprouts Analysis Definitions

- Each spot is given a unique name.
- The curves of the sprout graph divide the plane into *regions*.
- The loop of spots and curves surrounding a region is called a *boundary*.
- The sprout graph can be encoded in *set representation*:
  \[
  \{ \{ (abcdhfcb) (ijkl) (l) \} \{ (cfed) \} \{ (degh) \} \{ (efhg) \} \} 
  \]
Sprouts Analysis Move Definitions

A **two-boundary move** joins spot $x$ on boundary $B_1$ to spot $y$ on boundary $B_2$ and adds spot $z$. Boundaries $B_1$ and $B_2$ are now joined. No new regions are formed.

A **one-boundary move** connects spots on the same boundary of region $R$. Region $R$ is split into two subregions $R_1$ and $R_2$. All boundaries of and within $R$ must be designated for $R_1$, or $R_2$.

Note: $2^{k-1}$ ways to group $k$ boundaries
Sprouts Analysis Implementation

The String representation can be simplified/compressed:

- Spots of degree 3 can be thrown away
- Regions with fewer than 2 lives can be thrown away
- Boundaries with no live spots can be thrown away
- Spots of degree zero or one do not appear on more than one boundary, so they don’t need unique names, label them 0 and 1.
- If a degree two spot has no live spots between its two occurrences on one boundary, it only needs to be listed once.
Sprouts Analysis Pseudocode

function eval(P)
    S ← successors(P)
    if S is empty then return “L”
    for each position $P' \in S$ do
        if $P'$ is in the hash table return “W”
    for each position $P' \in S$ do
        if eval($P'$) is “L” then return “W”
    put $P$ into the hash table;
    return “L”
end eval
Sprouts Analysis Implementation

- Lexicographically sort the boundaries & regions to remove (significantly reduce) duplicate/isomorphic graphs.
- Use a hash table to store compressed set representation of all sprout graphs that we have previously examined & labeled “W” or “L”.
- Implementation notes with 1990 hardware: *may be different now*
  - Memory was a more significant resource limitation than CPU/time.
  - There are 10X more “W” configurations than “L” configurations.
  - Therefore, only store “L” configurations in the hash table & recompute successors when needed.
### Sprouts Perfect Play Winner - Normal Play

**Conjecture:** Player 1 wins if # spots $\% 6 = 3, 4, \text{ or } 5$

<table>
<thead>
<tr>
<th>Number of spots</th>
<th>Value of game</th>
<th>Cpu seconds (on a DEC 5000)</th>
<th># of positions in hash table</th>
<th>size of hash table (in bytes)</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>L</td>
<td>$&lt; 0.1$</td>
<td>1</td>
<td>101</td>
</tr>
<tr>
<td>2</td>
<td>L</td>
<td>$&lt; 0.1$</td>
<td>4</td>
<td>606</td>
</tr>
<tr>
<td>3</td>
<td>W</td>
<td>$&lt; 0.1$</td>
<td>7</td>
<td>606</td>
</tr>
<tr>
<td>4</td>
<td>W</td>
<td>0.2</td>
<td>33</td>
<td>1515</td>
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<td>1.1</td>
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<td>2828</td>
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<td>W</td>
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<td>24897</td>
<td>264756</td>
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<td>W</td>
<td>842.8</td>
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<td>W</td>
<td>10107.6</td>
<td>116299</td>
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</tbody>
</table>

Analysis now complete through $n = 44$ (2011)


- Almost 3 hours
- 1.5 MB, max memory for DEC was 8-480MB (after 1991?)

**Conjecture:** Player 1 wins if # spots $\% 6 = 3, 4, \text{ or } 5$
Sprouts Perfect Play Winner - Misère Play

Original Conjecture was later disproven

Current Conjecture: Player 1 wins when \( \# \text{ spots} \mod 6 = 0, 4, \text{ or } 5 \)

**EXCEPTIONS:** Player 1 wins if \( \# \text{ spots} = 1 \) and Player 1 loses if \( \# \text{ spots} = 4 \)

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**Analysis now complete through \( n = 20 \) (2011)**

<table>
<thead>
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<th>Number of spots</th>
<th>Value of game</th>
<th>Cpu seconds (on a DEC 5000)</th>
<th># of positions in hash table</th>
<th>size of hash table (in bytes)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>W</td>
<td>&lt; 0.1</td>
<td>1</td>
<td>202</td>
</tr>
<tr>
<td>2</td>
<td>L</td>
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<td>L</td>
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</tbody>
</table>

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8.5 hours

13 MB, max memory for DEC was 8-480MB (after 1991?)
Discussion: Interactive Sprouts Implementation

- I couldn’t find a good interactive, sketch-based digital Sprouts game
  *Many versions seem to require Flash, so I don’t know if they were any good*
- What is hard about making a computer/phone implementation of Sprouts?
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Brussel Sprouts Variant Game Rules

- Draw \( n \) crosses
- Players take turns:
  - Draw a line joining two cross “legs”.
  - Draw a hash mark across the new line (a new “cross” position).
  - No more than four lines can emerge from any cross.

https://en.wikipedia.org/wiki/Sprouts_(game)#/media/File:Brussel_Sprouts_Game.png
Brussel Sprouts Analysis

- We start with $n$ crosses and $4*n$ live ‘leg’s.
- Each move adds a line, uses up two legs, and adds two legs.
- Does it ever end?
Brussel Sprouts Analysis

- Yes, in fact it does end!
  - Every face contains at least one live leg
  - We are done when no face has more than one live leg
- It must follow Euler’s Characteristic for Planar Graphs:
  - Let $m = \# \text{ of moves}$
  - $\# \text{ of edges} = e = 2m$
  - $\# \text{ of vertices} = v = n + m$ (we start with $n$, and we add one each move)
  - $\# \text{ of faces} = 4n$, there is exactly 1 free end inside of each face at the end
  - $2 = f - e + v$
  - $2 = 4n - 2m + n + m$
  - $\# \text{ of moves} = m = 5n - 2$
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Hybrid Sprouts & Brussel Sprouts Game Rules

- Start with a mix of dots and/or crosses
- Draw a line connecting a dot or cross to itself or another dot or cross
Hybrid Sprouts & Brussel Sprouts Analysis

For $n = 1$

For $n > 1$: *Analysis is not completed!!!*