CSCI 2200
Foundations of Computer Science

Lecture 2:
Discrete Structures & Elements of Proofs

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Foundations of Computer Science

Lecture 2: Discrete Structures & Elements of Proofs

“Time’s Scar” from *Chrono Cross*, Yasunori Mitsuda

Quick administrative stuff

Links to important stuff:

- Main website: https://www.cs.rpi.edu/~gittea/old-site/teaching/spring2025/focs.html
- List of assignments: https://www.cs.rpi.edu/~gittea/old-site/teaching/spring2025/assignments.html
- Piazza: https://piazza.com/class/m5gy7aew1i72ka/
- My slides: https://www.cs.rpi.edu/~ditursi/FOCS/slides/

These links are also in Course Materials in Submitty.

Office hours

Prof. DiTursi – AE 123A – Tu 9:30-12:00, Th 14:00-16:30

TAs (all in AE 127)

Eric (Section 1): Tu 10:00-12:00
Mei (Section 2): M 12:00-14:00
Xingjian (Section 3): M 14:00-16:00
Jun (Section 5): Th 14:00-16:00 *(nb: change from Monday)*
Lilian (Head TA): F 14:00-16:00

Sunday office hours 14:00-18:00 in AE 118

Problem sets

- Assignments published Thursdays after class
  - Two groups of problems: Recitation problems and submission problems.
- Submission problems are due the following Thursday <u>before 9pm</u>
  - No late submissions accepted!
- Other than diagrams, work must be typeset and submitted as a single PDF file smaller than 10MB

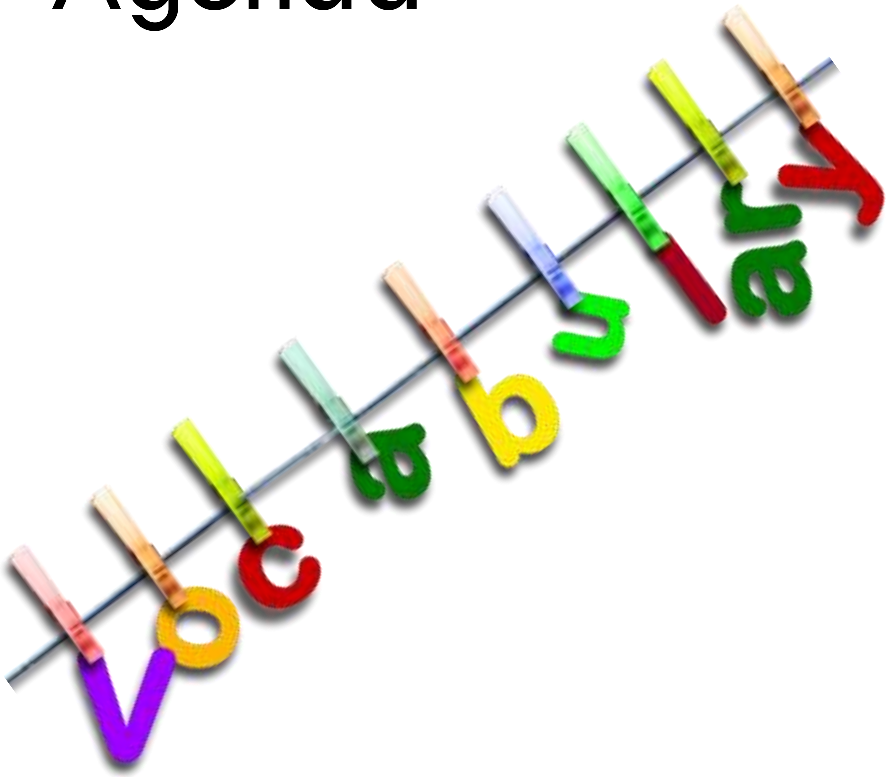
Recitations

- All recitations are in Troy 2018.
- During recitations, your TA will present solutions to the recitation problems.
- After that, your TA and an undergrad mentor will be available to answer student questions.
- There will be no recitations on exam days (1/29, 2/26, and 4/9).

A note on slides

- My slides are (intentionally) not designed as stand-alone reading material – they're meant to be *presented*.
- They are posted for you to look back on after class, in the hopes that they will help spark your memory; however, they are not a substitute for:
    - Attending class
    - Taking your own notes
    - Reading the textbook

Agenda



Building blocks of discrete math:
 * Sets
 * Sequences
 * Graphs

Mathematical / logical statements

Intro to propositional logic

The Building Blocks of Discrete Math

Sets – the fundamental unit of mathematics

Simply a collection of things – any objects you like: numbers, people, shapes, variables, other sets, etc.

The things in the set are *elements* or *members*. The number of elements may be zero, finite, or infinite.

The order within a set does not matter, and there are no duplicates – objects are either in the set or not.

Set notation

Sets => capital letters: A, B, WF, Q_3, …

Certain special sets have other notation: $\mathbb{Z}$

Generic elements => lowercase letters: a, p, x_1, …

If we have specific elements (names, numbers), we can just use those directly.

Curly braces are used as set containers:

S = {a, s, m, r}
N = {Alice, Bob, Charlie, Doug, Eve}
R = {2, 4, 8, …}

$\in$ is used for membership: $m \in S$, Eve $\in$ N

Special sets – everything and nothing

We generally reserve U (often $\mathcal{U}$ or $\mathbb{U}$ in texts) to indicate the *universe of discourse* – the set of every object we could be talking about in this context.

The set containing **no** elements is called the *empty set*, and is often of great importance. We represent the empty set with either { } or ∅

NOT {∅} – that would be the set containing the empty set, which is not empty!

Special sets – types of numbers

$\mathbb{N}$ - natural numbers: {1, 2, 3…}

$\mathbb{N}_0$ - whole numbers: {0, 1, 2, 3, …}

$\mathbb{Z}$ - integers: {…, -3, -2, -1, 0, 1, 2, 3, …}

$\mathbb{Q}$ - rational numbers: anything that can be written as a fraction of two integers

$\mathbb{R}$ - real numbers: the full (continuous) number line, rationals + irrationals (e, π, sqrt(2), etc.)

Infinite sets

We (obviously) can't list every element in an infinite set. If we are being quick (sloppy), we can use an ellipsis and **hope** everyone gets the idea:

$$E = \{2, 4, 6, \ldots\}$$

If this is the set of even positive integers, we're probably fine. If not… 😬

Set descriptions and set builder notation

It would be more precise to just go ahead and describe the set we intend:

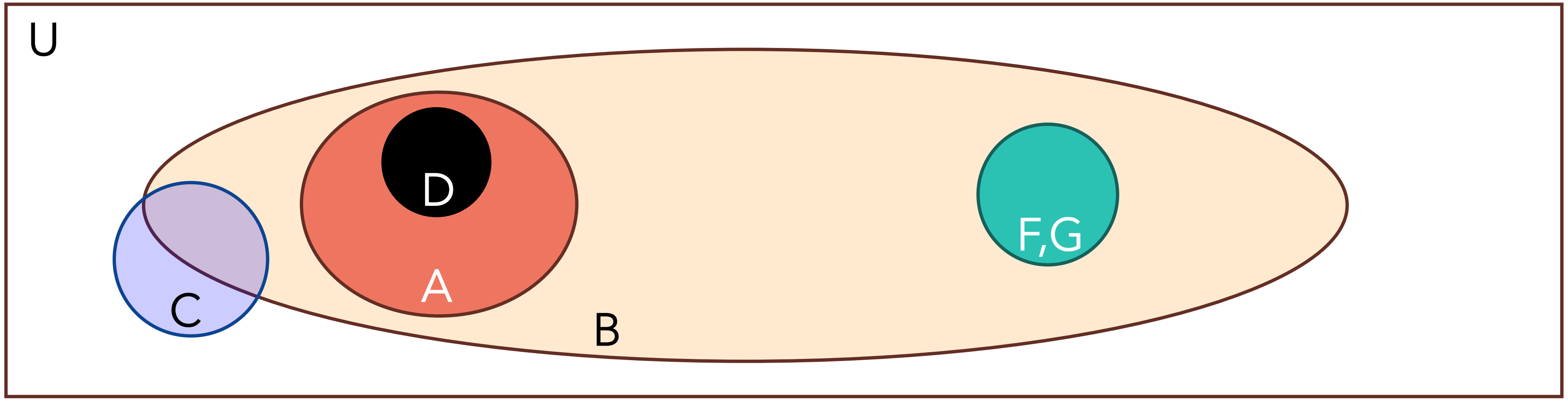
$$E = \{\text{all even positive integers}\}$$

We can also use variables in a form called *set builder notation* – this often still includes some English description as well:

$$E = \{\, n \mid n = 2k, \text{ where } k \in \mathbb{N} \,\}$$

We would read this as "the set of n such that n = 2k, where k is a positive integer."

Set relations (i.e. T/F statements)



Subset: Is the first set fully contained in the second set?
$A \subseteq B$, $D \subseteq A$, $D \subseteq B$, $C \not\subseteq B$

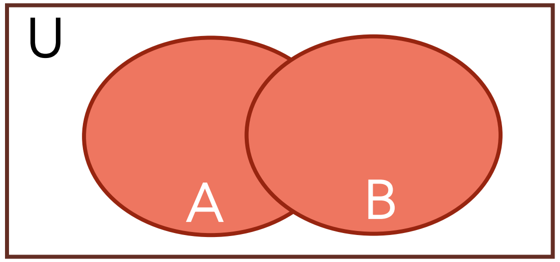
Superset: Does the first set fully contain the second set?
$B \supseteq A$, $A \supseteq D$, $B \supseteq D$, $B \not\supseteq C$

Proper Subset / Superset: Does the larger set have any elements outside the smaller set?
$F \subseteq G$, $F \not\subset G$, $F \subset B$

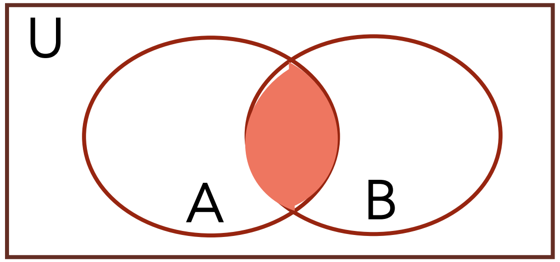
Set Equivalence: Do the sets contain precisely the same members? $F = G$

Set operations (new sets from known sets)

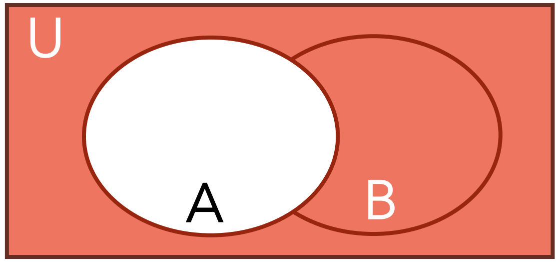
- *Union*: All elements in **either** set – $A \cup B$



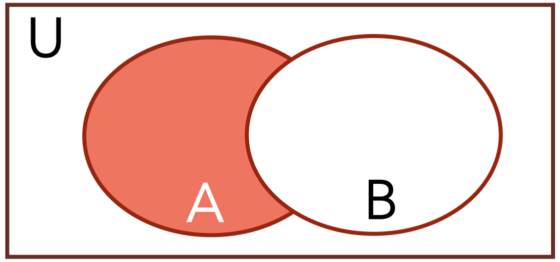
- *Intersection*: Only elements that are in **both** sets – $A \cap B$



- *Negation*: All elements from U not in a set – $\overline{A}$

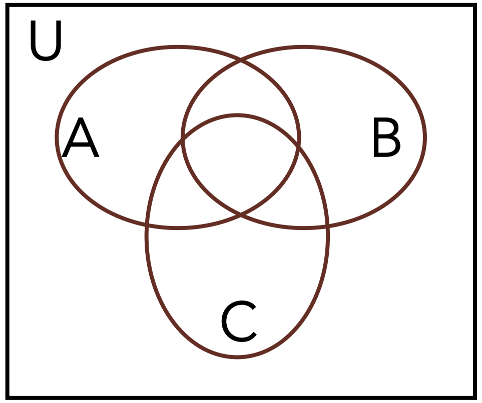


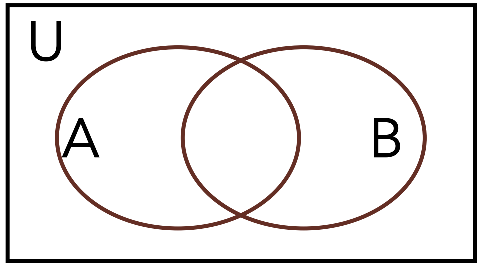
- *Set Difference*: All elements in one set and not in the other – $A - B$



Quick practice

- Draw diagrams for the following:
  - $A \cap B \cap C$
  - $A \cup B \cap C$
  - $\overline{A \cup B}$
  - $\overline{A} \cap \overline{B}$





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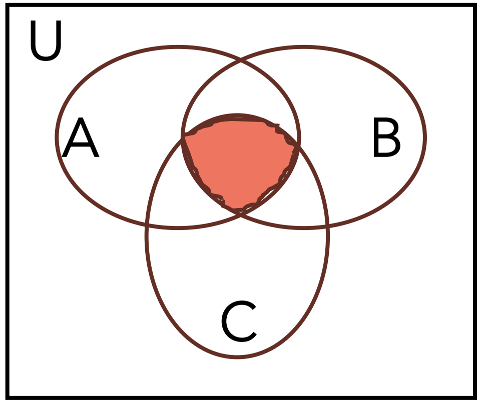

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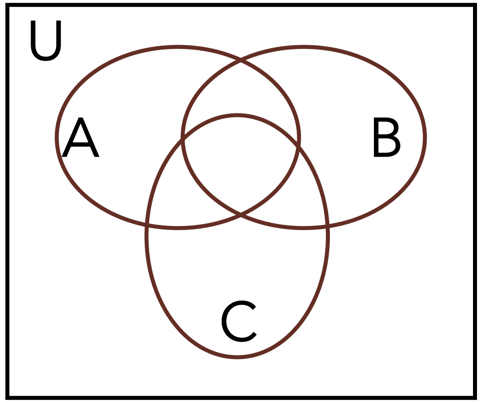

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Quick practice

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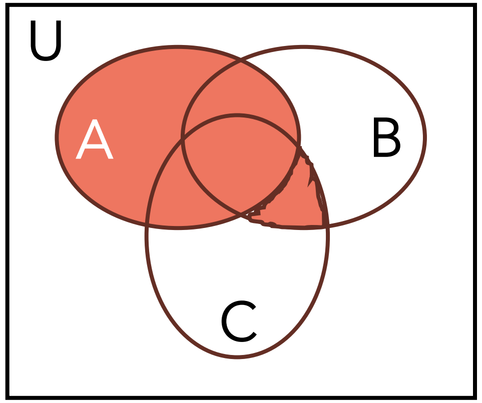


  - $A \cup B \cap C$

    $A \cup (B \cap C)$

    order of operations – negation*, intersection, union

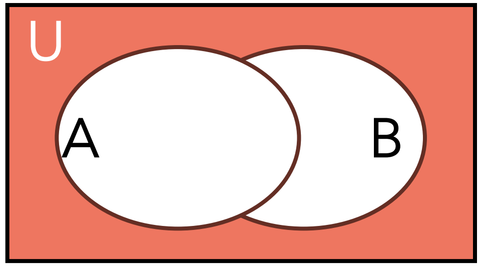
  - $\overline{A \cup B}$

  - $\overline{A} \cap \overline{B}$

Quick practice

- Draw diagrams for the following:
  - $A \cap B \cap C$
  - $A \cup B \cap C$

  - $\overline{A \cup B}$



  - $\overline{A} \cap \overline{B}$

Quick practice

- Draw diagrams for the following:
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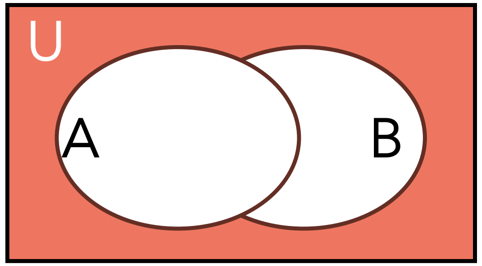
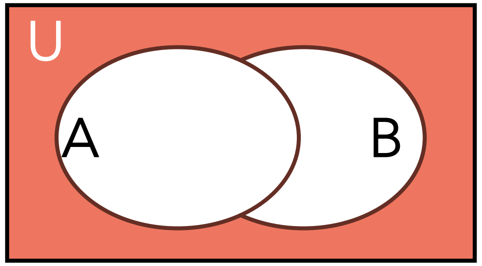
  - $A \cup B \cap C$
  - $\overline{A \cup B}$
  - $\bar{A} \cap \bar{B}$



DeMorgan's Laws

- To "distribute" a negation into an intersection or union, negate all of the individual pieces, then swap intersection for union and vice versa.

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$



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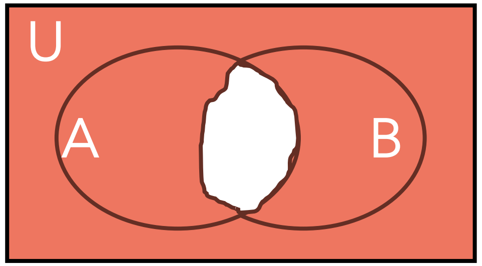


Power set

- The *power set* operation (written 2^S) creates the set of all subsets.

- A = {x, y, z}

- $2^A = \{ \{\}, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\} \}$

Sequences

- A structure where you care about the order.
- This also implies that repetition matters.
- If we are talking about a sequence of symbols, we will also call that a *string*
  - Strings of letters: focs, abbabaabaaa
  - Binary (or bit) strings: 010, 1111010
  - The length-zero string is the *empty string* or the *null string* and is written ε or λ. (We'll use ε.)

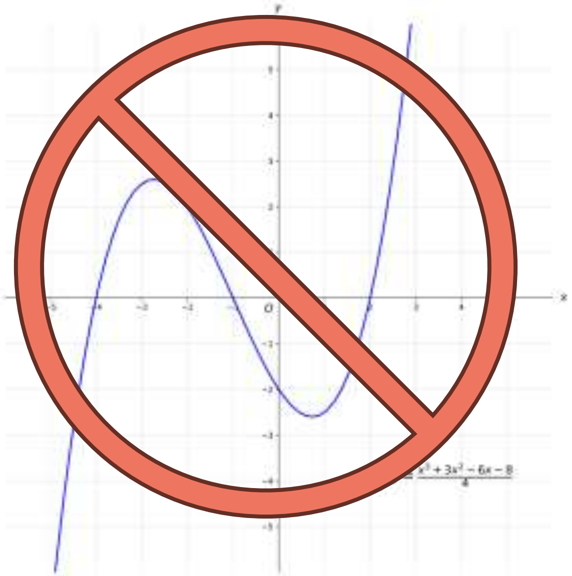
Well-known sequences

- P = {2, 3, 5, 7, 11, 13, 17, …}
- F = {0, 1, 1, 2, 3, 5, 8, 13, …}
  - What's F_{10}? (Note: $F_0 = 0$)

- H = {7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, …}

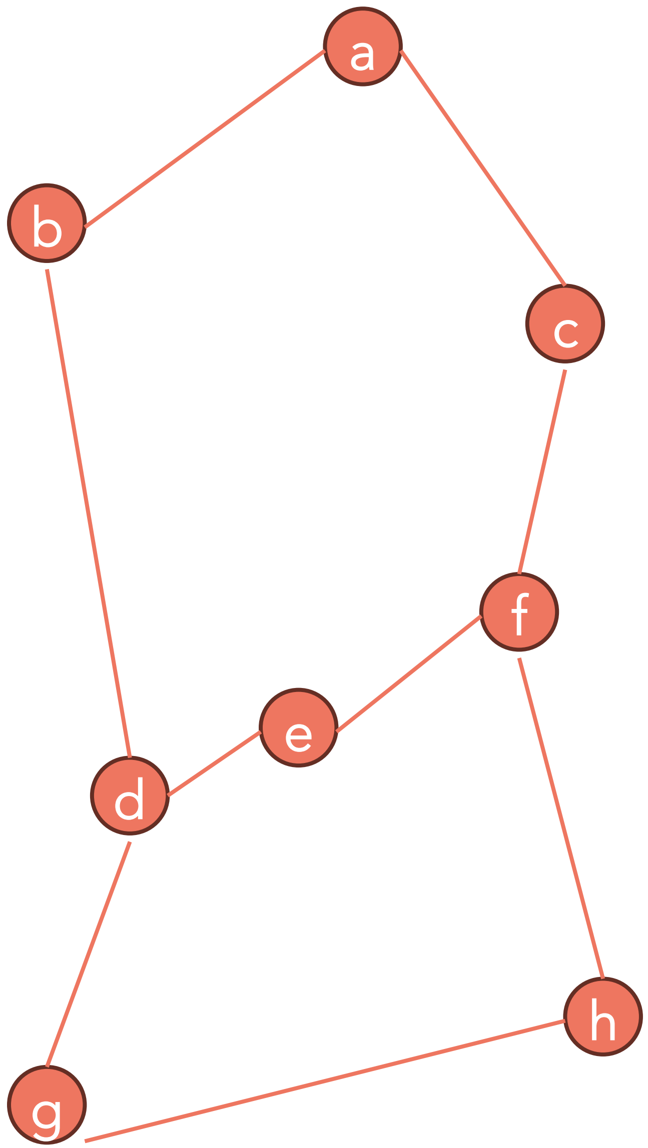
Graphs – modeling *connections*

- Sets do not model relationships between the elements, and sequences only have one type of relationship: precedes / follows
- If we want to be able to model relationships between arbitrary elements, we need a richer structure: a *graph*
- Note this is not a graph as the term is usually used in say, algebra or calculus. . .



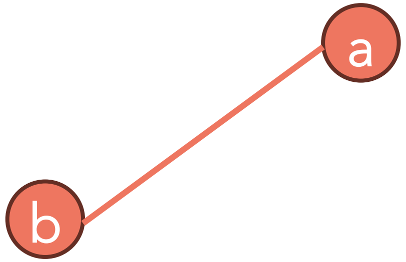
Components of a graph

- Graphs are defined as two sets:
  - *Vertices* (or *nodes*) are the objects we want to model. Often we'll use lowercase letters for these.
  - *Edges* (or *links*) are the connections between the nodes. An edge is written like this: (a,b)
- When drawn, the positions of the vertices DOES NOT MATTER!

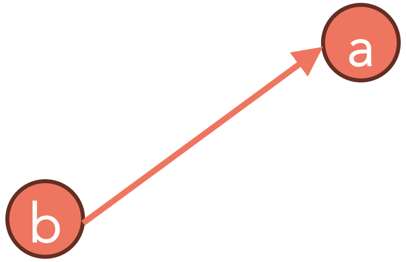


Types of edges

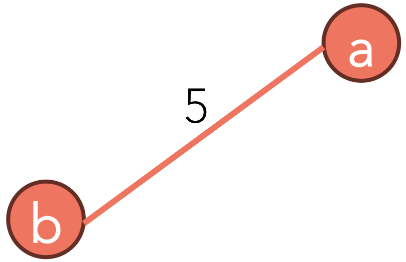
- Often, connection between vertices is transitive: that is, if *a* is connected to *b*, then *b* is automatically connected to *a*. This produces an *<u>undirected graph</u>*.



- If connections are **<u>not</u>** transitive, then the edge *(a,b)* is not the same is *(b,a)*, and we have a *<u>directed graph</u>*.

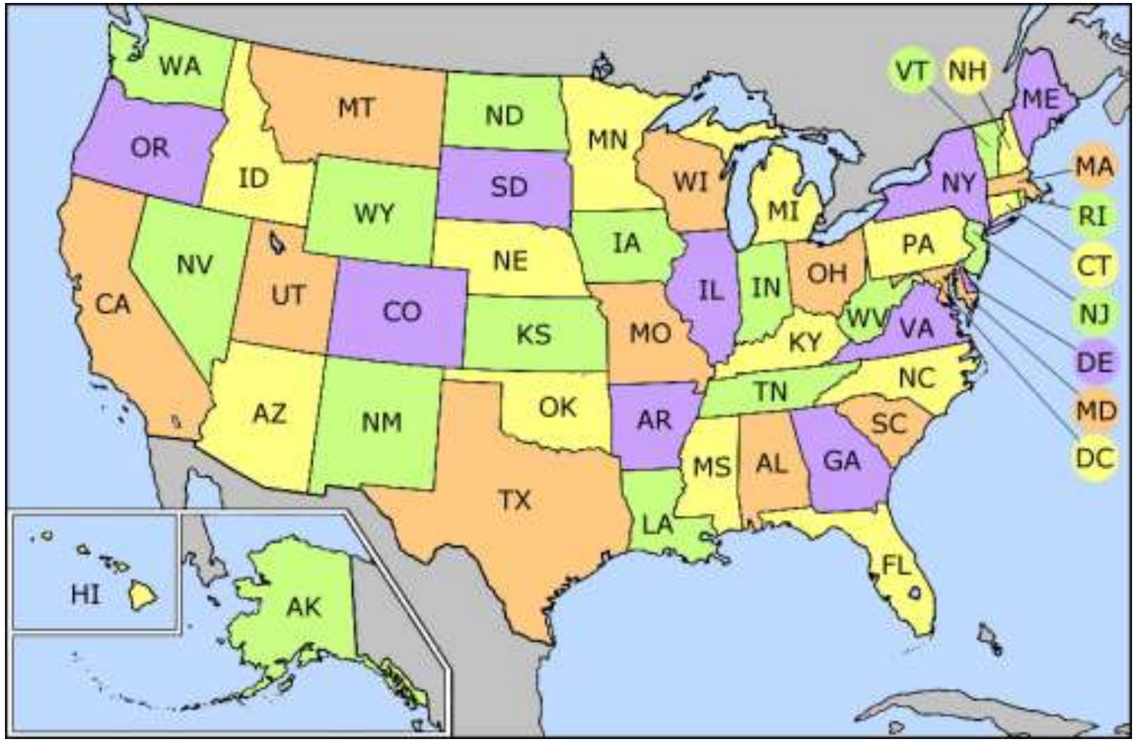


- Edges are sometimes labeled. Often these labels are numbers (indicating capacity, distance, what have you); these numbers are called *<u>weights</u>*.



Some types of graph models

- Social networks (note: social media not required!)
- Affiliation graphs (e.g. students & courses)
- Conflict graphs (edges = problem!)
- Similarity graphs (DNA / RNA analysis)
- Literal maps!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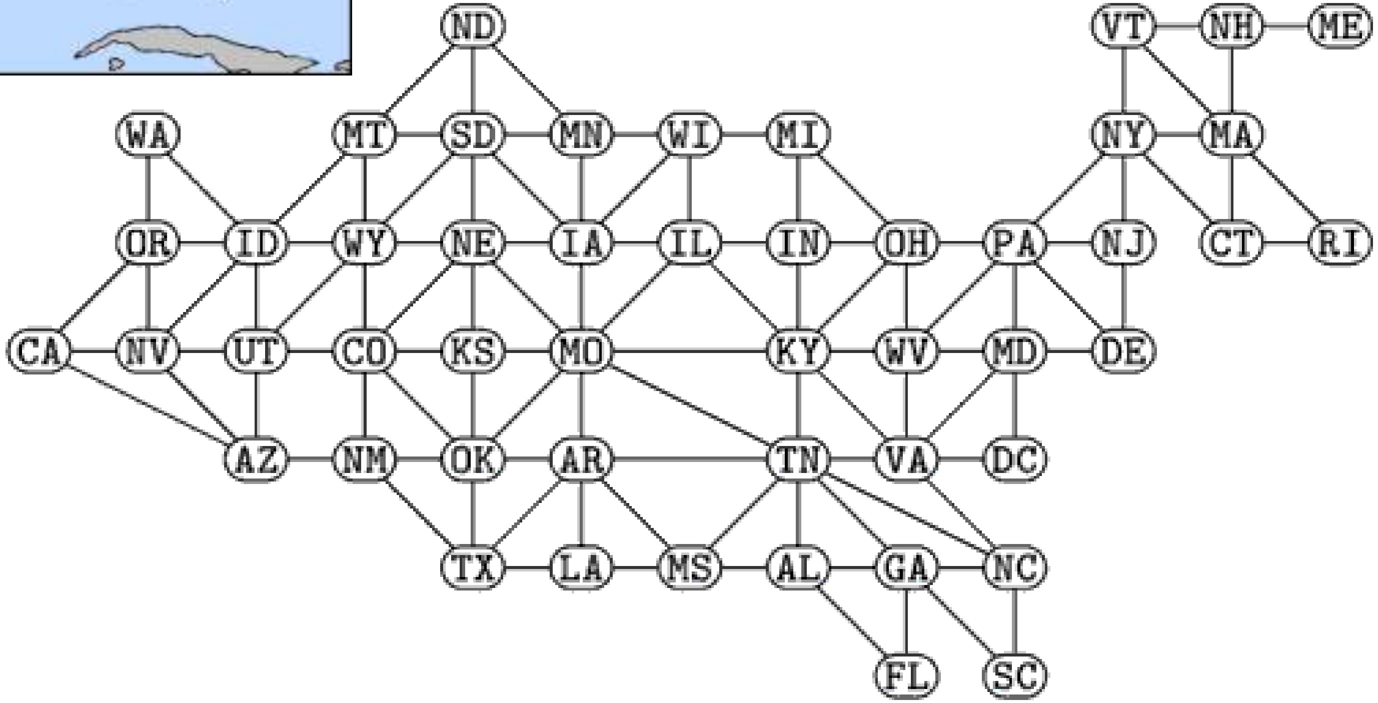
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The Elements of Proof

Statements in English

- Consider the sentence:
  "She said she didn't take his money."
- How you say it matters a lot!

Statements in English

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  "She said she didn't take his money."
- How you say it matters a lot!
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  "She said she didn't **take** his money."
  "She said she didn't take **his** money."
  "She said she didn't take his **money**."

Ambiguity

- Even without emphasis, common language can leave meaning unclear. What does "another" mean here?
  - "What drink are you having? I'll get another." "What shirt are you wearing? I'll wear another."
- "Everything that glitters is not gold."
  - Do we mean that nothing that glitters is gold? Or that there are some things that glitter that are not gold?

Avoiding ambiguity

- When talking to a computer, it is necessary to be very precise about what you mean. There is no such thing as ambiguity to a computer – it'll just decide what an instruction means based upon its own rules.
- When working in mathematics, we insist upon the same precision – even if it makes our statements much more complicated, it must be absolutely clear what we mean.

What, precisely, is a proof?

- A proof generally starts with a *claim* – the logical statement you wish to demonstrate.
- Every proof relies on one or more *axioms* – statements that we accept as true without proof. Sometimes these are given in the problem statement; sometimes these are just fundamental ideas in math. (1 + 1 = 2)
- We then give a sequence of **true** statements that are designed to **convince** the reader that our claim is true. Ultimately, this is the point: To ensure that "everyone" (in whatever context) agrees on the truth of our claim. It then becomes a *theorem* we can use in other proofs.

An example proof: Parity of perfect squares

- Consider the following claim: The square of every odd integer is also odd.
- One can't simply say "well, 1^2 is 1, 3^2 is 9, 5^2 is 25… looks like it works for everything!"
- Most proofs rely heavily on the definition of the terms involved. In this case: what does "odd" mean?
- A useful definition is this: A number is odd precisely when it is equal to 2k+1 for some integer k.

An example proof: Parity of perfect squares

- Consider the following claim: The square of every odd integer is also odd.
- Using our definition, we can write:
$$(2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$
- Next, we can note that whenever k is an integer, then so is $2k^2 + 2k$. (Why?)
  - The *closure properties* of integers are something that we take as axioms.
- Finally, we can state that the square of any odd integer can be written as two times an integer plus one, and therefore the square is also odd. *Quod erat demonstratum*.
- <u>Are you convinced?</u>

An example proof: Irrationality of $\sqrt{2}$

- Consider the following claim: The square root of 2 cannot be written as $\frac{a}{b}$, where a and b are integers.
- One often hears the phrase "you can't prove a negative." This is false, but what *is* true is that it's often more difficult to prove that something cannot exist than that it can exist.
- Often such proofs begin with the question "Well, what if it did exist?" Let's try that here…

An example proof: Irrationality of $\sqrt{2}$

- Imagine that $\sqrt{2} = \frac{a}{b}$ for some integers a and b, and that $\frac{a}{b}$ is in lowest terms. (If it's not, reduce it until it is.)
- We can square both sides and get $2 = \frac{a^2}{b^2}$ or $a^2 = 2b^2$.
- a^2 is therefore even, by the definition we just used.
- If a^2 is even, then a must also be even.
  - We actually just proved that in a sneaky way in our first example! More on that specific topic (the contrapositive) next class. For now, if you're not convinced, please take it on faith.

An example proof: Irrationality of $\sqrt{2}$

- If a is even, then $a = 2k$ for some integer k, and $a^2 = 4k^2$.
- Plugging into our previous equation, $4k^2 = 2b^2$, or $2k^2 = b^2$.
- This means b is also even, and $\mathrm{b} = 2m$ for some integer m.
- But if a and b are both even, then $\frac{a}{b} = \frac{2k}{2m}$ … which means that $\frac{a}{b}$ is not in lowest terms.

- But we specifically put it in lowest terms!
- We've got a big problem here!

Actually… the problem is solved!

- Creating an impossible situation is in fact the <u>goal</u> of proofs like this – it means that some hypothetical situation you proposed can't actually happen.
- Remember that we started this proof "Imagine that $\sqrt{2} = \frac{a}{b}$ for some integers a and b"
- Well, we did, and we got a *<u>paradox</u>*! So $\sqrt{2}$ must not be able to be written as a fraction!

Today's class survey

At the end of every class, I will ask you to complete a very brief survey about that day's lecture. The QR code at left contains the link to the survey – please complete it now.

Problem Set 1 will be posted around 10am; it is due on Thursday 16 January by 8:59pm.