CSCI 2200

Foundations of Computer Science

Lecture 5: Induction, Part 1

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Lecture 5: Induction, Part 1

"Tristram" from *Diablo* by Matt Uelmen

Today's tasks

- Questions from recent problem sets
- One more example of proof by contradiction
- A few caveats about proofs in general
- . . . INDUCTION

Questions from problem sets

Item #1: When to use $\wedge$ versus $\Rightarrow$

General guideline:

"All"/"every": Use $\Rightarrow$

"Some" / "There exists": Use $\wedge$

"Computers are annoying." (impl. all)

$$\forall x\ C(x) \Rightarrow A(x)$$

"Some computers are useful."

$$\exists x\ C(x) \wedge U(x)$$

Item #2: Watch the details.

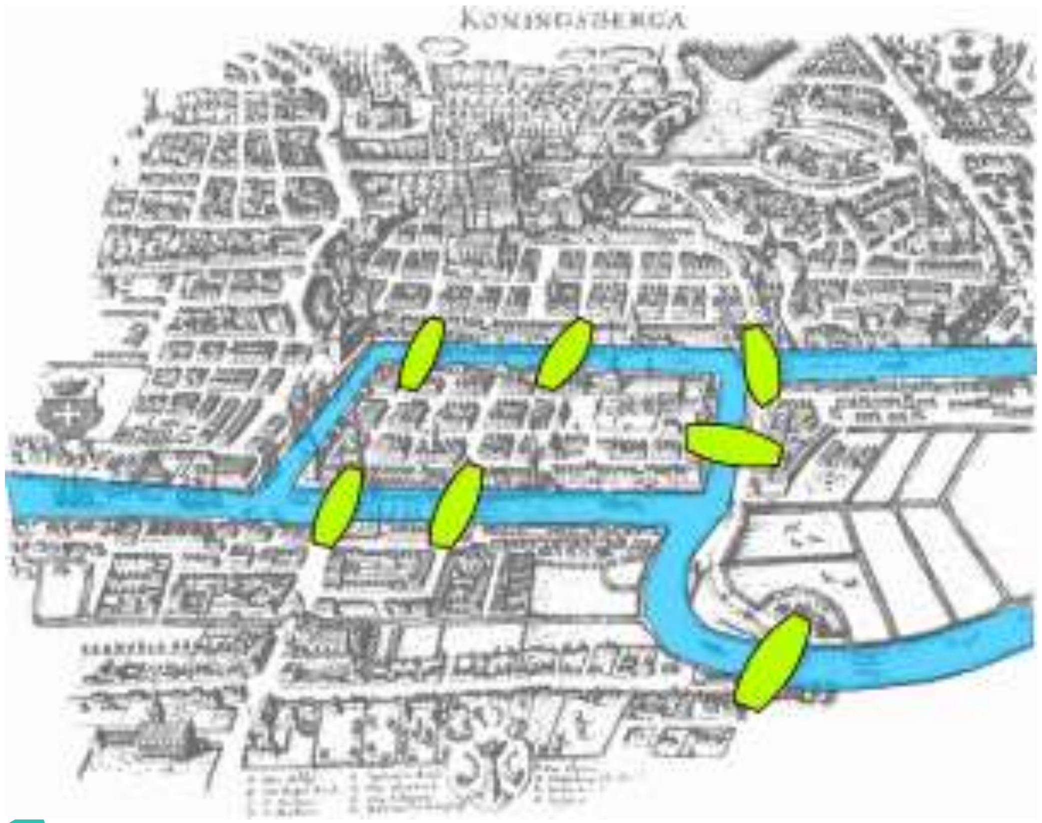
$$a + b = a + 2\sqrt{ab} + b$$

This is not a contradiction! Why not?

Other student questions?

General proofs cont'd.

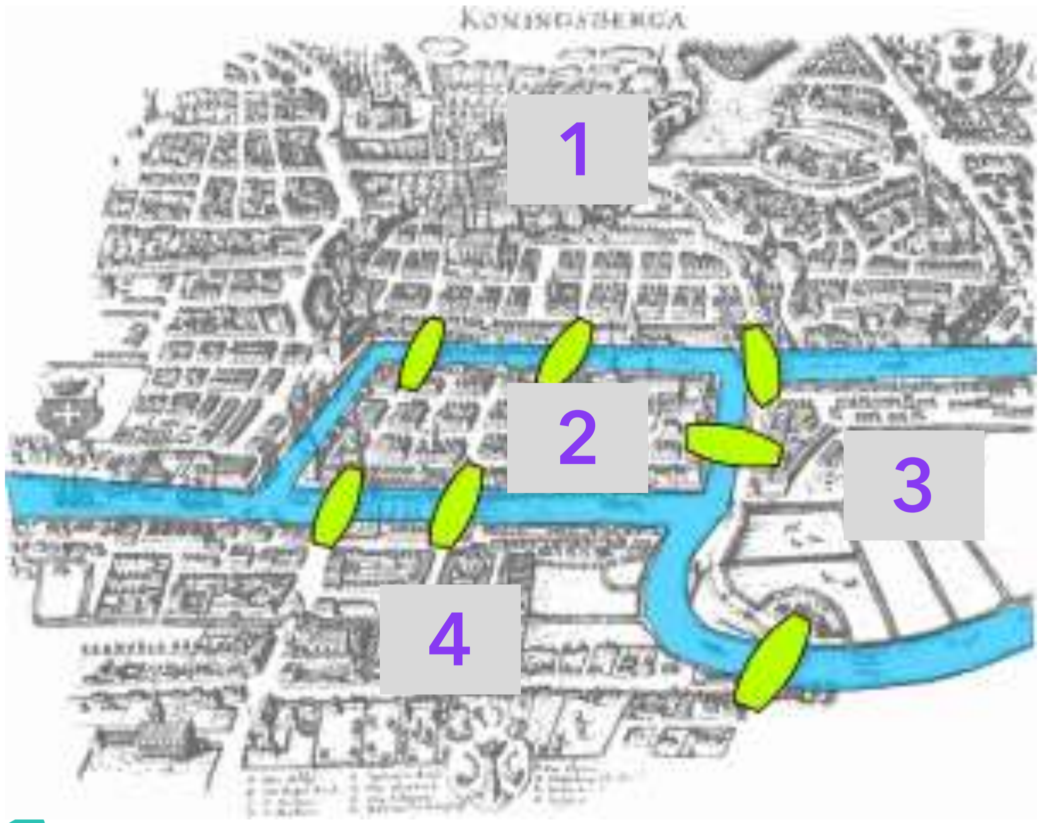
The Seven Bridges of Konigsberg



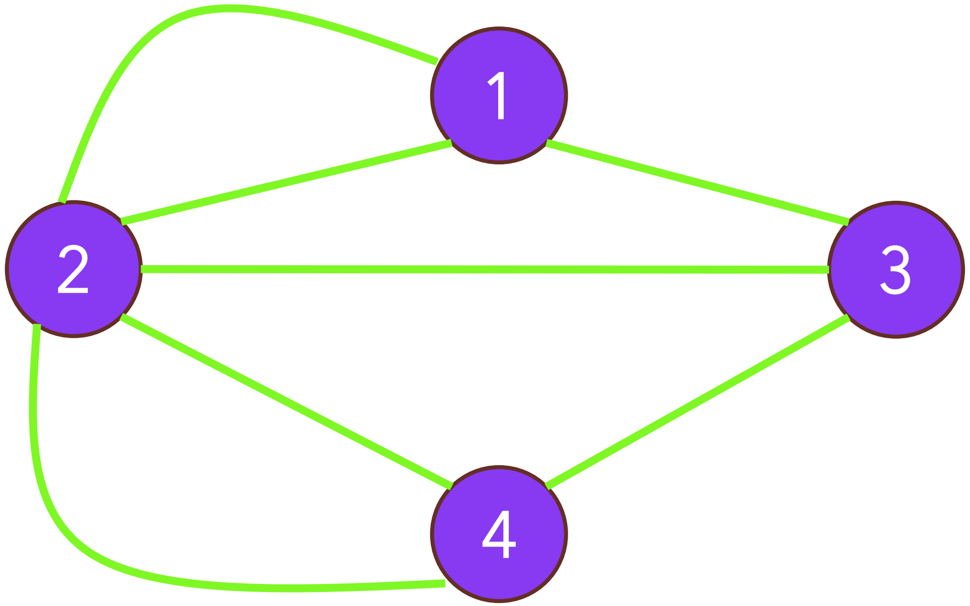
From Leonhard Euler (1736):

Find a path, starting and ending anywhere (possibly different locations), that crosses each bridge exactly once.

The Seven Bridges of Konigsberg



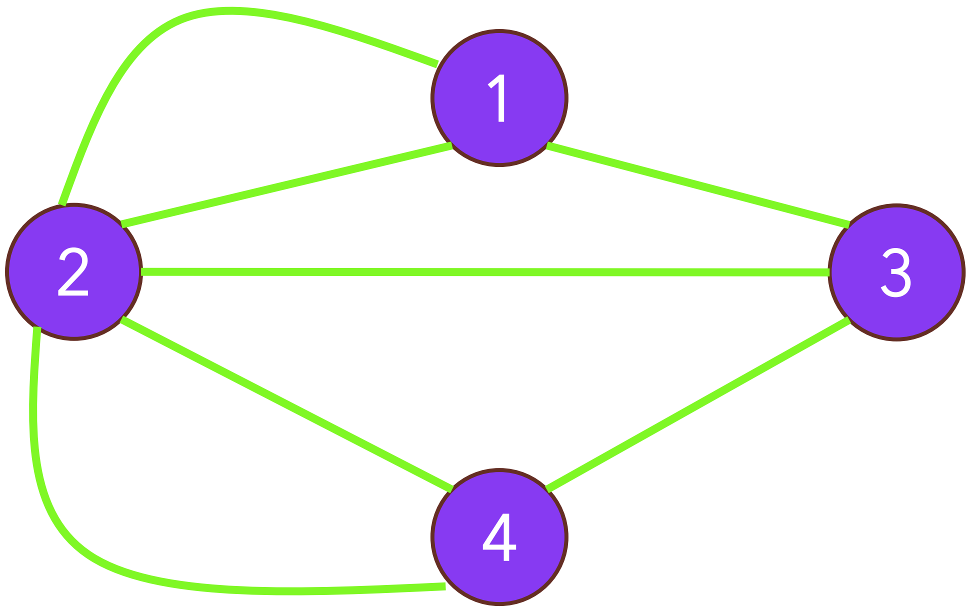
Observe that we can simplify our visual by constructing a *multigraph* (allows multiple edges between nodes):



No such path: proof by contradiction

Assume we can find an *Eulerian path*.

Then, except for our start and end points, each time we visit a node (landmass), we also must leave it via a different bridge. This means that the number of bridges connected to these intermediate nodes must be even!



Since there is only one starting point and one end point, there can be a maximum of two nodes with an odd number of edges. However, all four nodes have an odd number of edges.

This is a contradiction, and thus, our assumption is false – there is no Eulerian path.

Your turn!

Claim: If x is rational and y is irrational, then x+y is irrational.

Give an <u>indirect</u> proof (that is, proof by contradiction).

The rational approach…

Claim: If x is rational and y is irrational, then x+y is irrational.

Proof: Assume that the claim is false; that is, there is some x and y such that x is rational, y is irrational, and x+y is <u>rational</u>.

Since x is rational, we can write it as the ratio of two integers: $x = \frac{a}{b}$ $(b \neq 0)$

Similarly, $x + y = \frac{c}{d}$ $(d \neq 0)$

Therefore, $y = \frac{c}{d} - \frac{a}{b}$ (i.e. (x+y) – x)

Applying a common denominator gives $y = \frac{bc - ad}{bd}$

Because the integers are closed under subtraction and multiplication, we know that $bc - ad$ and bd are both integers.

Thus, y meets the definition of a rational number. This is a contradiction.

Therefore, our initial assumption must be wrong, and the claim is proven true. ■

Caveat lector…

$a = b$	Take two variables a & b and set them to the same positive number
$a^2 = ab$	Multiply both sides by a
$a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides
$(a+b)(a-b) = b(a-b)$	Factor
$a + b = b$	Divide by $(a - b)$
$b + b = b$	Because $a = b$
$2b = b$	Combine like terms
$2 = 1$	Divide by b

Induction

Chains of implications

- Consider the statements:
  - If we have mushrooms on the pizza, then we must also have anchovies.
  - If we have anchovies on the pizza, then we must also have pineapple.
  - $(p \rightarrow q) \wedge (q \rightarrow r)$
- Now I tell you p, that we have mushrooms on the pizza. What can you conclude?
  - Both q and r must be true. Mushroom, anchovy, and pineapple pizza it is!

Longer chains of implications

- Does it still work if I have three chained implications?
- Four?
- Ten?
- …
- Infinitely many?

Congratulations! You now understand induction!

Why do we need induction?

- Consider the statements:
  - $\forall n \in \mathbb{N}, n^2 - n + 41$ is a prime number. $\Leftarrow$ 41 doesn't work!
  - $\forall n \in \mathbb{N}, 4^n - 1$ is divisible by 3.

- Both of them work for 1, for 2, for 3…
- … for 12, for 13, for 14, for 15, …
- … for 37, for 38, for 39, for 40, …

- How many cases are enough? No. Such. Thing.

Mathematical induction, basic version

- A proof by induction is used when you want to show some predicate *P(n)* is true for every element in an infinite sequence starting with some baseline element n_0.
  - Quite often, this will be the set ℕ, possibly starting somewhere other than zero.
- Proofs by induction require two elements:
  - *Base case:* Proving $P(n_0)$
  - *Induction step:* Proving $\forall i, P(n_i) \rightarrow P(n_{i+1})$

Simple example: Gauss's formula

- Consider the triangle numbers: 1, 1+2, 1+2+3, …
  - Over 200 years ago, Gauss proved this formula for T(n): $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$. Let's prove it via induction.
  - Base case: n=1. The sum has only one term, 1. And ½ ·1 · (1+1) = 1. So we have proven T(1).
  - Induction step: We must prove the following implication:
    - IF $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, THEN $\sum_{i=1}^{n+1} i = \frac{1}{2}(n+1)(n+1+1)$
    - Take a moment and try to really grok that implication. It is the core of the entire concept of induction.

Gauss's formula, induction step

- IF $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, THEN $\sum_{i=1}^{n+1} i = \frac{1}{2}(n+1)(n+1+1)$
- As with every other direct proof of an implication, we assume the first part $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ is true. This is called the *inductive hypothesis.*
- And as with our other proofs, we will then try to move toward the second part of the implication. Let us add (n+1) to both sides of the equation:

$$\sum_{i=1}^{n} i + (n+1) = \frac{1}{2}n(n+1) + (n+1)$$

Gauss's formula, induction step

$$\sum_{i=1}^{n} i \ + \ (n+1) \ = \ \frac{1}{2}n(n+1) \ + \ (n+1)$$

- Observe that, on the left, we can incorporate the new term into the sum:

$$\sum_{i=1}^{n+1} i \ = \ \frac{1}{2}n(n+1) \ + \ (n+1)$$

- On the right, we can factor out an (n+1):

$$\sum_{i=1}^{n+1} i \ = \ (\frac{1}{2}n+1) \ (n+1)$$

Gauss's formula, induction step

$$\sum_{i=1}^{n+1} i = \left(\frac{1}{2}n + 1\right)\ (n+1)$$

- Finally, we can factor a ½ out of the first set of parentheses. Note that 1 = ½ x 2.

$$\sum_{i=1}^{n+1} i = \frac{1}{2}(n+2)\ (n+1)$$

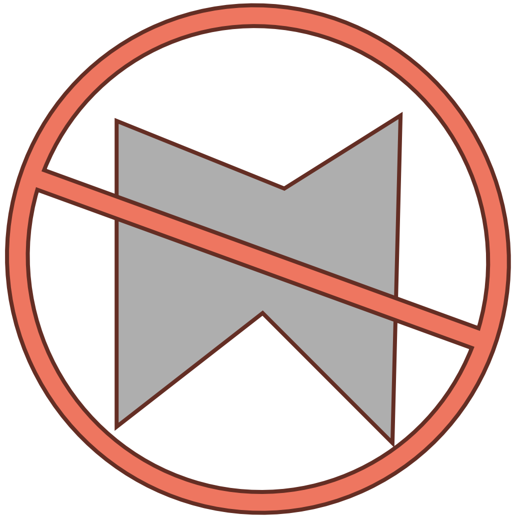
- This now matches the right side of our implication!
  - IF $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, THEN $\sum_{i=1}^{n+1} i = \frac{1}{2}(n+1)(n+1+1)$
- Thus we have proven the induction step.

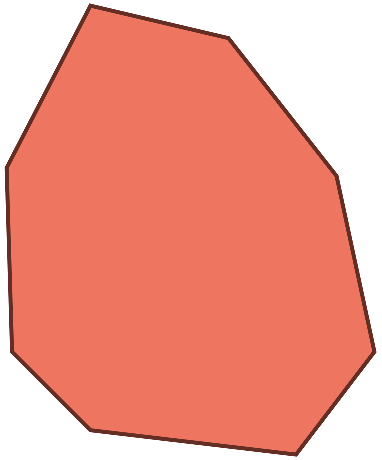
Gauss's formula, wrapping up

- To recap, we have proven:
  - $T(n) \Rightarrow T(n+1)$, for all $n \geq 1$. (i.e. We have set up a beautiful chain of dominoes…)
  - $T(1)$. (i.e. We have pushed over the first domino.)
- Therefore, by induction, we have proven $T(n)$ for all $n \geq 1$. ■

Next example: Polygons

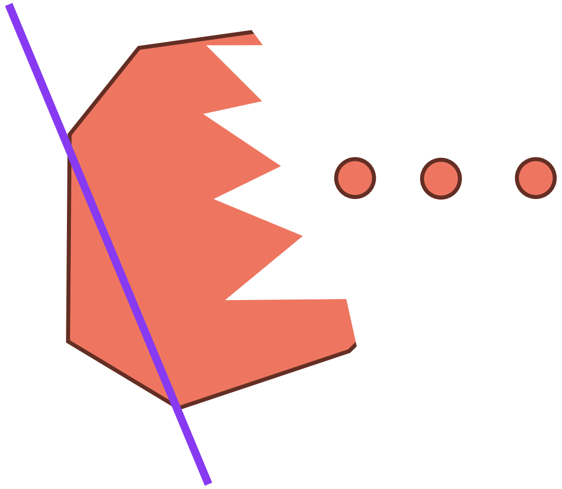
- A convex polygon is one where all of the interior angles are less than 180 degrees.
- Claim: In a convex polygon with *n* sides, the sum of the interior angles is (n-2) · 180°.
- Proof: We will use induction.
    - Base case: n=3 (Why?). This is a triangle. As an axiom, the sum of the interior angles of a triangle is 180°.



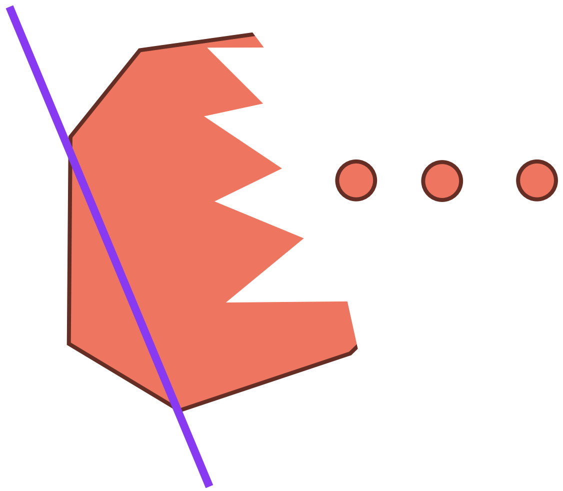


Polygons, induction step

- Induction step: We need to prove that
  - If the angles in a polygon with *n* sides adds up to $(n-2) \cdot 180°$, then the angles in a polygon with *n+1* sides adds up to $((n+1)-2) \cdot 180°$.
- Inductive hypothesis: Assume that the angles in a polygon with *n* sides DO add up to $(n-2) \cdot 180°$
- Consider a polygon with n+1 sides:
  - We can look at three adjacent vertices, and draw a line connecting the two outer ones.



Polygons, induction step



- Observe that the line divides the (n+1)-gon into two sections: a triangle and an n-gon.
- By axiom, the triangle's angles add up to 180°.
- By the inductive hypothesis, the n-gon's angles add up to $(n-2) \cdot 180°$.
- Adding these together, we get $(n-1) \cdot 180°$, which is the value we are trying to show.
- Since we have proven the base case and the induction step, we have proven the claim for all $n \geq 3$. ■

Your turn!

Claim: The sum of the first n perfect squares equals $\frac{1}{6}n(n+1)(2n+1)$

Prove using induction.

Claim:

$$\forall n \in \mathbb{N}, \sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$$

Proof by induction.

Base case: n=1. $1 = \frac{1}{6}(1)(2)(3)$.

Induction step:

IF $\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$, THEN

$$\sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)(n+1+1)(2(n+1)+1)$$

$$= \frac{1}{6}(n+1)(n+2)(2n+3)$$

Ind. Hypothesis: Assume that $\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$

Add $(n+1)^2$ to both sides, to make the left side look like the goal.

$$\sum_{i=1}^{n} i^2 + (n+1)^2 = \sum_{i=1}^{n+1} i^2 = \frac{1}{6} n(n+1)(2n+1) + (n+1)^2$$

Now factor out $\frac{1}{6}(n+1)$ on the right.

$$\frac{1}{6}(n+1)\big(n(2n+1) + 6(n+1)\big) =$$

$$\frac{1}{6}(n+1)(2n^2 + 7n + 6) =$$

$$\frac{1}{6}(n+1)(n+2)(2n+3) \blacksquare$$

Your turn again!

Claim: $\forall n \in \mathbb{N}, n^3 - n$ is divisible by 6.

Prove using induction.

Claim: $\forall n \in \mathbb{N}, n^3 - n$ is divisible by 6.

Proof by induction.

Base case: n=1. $n^3 - n = 0$, which is divisible by all positive integers, including 6.

Induction step: If $n^3 - n$ is divisible by 6, then $(n+1)^3 - (n+1)$ is divisible by 6.

Ind. Hyp.: Assume $n^3 - n$ is div. by 6.

Observe that

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - (n+1)$$

We can cancel the 1s and regroup:

$$(n^3 - n) + 3(n^2 + n)$$

Since $n^2 + n$ is always even, the second part is divisible by 2 & 3, and therefore by 6.

The first part is divisible by 6 by the I.H.

When you add two things that are divisible by 6, the result is also divisible by 6. Thus, we have proven the claim. ■

Common error #1

Among scholars, this is known as the Youtube Commentator's Fallacy.

Recall that to directly prove $p \Rightarrow q$, we begin by assuming that p is true, and then work our way to q.

If we instead begin by assuming q is true, we have committed a logical fallacy.

So, if you start by assuming that the n+1 case is true, your proof is broken before you even take it out of the box.

Common error #2

Forgetting the base case!

Common Error #3 – "There's no such thing as a horse of a different color."

Claim: In any set of $n \geq 1$ horses, all n horses are the same color.
Proof: By induction on n.

Base case: $n=1$; trivial.

Induction step: If every set of n horses are all the same color, then every set of n+1 horses are the same color.

Consider a set H of $n+1$ horses. Select two of them ($h_1 \neq h_2$). Consider the sets $H - \{h_1\}$ and $H - \{h_2\}$. By the I.H., each of them only contains one color of horse. Furthermore, since they only differ by one horse, they have a horse in common, e colors must be the same. Thus all of the horses in H are also that color. ■

DON'T BREAK THE CHAIN!

Class survey & reminders

- HW 2 due tonight
- No office hours this afternoon (moved to Friday afternoon)
- HW 3 posted later today
- Exam 1 next week!
  Wed. 1/29, 8-10am
  DCC 318 (Last names A-K)
  DCC 324 (Last names L-Z)