CSCI 2200

Foundations of Computer Science

Lecture 0x08: Counting, part 1


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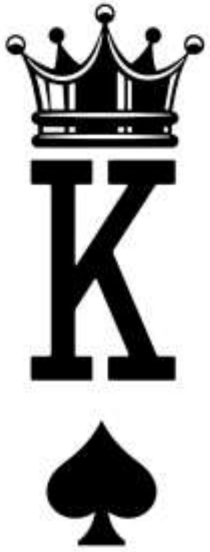
Lecture 0x08: Counting, part 1

“Pinball Number Count” composed by Walt Kraemer
performed by The Pointer Sisters

Tools of the trade…



Warm-up…

...0100011001
0101010101 0
1011011100 1
1011000010 1
0101101111 1
0110101010 1
0001010011 1
1011000101...

- How many bitstrings are there of length *n*?
- Let's say you didn't already know the answer. What would you do?
  - Right – play with it! Try small values of n, see what happens.
    - n = 1. There are 2: 0, 1
    - n = 2. There are 4: 00, 01, 10, 11
    - n = 3. There are 8: 000, 001, 010, 011, 100, 101, 110, 111.
  - Pattern seems clear – how do we prove it?

Yup. Induction.

...0100011001
01010101010
10110111001
10110000101
01011011111
01101010101
00010100111
1011000101...

- Base case: $n = 0$. There is only one way to write the empty string.
- Induction step: If there are 2^n bitstrings of length n, then there are 2^{n+1} bitstrings of length n+1.
- Take any string of length n. We can make a string of length n+1 by adding a 0 or 1.
- So there are 2^n strings of length n+1 that end in 0, and 2^n strings of length n+1 that end in 1. Combined, that's 2^{n+1}.

The Sum Rule

- Super basic / obvious: If a set S can be partitioned into disjoint subsets $A_1, A_2, \ldots, A_n$ (such that $\forall i, j\ A_i \cap A_j = \emptyset$ and $A_1 \cup A_2 \cup \cdots \cup A_n = S$), then $|S| = |A_1| + \ldots + |A_n|$
- In the previous example, we had "strings ending in 0" and "strings ending in 1"; there is obviously no overlap between those sets.

The Product Rule

- We can get to the answer another way…
- In an n-bit string, how many choices are there for the first bit?
  - For the second bit?
  - The third bit?
  - Do these choices impact each other?
- When a set is determined by a series of choices, the number of elements in the set is the number of options for the first choice times the number of options for the second choice <u>after the first choice is made</u>, times the number of options for the third choice <u>after the first two choices are made</u>, etc.

Quick check-in #1

- A standard New York license plate has three letters followed by four digits. How many standard license plate numbers are possible?

- *26 x 26 x 26 x 10 x 10 x 10 x 10 (or 26^3 x 10^4) ≃ 175M*
  - *Real-world note: it's less than this for a variety of reasons. But there are also other possible patterns of letters & digits that are allowed.*

Quick check-in #2

- A standard deck of cards has four suits (spades, hearts, diamonds, clubs) with 13 cards in each suit (ace, two, three, …, ten, jack, queen, king – these are the *ranks*).
- How many sequences of five cards will have no pairs (i.e. no cards of matching rank)?
- *52 x 48 x 44 x 40 x 36*
- Key notion: Once you pick the first card, that reduces the number of possibilities for all future cards by four; the second card similar eliminates four more options for each subsequent card, and so on.

Adding restrictions

How many bitstrings of length 10 are there…

… that have exactly three 1s?

How should we start this problem?

Right: Play with it. Try small cases.

How many bitstrings of length 1 are there with exactly three 1s?

Length 2? Length 3?

Length 4?

Adding restrictions

How many bitstrings of length 10 are there…

… that have exactly three 1s?

Length 3 (1): 111
Length 4 (4): 0111, 1011, 1101, 1110
Length 5 (10): 00111, 01011, 01101, 01110, 10011, 10101, 10110, 11001, 11010, 11100

The pattern is, perhaps, not obvious yet. We should play some more. To facilitate this, let's define a notation:

$\binom{n}{k}$ = # of length-n bitstrings with exactly k 1s

Further play

- Let's look at all bitstrings of length 3.

  000, 001, 010, 011, 100, 101, 110, 111

- What is $\binom{3}{0}$? $\binom{3}{1}$? $\binom{3}{2}$? $\binom{3}{3}$?
- Here are all of the length 4 bitstrings:

  0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111,
  1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111

- What is $\binom{4}{0}$? $\binom{4}{1}$? $\binom{4}{2}$? $\binom{4}{3}$? $\binom{4}{4}$?

Some observations

- What is $\binom{n}{0}$? $\binom{n}{n}$?
- What is $\binom{n}{1}$?
- What is $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$?

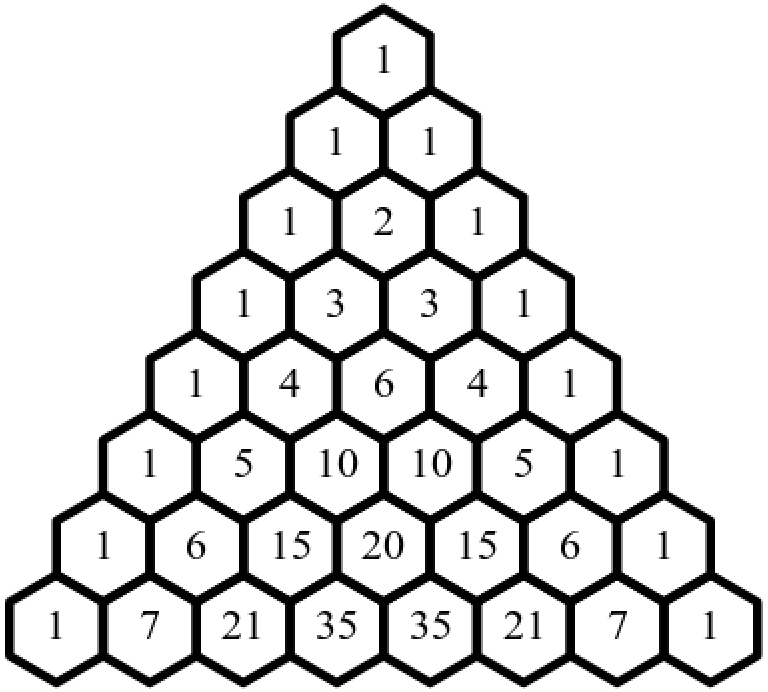
- Is there any relationship between $\binom{n}{k}$ and $\binom{n-1}{k}$? Or other values besides k on the bottom?
- Let's think inductively…

Pascal's Identity

An n-bit string has exactly k 1s in it. What is the last bit?

- 0. The k 1s are in the first n-1 bits.
- 1. The first n-1 bits have k-1 1s.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$



More counting

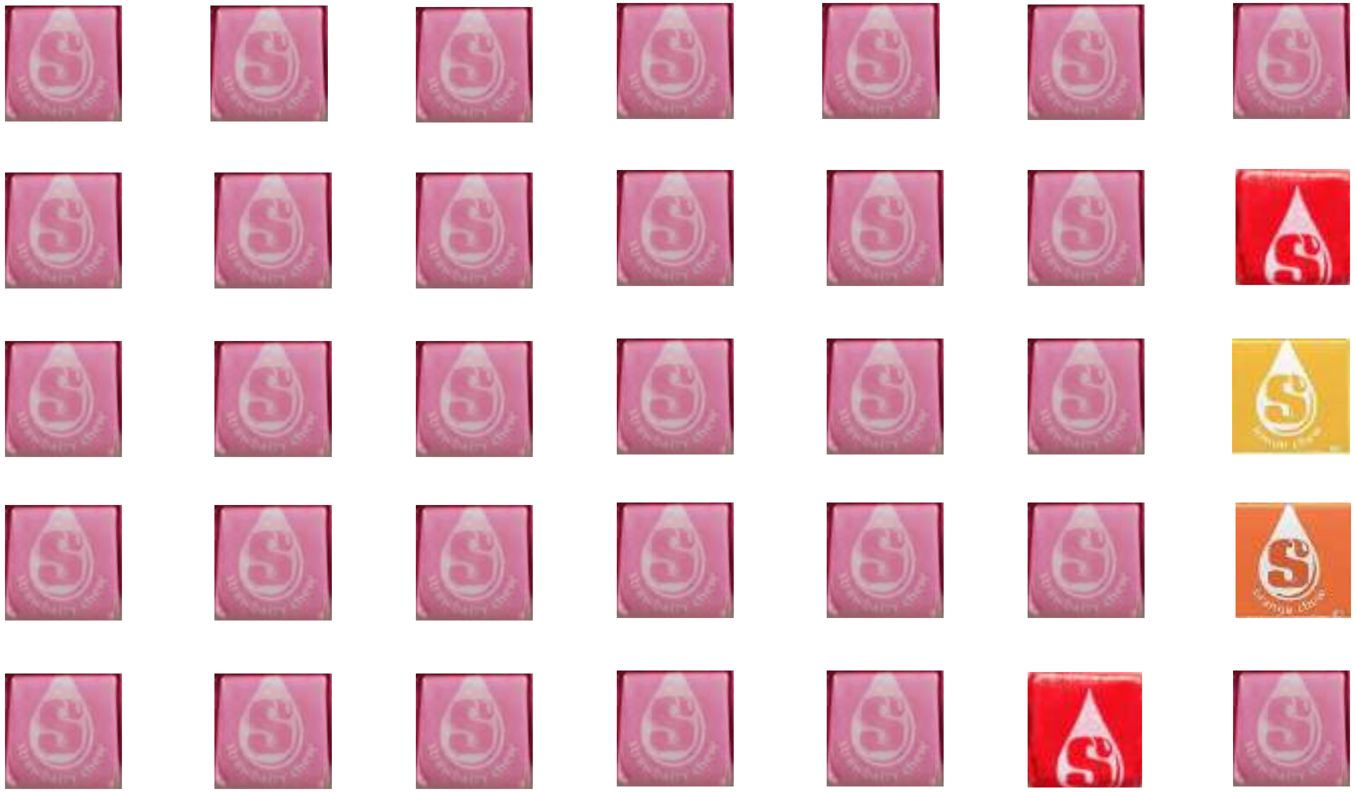
Starbursts originally had four colors: red, pink, orange, yellow

How many different packages of seven original Starbursts can be constructed?

How should we think about this?

Starburst counting

We *could* treat this as a base-4 value and enumerate…



… but this presumes we care about order. And it would take forever.

Better model

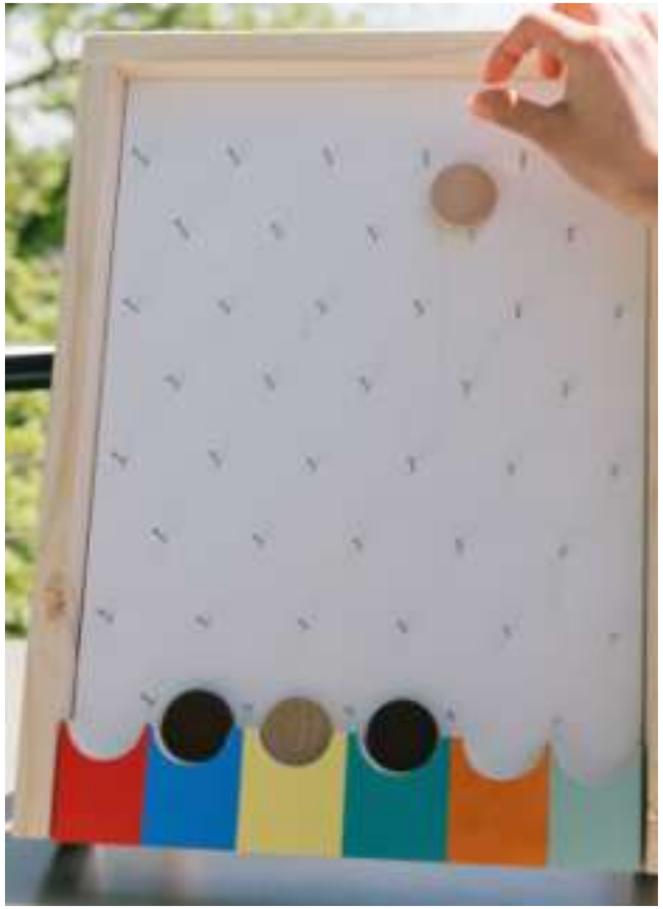
Four bins/buckets, one per color:

Better model

Four bins/buckets, one per color:

… ok, how does this help? Is this really easier to count?

Plinko



Disks are dropped from the top, hit the obstacles along the way, and eventually fall into one of the bins in the bottom.

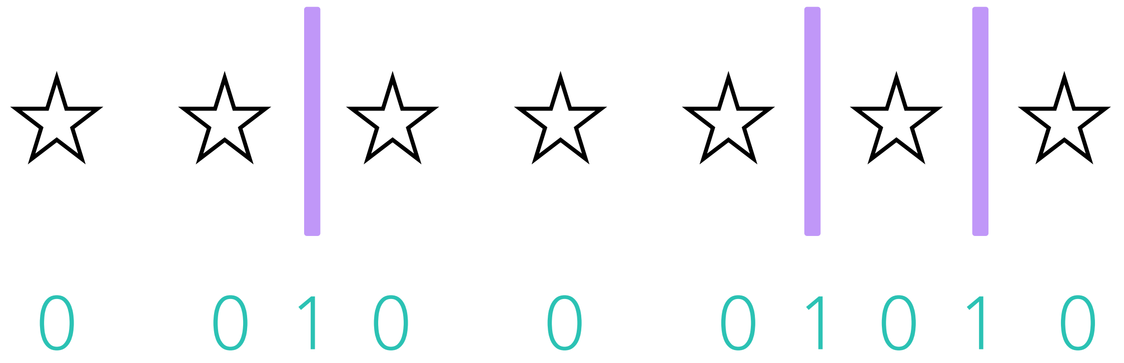
If we make the bins deep enough to hold multiple disks, they can stand in for our buckets.

Key insight: The important thing is the dividers at the bottom!

An encoding

Our first package had:

2 orange, 3 yellow, 1 red, 1 pink



Our second package had:

3 orange, 0 yellow, 4 red, 0 pink

The number of Starbucks packages is <u>precisely the same</u> as the number of 10-bit strings with exactly three 1s!! It's $\binom{10}{3}$

The Bijection Principle

- Timeout! That's a bold claim – can we verify it?
- We need to ask two questions:
  - Can we represent every Starbursts package with one of these binary strings?
  - Does every one of these binary strings encode a valid package of Starbursts?
  - $S \subseteq B \ \wedge \ B \subseteq S \ \Rightarrow \ S = B$ We're good!

- Any time we can establish a bijection, we can count the easier set, and that gives us the count of the harder set.

Sticks and stones

- Why did 7 Starbursts in 4 colors produce $\binom{10}{3}$? Can we establish a general formula?
  - Try smaller cases: 4 Starbursts, 2 colors; 5 Starbursts, 3 colors. What do you notice?
- When partitioning *n* objects ("stones") into *k* categories, you need to place *k-1* dividers ("sticks"), so…

$$\binom{n+k-1}{k-1}$$

Selecting a sequence from a set

- How many ways are there to draw a five-card flush in hearts off the top of a deck?
  - 13 x 12 x 11 x 10 x 9
  - This looks like 13! (factorial), but we're missing the last 8 factors off the end. Can we write that formulaically?
  - 13 x 12 x 11 x 10 x 9 = 13! ÷ 8!
    - … where did the 8 come from?
    - 13 (# of objects in the set) – 5 (number of selections)

Permutations

- When selecting a sequence of k objects from a set of n objects **without replacement**, the number of possible *permutations* is:

$$_nP_k = \frac{n!}{(n-k)!}$$

- Note that we are counting sequences here – that is, we care what order the objects are selected. What if we don't?

Selecting a subset from a set

- How many ways are there to draw a five-card flush in hearts off the top of a deck?
  - 13 x 12 x 11 x 10 x 9 … but we don't care what order the cards are drawn in, only that we have five hearts at the end.
- In how many different orders can five distinct cards appear?
  - Apply the Product Rule.
  - 5 x 4 x 3 x 2 x 1 = 5! = 120
- So there are 120 flush sequences for every flush subset.

Combinations

- When selecting a <u>**subset**</u> of *k* objects from a set of *n* objects <u>**without replacement**</u>, the number of possible *permutations* is:

$$_nC_k = \frac{n!}{(n-k)!\,k!}$$

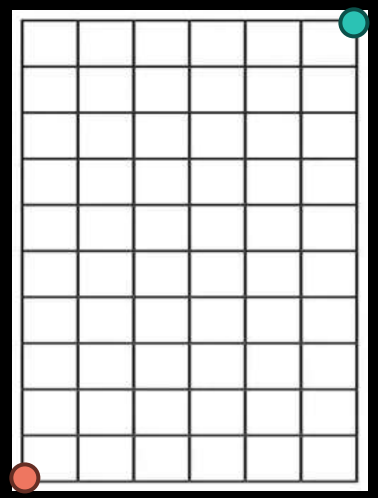
- Go back to our bitstring problem: "How many ways are there to select three positions to have 1s out of 10 total bit positions?"
  - It's the same thing!
  - $\binom{n}{k} = \quad {}_nC_k = \frac{n!}{(n-k)!k!}$

Your turn…

1) An Olympic race has 8 runners. How many different ways can the gold, silver, and bronze medals be awarded?

2) 26136000 factors into primes as $2^6 \times 3^3 \times 5^3 \times 11^2$. How many different integers is it divisible by?

3) In a city with a regular grid of streets, you need to travel 10 blocks north and 6 blocks east. If you never turn south or west, how many possible paths are there?



Olympic race

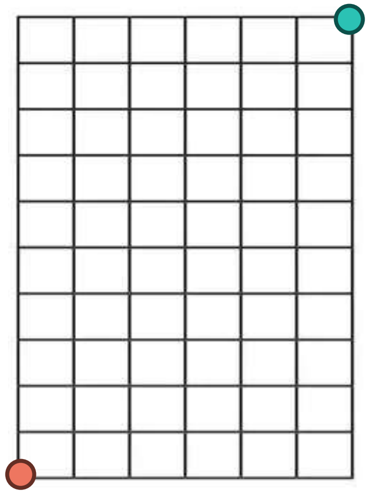
- An Olympic race has 8 runners. How many different ways can the gold, silver, and bronze medals be awarded?
- Can use Product Rule or the permutations formula – they get you to the same place…
  - $_8P_3$ = 8 x 7 x 6 = 336

Number of divisors

- 26136000 factors into primes as $2^6 \times 3^3 \times 5^3 \times 11^2$. How many different integers is it divisible by?
- This is another Product Rule problem. A divisor of 26136000 may contain any or all (or none!) of the prime factors listed above.
  - How many factors of 2 are in the divisor?
    - 0, 1, 2, 3, 4, 5, or 6
  - Similarly, there are 4 choices for the quantity of 3s, 4 choices for the quantity of 5s, and 3 choices for the quantity of 11s.
  - $7 \times 4 \times 4 \times 3 = 336$ divisors.

The grid

- In a city with a regular grid of streets, you need to travel 10 blocks north and 6 blocks east. If you never turn south or west, how many possible paths are there?
- Did you spot the connection to bitstrings?
- You need to drive 16 blocks. Six of those blocks will be eastward; they may occur anywhere along the route.
- $\binom{16}{6} = \frac{16!}{10!6!} = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8{,}008$



Questions?

REMINDER: Exam 2 on W 2/26

Last names A-K: DCC 318

Last names L-Z: DCC 324

Bring your ID card!