CSCI 2200
Foundations of Computer Science

Lecture 14: Counting, part 2

"Still Alive" composed by Jonathan Coulton
performed by Ellen McLain for *Portal*

Announcements

- [Final exam schedule](#) has been posted by the Registrar. We are on M 4/28, 11:30am. (Room(s) TBD.)

- Exam 2 is this Wednesday 2/26. No recitations that day.

  - Last names A-K DCC 318; last names L-Z DCC 324.

  - One-page (double-sided) crib sheet.

  - <u>Bring your ID!</u>

- Reminder: Prof. Gittens will be taking over lectures starting on Thursday.

Answering a question from last time

- What is $(x + y)^2$?

$$x^2 + 2xy + y^2$$

- What is $(x + y)^3$?

$$x^3 + 3x^2y + 3xy^2 + y^3$$

- What is $(x + y)^4$?

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Answering a question from last time

- What is $(x + y)^2$?

$$1x^2 + 2xy + 1y^2$$

- What is $(x + y)^3$?

$$1x^3 + 3x^2y + 3xy^2 + 1y^3$$

- What is $(x + y)^4$?

$$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

- What is $(x + y)^n$?

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

What we know so far

From a set of n objects, select k of them, and...

	Without replacement	With replacement
... we are choosing a <u>sequence</u>.	$_nP_k = \dfrac{n!}{(n-k)!}$	n^k
... we are choosing a <u>subset</u>.	$_nC_k = \dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$	$\dbinom{k + (n-1)}{(n-1)}$

The multinomial coefficient

- A small strand of Christmas lights has five red lights, four blue lights, and three green lights. In how many ways can they be arranged on the strand?

- Careful! This sounds like a sequence question, but consider a strand that instead had 8 white lights and 4 blue ones. How many ways could you arrange <u>that</u>?

  - That's just a bit string with 8 ones and 4 zeros! It would be $\binom{12}{8}$!

    - Though I advise you think of it as $\binom{12}{8}\binom{4}{4}$…

The multinomial coefficient

- A small strand of Christmas lights has five red lights, four blue lights, and three green lights. In how many ways can they be arranged on the strand?

  - Start simple: How many different ways could you select the locations for the red lights?

  $$\binom{12}{5}$$

  - After placing the red lights, how many ways could you select the locations for the blue lights?

  $$\binom{7}{4}$$

  - After placing both the red and blue lights, how many ways could you select locations for the green lights?

  $$\binom{3}{3}$$

The multinomial coefficient

- A small strand of Christmas lights has five red lights, four blue lights, and three green lights. In how many ways can they be arranged on the strand?

- Now we can apply the Product Rule: $\binom{12}{5} \cdot \binom{7}{4} \cdot \binom{3}{3}$

- $\binom{12}{5,4,3} = \dfrac{12!}{5! \cdot 7!} \cdot \dfrac{7!}{4! \cdot 3!} \cdot \dfrac{3!}{3! 0!} = \dfrac{12!}{5! \cdot 4! \cdot 3!}$

Quick check-in

- How many ways are there to rearrange the letters in RENSSELAER?

- $\dbinom{10}{3,2,2,1,1,1} = \dfrac{10!}{3! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1!} = 151{,}200$
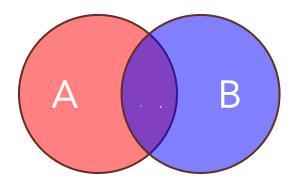
What we know so far
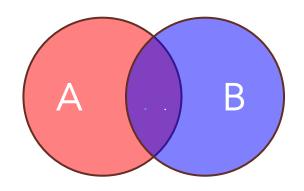
From a set of n objects, select k of them, and...

	Without replacement	With replacement
... we are choosing a <u>sequence</u>.	$_nP_k = \dfrac{n!}{(n-k)!}$	n^k
... we are choosing a <u>subset</u>.	$_nC_k = \dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$	$\dbinom{k+(n-1)}{(n-1)}$
... we are choosing a <u>sequence</u>, but some of the objects are <u>identical.</u>	$\dbinom{k_1+\cdots+k_r}{k_1,\ldots,k_r} = \dfrac{k!}{k_1!\ldots k_r!}$	

The Sum Rule

- Super basic / obvious: If a set S can be partitioned into disjoint subsets $A_1, A_2, \ldots, A_n$ (such that $\forall i, j \; A_i \cap A_j = \emptyset$ and $A_1 \cup A_2 \cup \cdots \cup A_n = S$), then $|S| = |A_1| + \ldots + |A_n|$

- What happens if the sets <u>aren't</u> disjoint? Let's start with just two sets…

Inclusion-exclusion

- Clearly we can't just say $|A \cup B| = |A| + |B|$ anymore.

- What's the problem?
  - We've double-counted the purple section – we need to correct for that.

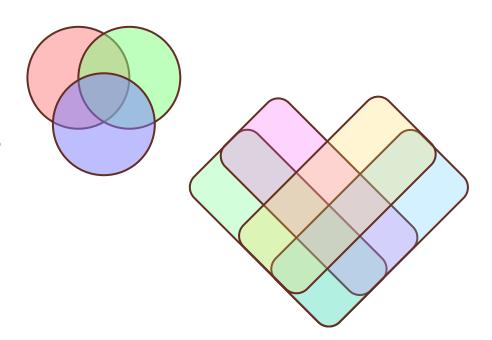- Our rule is $|A \cup B| = |A| + |B| - |A \cap B|$

Quick check-in

- How many integers from 1 to 100 inclusive are divisible by 2 <u>or</u> by 5?
  - A = integers 1-100 divisible by 2
  - B = integers 1-100 divisible by 5
  - We want $|A \cup B|$.
    - $A \cap B$ = integers 1-100 divisible by 2 <u>and</u> by 5
    - $|A| = 50, |B| = 20, |A \cap B| = 10$ (Why?)
    - Thus, $|A \cup B| = 50 + 20 - 10 = 60$
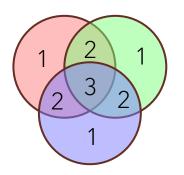
More inclusion-exclusion

What if there are three sets?

Or four sets?

Count the regions, then count how many times each region is covered by one of the main sets.

More inclusion-exclusion

- The outer sections are covered <u>once</u>, so we must start with $|A| + |B| + |C|$.

- The sections that are covered <u>twice</u> must be removed, giving $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$.

- But this adds the center section three times, and then removes it three times, and so we must add it back in: $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

- The pattern continues if there are more sets, alternating addition and subtraction.

Less-quick check in

- How many 8-bit strings have four consecutive 0s <u>or</u> four consecutive 1s? (*Note: The counting <u>before</u> you get to inclusion-exclusion will take some work.*)

  - Where do the consecutive 0s start?

    - 0000xxxx: 2^4=16 strings

    - 10000xxx: 2^3=8 strings

    - x10000xx: 2^3=8 strings

    - xx10000x: 2^3=8 strings

    - xxx10000: 2^3=8 strings

    - So there are 48 eight-bit strings with 4+ consecutive zeros.

Less-quick check in

- How many 8-bit strings have four consecutive 0s <u>or</u> four consecutive 1s? (*Note: The counting <u>before</u> you get to inclusion-exclusion will take some work.*)

  - There are 48 eight-bit strings with 4+ consecutive 0s.

  - By symmetry, there are 48 eight-bit strings with 4+ consecutive 1s.

  - How many strings have <u>both</u>?

    - Just two: 00001111 and 11110000

  - That makes our final answer 48 + 48 – 2 = 94 strings.

The most obvious theorem in FOCS

- The Pigeonhole Principle

  - If you have more objects than places to put them, then at least one of those places will end up with two or more objects.

  - That's it. That's the whole thing.

  - It is (almost) a straightforward contraposition of the Bijection Principle.

    - If you can put two sets into 1:1 correspondence, then they are the same size.

    - If two sets are not the same size, you cannot put them into 1:1 correspondence.

Some quick Pigeonhole Principle examples

- How many people do you need in a room before two of them were born in the same month?

- Every graph has a pair of vertices with the same degree.

- Any list of one hundred 27-digit positive integers will have two subsets that add up to the same value.

Some quick Pigeonhole Principle examples

- How many people do you need in a room to guarantee two of them were born in the same month?

  - 13. Once you have one per month, the next person must duplicate SOMEBODY.

Some quick Pigeonhole Principle examples

- Every graph with at least two vertices has a pair of vertices with the same degree.

  - With n vertices:

    - If the graph is connected, then the available degrees are $1 \ldots n-1$.

    - If the graph is disconnected, then the available degrees are $0 \ldots n-2$.

Some quick Pigeonhole Principle examples

- Any list of one hundred 27-digit positive integers will have two subsets that add up to the same value.

  - The maximum value of a 27-digit number is $< 10^{28}$.

  - Therefore, the maximum sum is $< 100 \cdot 10^{28}$, or 10^{30}.

  - There are 2^{100} possible subsets.

  - $2^{100} \approx 1.26 \times 10^{30}$

Questions?

REMINDER: Exam 2 on W 2/26

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