

Sample Problems in Discrete Mathematics

This handout lists some sample problems that you should be able to solve as a pre-requisite to Computer Algorithms. Try to solve all of them. You should also read Chapters 2 and 3 of the textbook, and look at the Exercises at the end of these chapters. If you are unfamiliar with some of these topics, or cannot solve many of these problems, then you should take a Discrete Math course before taking Computer Algorithms.

1 Using Mathematical Induction

The task: Given property $P = P(n)$, prove that it holds for all integers $n \geq 0$.

Base Case: show that $P(0)$ is correct;

Induction hypothesis: assume that for some fixed, but arbitrary integer $n \geq 0$, P holds for all integers $0, 1, \dots, n$

Induction step: prove that the induction hypothesis, $P(n)$, implies that P is true of $n + 1$: $P(n) \implies P(n + 1)$

Conclusion: using the principle of Mathematical Induction conclude that $P(n)$ is true for arbitrary $n \geq 0$.

Problem 1 Prove that for all $n > 10$

$$n - 2 < \frac{n^2 - n}{12}.$$

Proof: Define property $P(n)$ by

$$P(n) : \quad \forall k \leq n (k > 10; n > 10), \quad k - 2 < \frac{k^2 - k}{12}.$$

Then,

Base case: for $n = 11$,

$$11 - 2 < \frac{11^2 - 11}{12} \implies 9 < \frac{110}{12} = \frac{55}{6} = 9 + \frac{1}{6}.$$

Notice that the base of the induction proof start with $n = 11$, rather than with $n = 0$. Such a shift happens often, and it does not change the principle, since this is nothing more than the matter of notations. One can define a property $Q(m)$ by $Q(m) = P(n - 11)$, and consider Q for $m \geq 0$.

Induction step. Suppose, given $n \geq 11$, P holds true for all integers up n . Then

$$\begin{aligned} P(n) &\implies n - 2 < \frac{n^2 - n}{12} \\ &\implies (n + 1) - 2 - 1 < \frac{(n+1)^2 - (n+1) - 2n - 1 + 1}{12} \\ &\implies (n + 1) - 2 < \frac{(n+1)^2 - (n+1) - 2n - 1 + 1}{12} + 1 \\ &\implies (n + 1) - 2 < \frac{(n+1)^2 - (n+1)}{12} - \frac{2n}{12} + 1 \\ &\implies (n + 1) - 2 < \frac{(n+1)^2 - (n+1)}{12} \quad (\text{since } n > 10) \end{aligned}$$

The last inequality is $P(n + 1)$. □

Problem 2 For every integer $n \geq 1$,

$$\sum_{i=1}^n \sqrt{i} > 2n\sqrt{n}/3.$$

Proof: We prove it by induction on n .

Base. For $n = 1$, the left part is 1 and the right part is $2/3$: $1 > 2/3$.

Inductive step. Suppose the statement is correct for some $n \geq 1$; we prove that it is correct for $n + 1$.

Thus,

Given: $\sum_{i=1}^n \sqrt{i} > 2n\sqrt{n}/3$; **Prove:** $\sum_{i=1}^{n+1} \sqrt{i} > 2(n+1)\sqrt{n+1}/3$.

$$\sum_{i=1}^{n+1} \sqrt{i} = \left(\sum_{i=1}^n \sqrt{i} \right) + \sqrt{n+1} \tag{1}$$

Using the given inequality, we evaluate the right part of (1)

$$\left(\sum_{i=1}^n \sqrt{i} \right) + \sqrt{n+1} \geq 2n\sqrt{n}/3 + \sqrt{n+1}. \tag{2}$$

The following sequence of equivalent inequalities completes the proof:

$$2n\sqrt{n}/3 + \sqrt{n+1} \geq 2(n+1)\sqrt{n+1}/3 \tag{3}$$

$$2n\sqrt{n} + 3\sqrt{n+1} \geq 2(n+1)\sqrt{n+1} \tag{4}$$

$$2n\sqrt{n} \geq (2n-1)\sqrt{n+1} \tag{5}$$

$$4n^2n \geq (2n-1)^2(n+1) \tag{6}$$

$$4n^3 \geq (4n^2 - 4n + 1)(n+1) = 4n^3 - 3n + 1 \tag{7}$$

$$0 \geq -3n + 1. \tag{8}$$

The last inequality holds true for any $n > 0$. □

Problem 3 A set with an even number of elements is called even. Guess a formula for the number of even subsets of a set with n elements and prove the formula by induction.

Problem 4 For every integer $n \geq 0$,

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Prove.

Problem 5 For any integers a and b with $0 \leq a \leq b$, and every integer $n \geq 1$, $b^n - a^n$ is divisible by $b - a$.

Problem 6 Show that if n and k are integers with $1 \leq k \leq n$, then

$$\binom{n}{k} \leq \frac{n^k}{2^{k-1}}.$$

2 Recurrence Relations

Problem 7 Prove by induction that

$$F_1 + F_3 + \dots + F_{2n-1} = F_{2n},$$

where F_i denotes the i^{th} Fibonacci number.

Problem 8

Suppose that the sequence G_i is defined by $G_0 = 0$; $G_1 = 1$; $G_n = G_{n-1} + 2G_{n-2}$. Determine which of the following propositions are true (prove your answers):

- (a) $G_n = O(n^2)$
- (b) $G_n = G_0 + G_1 + \dots + G_{n-1}$
- (c) $G_n < 2^n$ ($n = 0, 1, \dots$)

3 Order of Functions

See Chapter 2 of the textbook, and the exercises therein.

Problem 9 Which of the statements below is correct and which is false? Prove your answers.

1. If $[f(n)]^2 = O(n)$ then $[f(n)]^4 = O(n^2)$;
2. If $2^{f(n)} = \Omega(2^n)$ then $f(n) = \Omega(n)$;
3. There are two monotonically increasing function $f(n)$ and $g(n)$ such that $f(n) \neq \Omega(g(n))$ and $g(n) \neq \Omega(f(n))$.

Solution.

All three statements are correct.

For (1). If $[f(n)]^2 = O(n)$, then $\exists C > 0, n_0 > 0$, such that $\forall n > n_0$, $[f(n)]^2 < Cn$. Then $\forall n > n_0$, $[f(n)]^4 < C^2n^2$, which proves statement 1.

For (2). If $2^{f(n)} = \Omega(2^n)$ then $\exists C > 0$ and $n_0 > 0$, such that $2^{f(n)} > C2^n$ for all $n > n_0$. We prove statement (2) by taking logarithm of the latter. \blacksquare

Problem 10 Determine whether the following is true or false:

(a) $(n+1)! = O(n!)$; (b) $n^3 = O(500n^2)$; Explain.

Problem 11

Rank the functions

$$h_1(n) = 2^{2^n}, \quad h_2(n) = (\sqrt{2})^{\ln n}, \quad h_3(n) = n!, \quad h_4(n) = n^{1/\ln n},$$

by order of growth, that is find an arrangement $h_{i_1}, h_{i_2}, h_{i_3}, h_{i_4}$ ($i_1, i_2, i_3, i_4 = \{1, 2, 3, 4\}$), satisfying

$$h_{i_1} = \Omega(h_{i_2}); \quad h_{i_2} = \Omega(h_{i_3}); \quad h_{i_3} = \Omega(h_{i_4}),$$

Problem 12

What is the complexity of the algorithm below which computes the binomial coefficients $\binom{n}{k}$? Is it exponential, linear, polynomial, ...?

The **binomial coefficient** $\binom{n}{k}$ —the number of distinct k -subsets of an n -set—is defined by

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n; \\ \binom{n-1}{k-1} + \binom{n-1}{k} & \text{if } 0 < k < n; \\ 0 & \text{otherwise.} \end{cases}$$

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function C(n, k);
1.  if (k > n)
2.      return 0;
3.  else
4.      if (k == 0) or (k == n)
5.          return 1;
6.      else
7.          return C(n - 1, k - 1) + C(n - 1, k);
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Problem 13

Prove the following statements.

1. $\lg(n!) = \Omega(n \lg n)$.
2. For all $a, b > 0$, $(\log n)^a = O(n^b)$.
3. If $f(n)$ is a polynomial of degree > 0 , then $O(\log(f(n)))$ and $O(\log n)$ are identical sets.

Problem 14

Let $p(n) = \sum_{i=0}^d a_i n^i$, where $a_d > 0$ and let k be a constant. Prove the following statements.

1. if $k \geq d$, then $p(n) = O(n^k)$.
2. if $k \leq d$, then $p(n) = \Omega(n^k)$.
3. if $k = d$, then $p(n) = \Theta(n^k)$.

4 Graph Theory

See also Chapter 3 of the textbook and the exercises therein.

Problem 15 A binary tree T is a rooted tree in which each node has at most two children. Let n be the total number of nodes of T , and let $g(T, i)$ be the number of nodes in the tree T with exactly i children, for $i = 0, 1$, or 2 . Show by induction that in any binary tree the number of nodes with two children is exactly one less than the number of leaves. In other words, show that $g(T, 2) = g(T, 0) - 1$.

Solution.

Base Case: If $n = 2$, then $g(T, 2) = 0$ but $g(T, 0) = 1$, as desired.

Inductive Step: Assume that $g(T, 2) = g(T, 0) - 1$ for all trees of size less than n . For a tree T of size n with root node r , we consider 2 cases.

Case 1: r has exactly 1 child. Let T_1 be the subtree below r . By the inductive hypothesis, we know that $g(T_1, 2) = g(T_1, 0) - 1$. Since r has exactly 1 child, we have that $g(T, 2) = g(T_1, 2)$ and $g(T, 0) = g(T_1, 0)$.
 Case 2: r has 2 children. Let T_1 be the left subtree below r , and T_2 be the right subtree below r . By the inductive hypothesis, we know that $g(T_1, 2) = g(T_1, 0) - 1$ and $g(T_2, 2) = g(T_2, 0) - 1$. In this case, $g(T, 2) = g(T_1, 2) + g(T_2, 2) + 1$, and $g(T, 0) = g(T_1, 0) + g(T_2, 0)$. Therefore, $g(T, 2) = g(T_1, 2) + g(T_2, 2) + 1 = g(T_1, 0) - 1 + g(T_2, 0) - 1 + 1 = g(T, 0) - 1$. \blacksquare

Problem 16

By analyzing the algorithm for Breadth-First-Search (BFS), show that in the case of a connected, undirected graph G , its every edge is either a tree edge (belongs to the BFS tree), or a cross edge (connects two vertices, neither is a descendant of the other in the BFS tree.)

Problem 17 Give an algorithm to detect whether a given undirected graph contains a cycle. If the graph contains a cycle, then your algorithm should return “true”, otherwise it should return “false”. (It does not need to output the cycle, just determine if one exists.) The running time of your algorithm should be $O(m+n)$ for a graph with n nodes and m edges.

Problem 18 Give a linear-time algorithm for the following task. Input: A directed acyclic graph G . Question: Does G contain a directed path that touches every vertex exactly once? (Hint: Topological Sort.)

Problem 19 The transitive closure of a directed graph $G = (V, E)$ is the graph $G^*(V, E^*)$ whose edge set is defined as follows

$$E^* = \{ij : \text{such that } G \text{ contains a directed path from } i \text{ to } j.\}$$

Construct an algorithm which in $O(|V|^2)$ time updates the transitive closure G^* of a graph $G(V, E)$ when a new edge xy is added to G .

Hint. Your algorithm will have to add to the graph the edge xy and some other edges. You might think of how these new edge are related to xy .

5 Additional Problems in Discrete Math and Logic

Problem 20 How many eight digit numbers are there that contain a 5 and a 6? Explain.

Problem 21 Find the smallest value of n for which $(n!)! > 1,000,000$. Explain.

Problem 22 What are the coefficients of the terms $\frac{1}{x}$, $\frac{1}{x^2}$ in the expansion of $(x + \frac{1}{x})^n$, where $(n > 2)$? Explain.

Problem 23 Let $A = \{a_1, a_2, \dots, a_p\}$ ($p \geq 3$) and let N_k denote the number of subsets of A whose size is k ($k = 1, 2, \dots, p$). Which of the following three statements is correct:
 (a) $N_2 > N_{p-2}$; (b) $N_2 < N_{p-2}$; (c) $N_2 = N_{p-2}$? Explain.

Problem 24 Describe an algorithm which computes x^{96} using at most seven multiplications.

Problem 25 Which of the following statements are true?

Dogs have wings only if cats have wings.	
Birds have wings only if cats have wings.	
If dogs have wings, then birds have wings.	
Snakes have legs iff cats have wings.	
If frogs have hair or mice have eyes, then sharks do not have teeth.	
If frogs have hair and mice have eyes, then sharks do not have teeth.	

Problem 26 Fill in the truth tables:

$\mathbf{P} \wedge \mathbf{Q}$	\mathbf{P} is true	\mathbf{P} is false	$\mathbf{P} \vee \mathbf{Q}$	\mathbf{P} is true	\mathbf{P} is false
\mathbf{Q} is true			\mathbf{Q} is true		
\mathbf{Q} is false			\mathbf{Q} is false		

If \mathbf{Q} then \mathbf{P}	\mathbf{P} is true	\mathbf{P} is false		true	false
\mathbf{Q} is true			\mathbf{P}		
\mathbf{Q} is false			$\neg \mathbf{P}$		