CSCI 6964 Fall 2007 | Advanced Algorithm Design

Problem Set 7 – Primal-Dual

Due November 30

(1) Consider the maximum weight matching problem. In this problem, we are given an undirected graph $G = (V, E)$, with nonnegative weights c_e on the edges. The goal is to find a matching (not necessarily perfect) of largest possible weight.

Here we will analyze the following greedy algorithm for maximum-weight matching. Start with an empty matching. At every step, find the edge of maximum weight that does not intersect any edges that we have already chosen, and add this edge to the matching. Stop when no more edges can be added. Let M_g be the matching found by this greedy algorithm, and let Z_g be its cost.

(a) Show that the optimal value Z_{LP} of the following LP is an upper bound on Z_{OPT}, the value of the optimum matching.

$$\min \sum_{i \in V} x_i$$
$$\forall (i,j) \in E \quad x_i + x_j \geq c_e$$
$$\forall i \in V \quad x_i \geq 0$$

(b) Construct a feasible solution to the above linear program from the greedy matching M_g, and show that the value of this solution is $2Z_g$. Conclude that the greedy algorithm is a 2-approximation algorithm by showing that $Z_g \geq \frac{1}{2} Z_{OPT}$.

(2) Consider the 2-approximation algorithm seen in class for the Steiner forest problem. In this problem, we are given a set $\{s_i, t_i\}$ of pairs of vertices and costs on the edges of a graph, and the goal is to find a subgraph (a forest) of minimum cost in which every pair of vertices (s_i, t_i) is connected.

(a) Argue that this problem is a generalization of the shortest s-t path problem (in an undirected graph with nonnegative edge weights).

(b)
- Does the algorithm seen in class produce a shortest s-t path in that case? If so prove it; if not, give a counterexample.
- Is the value ($\sum_S y_S$) of the dual solution $\{y_S\}$ constructed by this algorithm equal to the shortest path value? If so, prove it; if not, give a counterexample.

(3) Recall the min-cost spanning arborescence problem. In this problem, we are given a *directed* graph $G = (V, E)$ with a root node r, and edge costs $c_e \geq 0$. Our goal is to find a minimum-cost arborescence, i.e., a set of edges $T \subseteq E$ of minimum cost that contains a directed path to r from every other node v. We will now develop a primal-dual algorithm for this problem, and in fact show that the algorithm we saw in class was a primal-dual algorithm in disguise.

(a) Write an integer linear program equivalent to the min-cost spanning arborescence problem. You should have a variable for every edge e, but you are allowed to have an exponential number of constraints.

(b) Take the dual of the LP-relaxation.

(c) Give an optimal primal-dual algorithm for this problem, and prove that it is a "1-approximation", i.e, it gives an exact solution. This algorithm should follow the primal-dual template we gave in class: it should consist entirely of increasing the appropriate primal and dual variables, with a possible cleanup step at the end. *Hint:* In your proof, you are allowed to use results and lemmas that we proved in class.