Instructions: To solve these problems, you are allowed to consult your classmates, as well as the class textbook (The Design of Approximation Algorithms by Williamson and Shmoys) and notes, but no other sources. As with all future homeworks, you are allowed to write up your solutions in pairs, and turn in only one copy for each pair. Please write clearly and concisely.

[20 points] Consider the metric traveling salesman problem (see Section 2.4) with $n$ cities. We have seen several approximation algorithms for this problem, but now we will analyze another natural greedy algorithm, as follows. Let $x$ be an arbitrary city $i$ to start with. Now find the city closest to $x$ which we have not yet visited (say this city is $j$), and go there. In other words, set $x$ to be $j$, and add $c_{ij}$ to our tour. Continue in this manner; once all cities have been visited return to the starting city.

(a) Argue that, for any subset of cities $S$, if each pairwise distance between cities in $S$ is greater than some value $z$, then the cost of the optimum tour visiting all $n$ cities must be greater than $z|S|$. 

(b) Let $c(j)$ denote the cost of the edge added by this algorithm when $x = j$, i.e., it is an edge with minimum cost leaving $j$ to an unvisited city. Renumber the cities so that $c(1) \geq c(2) \geq \ldots \geq c(n)$. Prove the following claim: For any fixed $j$, we have that $c(j) \leq OPT/j$, where OPT is the cost of the optimum tour. Hint: use proof by contradiction and part (a).

(c) Use the claim from part (b) to prove that the above greedy algorithm gives a $O(\log n)$ approximation, where $n$ is the number of cities. Recall that TSP without the triangle inequality is inapproximable: at which point in the proof are you using the metric assumption?