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Strategic Multiway Cut and Multicut Games

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Abstract We consider cut games where players want to cut themselves off from different parts of a network. These games arise when players want to secure themselves from areas of potential infection. For the game-theoretic version of Multiway Cut, we prove that the price of stability is 1, i.e., there exists a Nash equilibrium as good as the centralized optimum. For the game-theoretic version of Multicut, we show that there exists a 2-approximate equilibrium as good as the centralized optimum. We also give poly-time algorithms for finding exact and approximate equilibria in these games.

Keywords Network Formation Games · Price of Stability · Approximate Nash Equilibrium · Multiway Cut · Multi-cut

1 Introduction and Model

Networked systems for transport, communication, and social interaction have contributed to all aspects of life by increasing economic and social efficiency. However, increased connectivity also gives intruders and attackers better opportunities to maliciously spread in the network, whether the spread is of disinformation, or of contamination in the water supply [27]. Anyone participating in a networked system may therefore desire to undertake appropriate security measures in order to protect themselves from such malicious influences.

We introduce a *Network Cutting Game*, which is a game-theoretic framework where a group of self-interested players protect vertices that they own by cutting them off from parts of the network that they find untrustworthy. Cutting an edge should not be interpreted as destroying a part of the network: instead it can correspond to taking security measures on that edge such as placing sentries on lines of communication. These notions are applicable in areas such as airline security. Consider a situation where country A requires extra security screening of passengers or cargo from country B. Due to the networked multi-hop structure of international air travel, the optimal locations for carrying out such screenings may lie somewhere in between the two countries. In general, the goal of each player is to make sure that nothing can get to her from an "untrustworthy" part of the network without passing through an edge with installed security measures.

The purpose of each player is to protect herself while spending as little money as possible for her security. We investigate the efficiency of the security actions taken by a group of agents by studying the *price of*

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anarchy and the *price of stability* — the ratios between the costs of the worst and best Nash equilibrium[1] respectively, and that of the globally optimal solution. While the price of anarchy can be extremely high (see Section 5), we prove nice bounds on the price of stability.

Game Definition We now formally define the *Network Cutting Game* as follows. We are given an undirected graph $G = (V, E)$, and a set P of k players. Each player i corresponds to a single *player node* in the graph G, which we will also denote by i. We alternatively refer to them as 'terminal nodes'. The strategy set of every player i is 2^E; a strategy $S_i \subseteq E$ of player i is the set of edges that player i will cut. The outcome of the game is G_S, which is a subgraph of G obtained by removing the edges of $\bigcup_i S_i$. Since there is a one to one correspondence between the cut $\bigcup_i S_i$ and resulting subgraph G_S, we abuse notation by also referring to the cut $\bigcup_i S_i$ as outcome of the game.

The objective of each player i is to protect her node i from a given subset of nodes T_i of V. We say that player i *satisfies her cut requirement* if i is disconnected from all nodes of T_i in G_S. Every player wants to satisfy her cut requirement, but also wants to minimize the number of edges she cuts, which we denote by $|S_i|$. If a player i does not satisfy her cut requirement, she faces a penalty cost of β_i. We can think of β_i as the maximum number of edges that i would be willing to cut in order to satisfy her cut requirement. We conclude the definition of our game by defining the cost function for each player i as:

- $cost(i) = |S_i|$ if player i satisfies her cut requirement,
- $cost(i) = |S_i| + \beta_i$ otherwise.

Nash Equilibrium and OPT A pure Nash equilibrium (NE) of the *Network Cutting Game* is a strategy vector $S = (S_1, \ldots, S_k)$ such that no player i has an incentive for unilateral deviation from her strategy, i.e., no player can reduce her cost by changing her strategy from S_i to another strategy S'_i, assuming all other players stay with their existing strategies. Notice that in an equilibrium no player will cut more than β_i edges, since this player could change her strategy to $S'_i = \emptyset$, and reduce her cost to at most β_i. By the same reasoning, all players that do not satisfy their cut requirements will play $S_i = \emptyset$ at equilibrium. Therefore, in a Nash equilibrium, all edges must be cut by players that satisfy their cut requirements.

We analyze the quality of Nash equilibrium solutions by comparing them with the cost of the socially optimal solution, which we refer to as OPT. The socially optimal solution is an outcome of the *Network Cutting Game* that minimizes the total cost of all the players (equivalently, maximizes social welfare). Let $Q(S)$ denote the set of players whose cut requirements are not satisfied in G_S. The cost of solution S is given by:

$$\text{cost}(S) = \left|\bigcup_i S_i\right| + \sum_{j \in Q(S)} \beta_j,$$

which is the same as $\sum_i cost(i)$ since for any equilibrium solution we can assume that all sets S_i are disjoint. When all β_i are large, OPT is exactly the smallest set of edges that satisfies all the cut requirements.

Let Θ be the set of all strategy vectors S that result in a Nash equilibrium. Then, price of anarchy is the ratio of the worst Nash equilibrium to the socially optimal solution:

$$PoA = max_{S \in \Theta} \frac{\text{cost}(S)}{OPT}.$$

Similarly, price of stability is the ratio of the best Nash equilibrium to the socially optimal solution:

$$PoS = min_{S \in \Theta} \frac{\text{cost}(S)}{OPT}.$$

[1] Recall that a (pure-strategy) Nash equilibrium is a solution where no single player can switch her strategy and become better off, given that the other players keep their strategies fixed.

Our Results In general, the *Network Cutting Game* may not have any pure Nash equilibrium [23], and the price of anarchy for even very simple instances can be as large as the number of players. Our main results are about the existence and computation of cheap, exact, and approximate equilibria for several important special cases of the *Network Cutting Game*. By an α-approximate Nash equilibrium, we mean that no player in such a solution can reduce her cost by a factor of more than α by deviating. Specifically, we consider the following special cases of cut requirements.

In the *Single-Source Network Cutting Game,* each player i wants to disconnect her node from a common node t, i.e., $T_i = \{t\}$ for all players i.

In the *Network Multiway Cut Game,* each player i wants to disconnect her node from the nodes of all other players, i.e., $T_i = P \backslash \{i\}$ for all players i. Notice that when β_i for all players is large enough, the socially optimal solution is exactly the *Minimum Multiway Cut.*

In the *Network Multi-Cut Game,* each player i wants to disconnect her node i from some specific node t_i. In other words, this is the case where $|T_i| = 1$ for all players. When β_i for all players is large enough, the socially optimal solution is exactly the *Minimum Multicut* for the set of pairs (i, t_i).

Our main results are as follows:

- In Section 2, we study the *Single Source Network Cutting Game* and show that there always exists a Nash equilibrium as cheap as OPT, i.e., the price of stability is 1. Furthermore, we show that Nash equilibrium can be computed in polynomial time. This analysis easily generalizes to the case where T_i are not singleton sets, but $T_i = T_j$ for all i, j, and also follows as a consequence of [20].
- In Section 3, we study the *Network Multiway Cut Game* and show that there always exists a Nash equilibrium as good as OPT, i.e., the price of stability is 1. Given an approximate solution to Multiway Cut (for example, a 1.34-approximation found by [24]), we show how to compute a Nash equilibrium with the same or smaller cost than this solution in polynomial time.
- In Section 4, we study the *Network Multi-Cut Game* and show that there always exists a 2-approximate Nash equilibrium as cheap as OPT.

In Section 5, we consider the above games on graphs with non-uniform edge costs, i.e., where each edge has some fixed cost $w(e)$ to cut it. If every edge must be bought entirely by a single player, we show that there are simple instances for all these games where all Nash equilibria are expensive. Because of this, we allow players to share the cost of cutting edges. If the players are allowed to pay for cutting only a portion of an edge, we prove that all the above results extend to non-uniform edge costs.

Related Work An extensively studied game related to cuts is the max-cut game [11,15,19,7]. In this game players are forming a bipartition on the graph, where the utility of each player is the total weight of the edges of the cut incident to her. The Max-cut game always admits Nash equilibria since it is a potential game [30] and it is recently shown that it always admits strong Nash equilibria [17].

A contrasting approach to the Network Cutting Game are Network Formation Games [1,14,28], where a set of players is collectively trying to built a network, i.e., players want to connect some subset of nodes, rather than cutting them apart. Players connect nodes to each other by building edges and sharing the cost of the edges. The most relevant network formation game to our model is the Connection Game [4,6, 13], where each player i wants to connect a pair of nodes to each other. Existence and quality of equilibria depend on the cost-sharing method used [10]. The cost-sharing scheme used for the Connection Game in [6] is commonly referred to as 'arbitrary-sharing' [3,20–22], which will be explained in detail in Section 5. Another popular cost-sharing scheme in Network Formation Games is commonly referred to as 'fair-sharing', which will also be explained in detail in Section 5. The Connection Game with fair sharing was first studied in [5] and later addressed in [9,16], among others.

There have been many interesting applications of game theory to network security models (eg. [18,29, 31]). A notion of *interdependent security* (IDS) games was introduced by Kunreuther and Heal [26]. In these games, the decision to adopt a security measure by a player affects other players in the network. An algorithm for finding an approximate equilibrium on this model was later given by Kearns and Ortiz [25]. Work on a similar model by Aspnes et al. [2] deals with players immunizing their nodes against infections that can spread in the network. There are several major differences between these models and the games we consider: (i) players in IDS games can only immunize themselves, while in our games players are allowed to add security to different parts of the network, and (ii) these models consider that an attack can occur randomly anywhere in the network, while in our games the players are trying to protect themselves from specific areas of the network which might be different for different players.

Some of our questions are related to ones studied by Engelberg et al. [12], who look at bugeted versions of cut problems like Multiway Cut, Multicut, and k-cut. This work considers centrally optimal solutions, not equilibria, and while our values β_i can be thought of as budgets, they are budgets for individual players, not global budgets on the cost of the network.

Finally, similar problems are examined by Hoefer [23] where he gives examples for *directed* Multi-Cut Games with no Nash equilibrium, and those examples can be extended to show high price of stability as well. However, the price of stability, as well as existence of Nash equilibrium, in *undirected* Network Multi-Cut Games (the ones considered in our paper) are unknown. For more general Network Cutting Games (not just Multi-Cut Games) defined in the Introduction, [23] gives examples where no Nash equilibrium exists, even for undirected graphs.

2 Preliminaries and Basic Results

Given a graph $G = (V, E)$, and two disjoint subsets $A \subseteq V$ and $B \subseteq V$, denote by $M(G, A, B)$ the set of minimum-size $A - B$ cuts. By an $A - B$ cut, we mean a set of edges E' such that A and B are disconnected by removing E' from G. Denote by $m(G, A, B)$ the size of a minimum $A - B$ cut in graph G. We will also abuse notation slightly, and for a strategy vector $S = (S_1, \dots, S_k)$, we will let S_{-i} denote $\cup_j S_j - S_i$, i.e., the set of edges cut by players other than i.

Proposition 1 *A strategy vector $S = (S_1, \dots, S_k)$ of the Network Cutting Game is a Nash equilibrium if and only if S_i are pairwise disjoint, and*

- $|S_i| = m(G - S_{-i}, i, T_i) \leq \beta_i$ *if cut requirement of player i is satisfied on G_S,*
- $S_i = \emptyset$ *and* $m(G - S_{-i}, i, T_i) \geq \beta_i$ *otherwise.*

Proof. It is easy to see that all S_i must be disjoint, since otherwise a player could reduce her cost without altering the outcome G_S.

Consider the best response of player i to the strategy vector S. The minimum number of edges that i must cut in order to fulfill its cut requirements is exactly $m(G - S_{-i}, i, T_i)$. Therefore, the cost of i's best response is exactly $\min\{m(G - S_{-i}, i, T_i), \beta_i\}$. This is because if $\beta_i < m(G - S_{-i}, i, T_i)$, then player i could achieve smaller cost by cutting nothing and incurring a cost of β_i.

Let i be a player such that G_S satisfies the cut requirement of i. First consider the case where $m(G - S_{-i}, i, T_i) \leq \beta_i$. Then, player i will be stable in solution S if and only if $|S_i| = m(G - S_{-i}, i, T_i)$, as desired. If instead $\beta_i < m(G - S_{-i}, i, T_i)$, then S cannot possibly be a Nash equilibrium, since for i's cut requirements to be satisfied in S, it must be that $|S_i| \geq m(G - S_{-i}, i, T_i) \geq \beta_i$, which means that player i could switch her strategy to $S'_i = \emptyset$ and reduce her cost to at most β_i.

Now, let i be a player such that G_S does not satisfy her cut requirement. As argued above, i will not cut any of the edges of G in an equilibrium, so it must be that $S_i = \emptyset$. Moreover, i is stable in solution S if and only if $m(G - S_{-i}, i, T_i) \geq \beta_i$, since otherwise player i can reduce her cost to $m(G - S_{-i}, i, T_i)$ from β_i by changing her strategy from S_i to $S'_i \in M(G - S_{-i}, i, T_i)$. ■

To add intuition about the structure of Nash equilibria, notice that when β_i are large (say $\geq |E|$) for all players, then the above proposition says that a solution S is a Nash equilibrium exactly when $|S_i| = m(G - S_{-i}, i, T_i)$ for all players.

2.1 Single-Source Network Cutting Game

In this section, we study the *Single Source Network Cutting Game*. Since for single-source cut games, the LP to compute OPT has an integrality gap of 1, the theorem below also follows from [20]. This analysis easily generalizes to the case where T_i are not singleton sets, but $T_i = T_j$ for all i, j.

Theorem 1 *For the Single Source Cutting Game, a Nash equilibrium is guaranteed to exist and the price of stability is* 1.

Proof. We show that there exists an assignment of the edges in OPT to the players such that it is a Nash Equilibrium. Add a super-source s to the input graph G, so that s is connected only to each player node $i \in P$. If we set the capacities of edges (s,i) to be β_i, and the capacities of all other edges to be 1, then it is easy to see that OPT is an $s-t$ min-cut in this new graph. More precisely, if we let S^* be the set of edges in OPT in the original graph G, and Q^* be the set of players that are still connected to t in OPT, then $S^* \cup \{(s,i)|i \in Q^*\}$ is a minimum s-t cut. This is simply because if we take a set of edges S in G so that the players Q are still connected to t after removing S, then the cost of the s-t cut $S \cup \{(s,i)|i \in Q\}$ is exactly the social cost of solution S.

Let the set of edges in this s-t min-cut be $M = S^* \cup \{(s,i)|i \in Q^*\}$ and let $|M|$ denote the cardinality of M. This also means that an integral flow of size $|M|$ can be pushed from s to t saturating all edges of M, since all capacities are integral (we can assume that all β_i are integral without loss of generality). The assignment of edges of S^* to the players is as follows. Note that we only assign edges to terminals i if M did not contain the edge of weight β_i. Now consider the flow leaving the super-source s. Because of the construction, all of this flow passes through the terminal nodes. Since all the flows are integral we can decompose them into unit flows. Now we can categorize these unit flows depending upon the terminals they pass through. A flow passing through edge (s,i) is marked as f_i. Since all edges of the original graph G are of unit capacity, any edge $e \in S^*$ will receive flow from exactly one terminal in $P \backslash Q^*$. We set S_i to be the set of edges of S^* that receive a flow f_i. We now show that such an assignment results in a NE.

We use Proposition 1 to show that the resulting strategy vector $S = (S_1, \ldots, S_k)$ is a Nash equilibrium. First consider a player $i \in Q^*$, i.e., a player that is connected to t in $G-S^*$. We did not assign it to pay for any edges of OPT, so $S_i = \emptyset$, as desired. Now suppose to the contrary that $m(G-S_{-i}, i, T_i) = m(G-S^*, i, t) < \beta_i$. Then consider the set of edges $S^* \cup C$, for $C \in M(G-S^*, i, t)$. The social cost of this set of edges is at most $|S^*| + |C| + \sum_{j \in Q^* \backslash i} \beta_j$ which is cheaper than OPT and hence a contradiction. Therefore, by Proposition 1, i will not deviate in solution S.

Now consider a player $i \notin Q^*$. To show that she will not deviate, we must prove that $|S_i| = m(G - S_{-i}, i, T_i) \leq \beta_i$. Consider $m(G - S_{-i}, i, t)$. This is the size of the maximum flow that can be sent from i to t in the graph $G - S^* + S_i$. Since M is a min-cut, and thus saturated by a maximum flow, we know that the flow f_i does not use any edges of $M - S_i$, and so it is still a valid i-t flow in $G - S^* + S_i$. Therefore, $|S_i| \leq m(G - S_{-i}, i, t)$. On the other hand, S_i is a i-t cut in the graph $G - S^* + S_i$, since S^* is an i-t cut in G. Therefore, $|S_i| \geq m(G - S_{-i}, i, t)$, as desired. Finally, notice that $|S_i| \leq \beta_i$, since $|S_i|$ is the size of a flow passing through edge (s,i), and the capacity of that edge is β_i. Therefore, by Proposition 1, i will not deviate in solution S. Thus this assignment results in a NE and the price of stability is 1. ∎

Notice that the equilibrium constructed in the proof can be easily found in poly-time using standard flow algorithms.

3 Network Multiway Cut Game

In the Network Multiway Cut Game (NMCG) each player i wants to disconnect itself from every other player provided that the cost of cutting edges does not exceed the player's budget β_i. As mentioned earlier, when β_i for every player is very large the socially optimal solution is the well-studied Multiway Cut (MWC) problem.

When the values of β_i are bounded, the socially optimal solution becomes the *prize-collecting* Mutiway Cut (P-MC) problem. Let T be the set of terminal nodes associated with the players. In this problem, every vertex $i \in T$ has an associated budget β_i. Given any set of edges X, a vertex i incurs a cost of 0 if she is disconnected from every other vertex in T by removing X, and a cost of β_i if she is not. Let $R(X)$ be the set of vertices that are disconnected from all other vertices of T by X. Then the total cost associated with X is given by

$$c(X) = \sum_{i \in T \backslash R(X)} \beta_i + |X|.$$

The optimal P-MC is a cut X that minimizes $c(X)$. Since the P-MC problem is a strict generalization of the Multiway cut problem, it is also NP-hard.

An instance of the prize-collecting Multiway cut problem (G, T, β) where β is the set of weights assigned to the terminals T, can be reduced to an instance of the Multiway cut problem in the following manner.

Construct a graph G' from G by adding vertex i' for every terminal $i \in T$ and an edge (i, i') with weight β_i. Let the set of i' vertices be T'. Now it is very easy to show that the weight of the optimal Multiway cut for the instance (G', T') will be the same as the cost of the optimal P-MC for instance (G, T, β). Using this reduction, an algorithm that computes an approximate Multiway cut can be used to compute a P-MC with the same approximation factor.

For the Network Multiway Cut game (NMCG), a player will not pay for cutting any edge in a payment strategy which does not satisfy its cut requirement. She will instead incur a cost of β_i for not being disconnected from all other terminals. $c(X)$ can thus be interpreted as the total cost to players when they pay for the cut X, including the cost to players that are not isolated.

Analysis Technique As we saw in Section 2.1 for the Single-Source Network Cutting Game, we can use flows to determine the payments for players in the game. One approach would be to create an analogous multi-commodity flow for the Network Multiway Cut Game, and assign payments based on this flow. Unfortunately, this approach *does not* yield the desired results, since for the payments to be stable, we would need that the size of the maximum flow is equal to the size of the minimum cut, which does not hold for multi-commodity flow problems.

Instead, in our approach we use the key observation proven in Lemma 1 to construct a *single-commodity* flow that lets us determine stable payments for the optimum solution. It should be noted that this construction does not find the minimum prize-collecting multiway cut, but only calculates stable payments for this cut if it is given. We can, however, use the same ideas to find cheap Nash equilibria in polynomial time (see Theorem 3).

Properties and Terminology The problem input consists of graph $G = (V, E)$, and a set of k terminals $T \subseteq V$ such that every player $i \in P$ is assigned one terminal. Each terminal\player i has an associated budget $\beta_i \in \beta$. For every player i, $T_i = T \backslash \{i\}$. We assume G is connected, since otherwise we can consider each connected component separately. We define the following terms with respect to a prize-collecting multiway cut X of (G, T, β).

Let the component containing terminal i be termed as $C_i(X)$. Note that if $i \in R(X)$ then the only terminal present in $C_i(X)$ will be i, whereas if $i \in T \backslash R(X)$ then $C_i(X)$ will contain other non-isolated terminals. We define $\delta_i(X)$ as the set of edges (u, v) such that $|C_i(X) \cap \{u, v\}| = 1$ and $\delta_{ij}(X)$ as the set of edges (u, v) such that $u \in C_i(X)$ and $v \in C_j(X)$. Terminal j is a neighbor of terminal i (denoted by $j \in N_i(X)$) if $\delta_{ij}(X) \neq \emptyset$. Let $G_i(X)$ be the subgraph defined as follows: $G_i(X) = G - X + \delta_i(X)$.

Let OPT be the set of edges in the optimal P-MC. For the sake of brevity, when defined for OPT we will refer to the above terms simply as C_i, δ_i, δ_{ij}, N_i and G_i respectively. Notice that OPT does not form any connected components without at least one terminal node (otherwise there would be a P-MC with smaller cost), and so every terminal node must be contained in some C_i. Also, given a graph G, let $E[G]$ correspond to the set of edges of G.

A crucial observation about the model is stated in the following lemma.

Lemma 1 *Recall that $M(G_i, i, N_i)$ is the set of minimum edge cuts between terminal i and its neighboring terminals in graph G_i. Then for every $i \in R(OPT)$, there exists $M \in M(G_i, i, N_i)$, such that the set of edges in M is a subset of $E[C_i] \cup \delta_i$.*

Proof. Consider $M'_i \in M(G_i, i, N_i)$ to be a cut such that it is not a subset of $E[C_i] \cup \delta_i$. Since M'_i does not completely lie in the subgraph $C_i \cup \delta_i$, there is at least one $j \in N_i$ such that $M'_i \cap E[C_j] \neq \emptyset$. Consider such a terminal j. Let M'_i divide the graph G_i into W_i and W_j such that W_i contains terminal i and W_j contains terminal j (see Figure 1). Let set $A = \delta_{ij} \cap E[W_i]$ be the set of edges of δ_{ij} that lie inside W_i, and set $B = M'_i \cap E[C_j]$ be the set of edges of M'_i that lie in the component C_j. We make the following observations on graph G_i.

Observation 1 *The set of edges $Y = (M'_i \backslash B) \cup A$ is a valid (i, N_i) cut in G_i.*

Proof. Consider an arbitrary path p from j to i in G_i. As we trace this path starting from terminal j, consider the edge e when p enters the component W_i for the last time. By definition $e \in M'_i$.
Case 1: If e belongs to the set $M'_i \backslash B$ then e also belongs to Y. This means that all such paths p will be cut by Y.

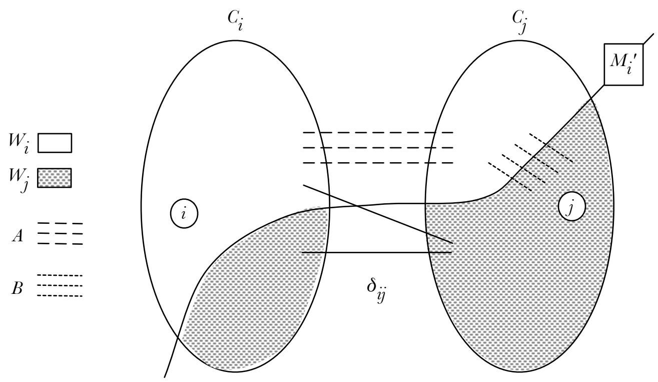

Fig. 1 Illustration of the Cuts and Components in the proof of Lemma 1.

Case 2: If e belongs to B then the path p enters the component $C_j \cap W_i$. Since there is no edge in G_i between the components C_j and C_d for $d \in N_i \backslash j$ and path p does not enter side W_j again, then the path has to travel across an edge of set A in order to reach terminal i. But $A \subseteq Y$ and hence all such paths p will also be cut by Y.

Therefore all (i, j)-paths are cut by Y. We still need to prove that Y also cuts all (i, d)-paths. As we trace a path starting from terminal d, consider the edge e when it enters the component W_i for the last time. Then the two cases considered above hold in the same fashion thus proving that Y is a valid (i, N_i) cut. ■

By a symmetric argument we can also prove the following claim.

Observation 2 *The set of edges $OPT' = (OPT \backslash A) \cup B$ is a P-MC such that $R(OPT') = R(OPT)$.*

Let us now compare the size of the sets A and B. If $|A| < |B|$ then by using Observation 1 we can construct a (i, N_i) cut that is smaller in size than M_i'. This contradicts our assumption that M_i' is a minimum (i, N_i) cut. If $|B| < |A|$ then according to Observation 2 we can construct a P-MC that costs less than OPT which is also a contradiction.

These observations lead us to the conclusion that $|A| = |B|$. As described in Observation 1, replacing the edges of B with A in the cut M_i' gives a valid (i, j) cut. And since $|A| = |B|$, the size of the new cut is the same as the original cut. This procedure can be applied to all $j \in N_i$ thereby resulting in a cut which belongs to the set $M(G_i, i, N_i)$ and also lies in the subgraph $C_i \cup \delta_i$. This completes the proof of Lemma 1. ■

Following is a useful observation about the optimal solution of the prize-collecting Multiway Cut.

Observation 3 *If OPT is the optimal P-MC for the instance (G, T, β), then for any $S \subseteq OPT$, $OPT - S$ is an optimal P-MC for the instance $(G - S, T, \beta)$.*

Proof. As we will be referring to two different graphs (G and $G - S$) in this discussion, we use the notation for cost and set of isolated terminal nodes w.r.t. to a particular graph by including it in the subscript: $c_G()\,, R_G()$.

Supposing to the contrary, let $OPT_S \neq OPT - S$ be the optimal P-MC in the graph $G_S = G - S$ with terminals T such that $c_{G_S}(OPT_S) < c_{G_S}(OPT - S)$. Now consider the P-MC $OPT_S + S$ for the original prize-collecting multiway cut problem (G, T, β). Since all edges of S are included in the cut $OPT_S + S$, the set of terminals in $R_{G_S}(OPT_S) = R_G(OPT_S + S)$. This leads us to the following set of inequalities:

$$\begin{aligned} c_G(OPT_S + S) &= c_{G_S}(OPT_S) + |S| \\ &< c_{G_S}(OPT - S) + |S| \\ &= c_G(OPT) \qquad \ldots \text{Since } R_{G_S}(OPT - S) = R_G(OPT). \end{aligned}$$

But this is a contradiction, hence no such OPT_S exists. ■

We can now state the main theorem of the section.

Theorem 2 *The price of stability for the Prize-collecting Network Multiway Cut problem is* 1*, i.e. there exists an assignment of the edges of OPT to the players that forms a Nash Equilibrium.*

In order to prove the theorem we make use of the following construction. For a P-MC X of an instance (G, T, β), we construct a directed flow graph $F(G, T, X)$ as follows: Construct source node s and sink node t. For every edge $e = (u, v) \in X$, construct vertices m_u, n_v and a directed edge (m_u, n_v) of capacity 1. Construct a directed edge (n_v, t) of capacity ∞ for every such n_v. For all $i \in R(X)$, add components $C_i(X)$ to this graph and make all of their edges bi-directed with capacity 1. Add an edge (s, i) for every terminal $i \in R(X)$ with capacity β_i. For every $e = (u, v) \in \delta_i(X)$, construct a directed edge (u, m_u) with capacity ∞. Notice that the edges in this flow graph $F(G, T, X)$ have capacities whereas our original graph G does not. Hence for this graph given a set of edges A, let $w(A)$ be the sum of capacities of edges in A.

Now consider a finite capacity $s - t$ cut χ for $F(G, T, X)$. Observe that except for (s, i) edges, all edges of $F(G, T, X)$ with finite capacity have a corresponding edge in the original graph G. This means that all the non-(s, i) edges of χ can be mapped to the original graph G. Denote these edges to be the set χ' in G.

Lemma 2 *Let χ be a finite capacity $s - t$ cut in $F(G, T, X)$. Then χ' is a P-MC for (G, T, β) such that if $i \in R(X)$ and $(s, i) \notin \chi$ then $i \in R(\chi')$ and if $i \notin R(X)$ then $i \notin R(\chi')$. Moreover,*

$$c(\chi') \leq \sum_{i \in T \setminus R(X)} \beta_i + w(\chi).$$

Proof. Note that the edges corresponding to X in $F(G, T, X)$ form a valid $s - t$ cut and hence there always exists a finite capacity $s - t$ cut in $F(G, T, X)$. Consider a terminal $i \in R(X)$ such that $(s, i) \notin \chi$. Also consider an edge $e = (u, v) \in \delta_i(X)$. Since the edges (u, m_u) and (n_v, t) have infinite capacities, χ will cut all (i, u) paths or the edge (m_u, n_v) itself. When mapped to the original graph G, this observation implies that for any such terminal i and any $(u, v) \in \delta_i(X)$, every (i, v) path in $C_i(X)$ will be cut by χ'. Since for any $i \in R(X)$ and $j \in N_i(X)$, an (i, j) path traverses through at least one edge in $\delta_i(X)$, all (i, j) paths will be cut by χ' in $F(G, T, X)$. This implies that $i \in R(\chi')$. Since all $i \notin R(X)$ are not affected by the cut χ', it also implies that if $i \notin R(X)$ then $i \notin R(\chi')$.

All terminals that are not isolated by the P-MC X do not feature in the flow graph $F(G, T, X)$ and hence remain non-isolated in the cut made by χ'. Let $\gamma(\chi)$ be all terminals j such that the $(s, j) \in \chi$. The terminals in $\gamma(\chi)$ may or may not be isolated by the cut χ' in (G, T, β). We can also observe that $w(\chi) = |\chi'| + \sum_{i \in \gamma(\chi)} \beta_i$. Now consider the cost of the cut χ':

$$\begin{aligned} c(\chi') &= \sum_{i \in T \setminus R(\chi')} \beta_i + |\chi'| \\ &\leq \sum_{i \in T \setminus R(X)} \beta_i + \sum_{i \in \gamma(\chi)} \beta_i + |\chi'| \\ &= \sum_{i \in T \setminus R(X)} \beta_i + w(\chi). \end{aligned}$$

Hence proved. ■

Lemma 3 *Consider the flow graph $F(G, T, OPT)$ for the socially optimal solution, OPT of (G, T, β). If θ is the set of edges corresponding to the optimal cut OPT in $F(G, T, OPT)$, then θ is a minimum $s - t$ cut for $F(G, T, OPT)$.*

Proof. Supposing to the contrary, let χ be a minimum $s - t$ cut of $F(G, T, OPT)$ such that $w(\chi) < w(\theta)$. Notice that $w(\theta) = |\theta|$ since capacities of all edges of θ are equal to 1. It follows that total cost to all players in the optimal solution is

$$c(OPT) = \sum_{i \in T \setminus R(OPT)} \beta_i + w(\theta).$$

Now if we consider the cut χ' corresponding to χ in (G, T, β) then by Lemma 2, the total cost to players for this cut will be given by

$$c(\chi') \leq \sum_{i \in T \setminus R(OPT)} \beta_i + w(\chi).$$

Since we assumed that $w(\chi) < w(\theta)$, it implies that χ' is a better solution to (G, T, β) than OPT which is a contradiction. Hence, any minimum $s-t$ cut of $F(G, T, OPT)$ will be of capacity at least $w(\theta)$. Also the capacity of the minimum $s-t$ cut cannot be bigger than $w(\theta)$ since we know that θ forms a $s-t$ cut in the graph $F(G, T, OPT)$. ■

Lemma 3 implies that the maximum $s-t$ flow in $F(G, T, OPT)$ will saturate edges of θ. Then the assignment of edges S_i to terminal/player i can be made as shown in Algorithm 1.

Data: Flow Graph $F(G, T, OPT)$
Result: Assignment of edges to players
Find the maximum integral $s-t$ flow;
Mark the flows that pass through terminal i as f_i;
All edges of OPT that carry flows marked f_i will be assigned to S_i;

Algorithm 1: Algorithm that assigns edges of OPT to players

Since OPT forms the minimum capacity $s-t$ cut, we know that all edges of OPT will be assigned to players. Note that by construction of $F(G, T, OPT)$, the only edges assigned to player i will be edges in δ_i, since these are the only edges that can carry flow passing through node i. Now all we need to prove is that this assignment will form a Nash Equilibrium, which we do in the following lemma.

Lemma 4 *The assignment made by Algorithm 1 forms a Nash Equilibrium.*

Proof. We first analyze the payments made by players $i \in R(OPT)$. Consider the graph $G'_i = G - OPT + S_i$. Since $S_i \subseteq \delta_i$, we know that i is disconnected from all terminals in G'_i, except possibly those in N_i. Thus the best response for player i, given a strategy decided by the above algorithm, will be a cut $M \in M(G'_i, i, N_i)$. The assignment algorithm chooses S_i that is a subset of OPT and consequently Observation 3 tells us that S_i will be an optimal solution for the P-MC instance (G'_i, T, β). Let δ_i and C_i defined for the optimal P-MC of instance (G'_i, T, β) be represented as δ'_i and C'_i respectively. It is clear that δ'_i is in fact the set S_i and so C'_i is same as original component C_i. Using Lemma 1, we can now show that there exists an $M \in M(G'_i, i, N_i)$ which lies in the subgraph $\delta'_i \cup C'_i$. But according to the assignment algorithm there exists a flow of capacity $|S_i|$ from terminal i to δ_i in the subgraph $C_i \cup S_i$. Hence $m(G'_i, \{i\}, N_i)$ is at least $|S_i|$. It follows that if player i chooses to deviate then it will have to pay at least $|S_i|$ to disconnect itself. Therefore the assignment forms a Nash Equilibrium for all such players.

Now consider the players $i \in T \backslash R(OPT)$. For all such players $S_i = \emptyset$. This means that $G'_i = G - OPT$ and their best response would be a cut of capacity $m(G, \{i\}, N_i)$. If $m(G, \{i\}, N_i) < \beta_i$ then by adding any cut $M \in M(G, i, N_i)$ to OPT, the total cost faced by all players would strictly reduce thus giving us a solution better than OPT. This is clearly a contradiction. Hence $\beta_i \leq m(G, \{i\}, N_i)$ for all such players. It follows that there does not exist an improving deviation for such players.

Hence the assignment by Algorithm 1 forms a Nash equilibrium. ■

3.1 Poly-time computable Nash Equilibrium

Our algorithm in the previous section depends on the knowledge of the optimal prize-collecting Multiway cut for a given problem. Since this is often computationally infeasible due to the NP-hardness of the Multiway Cut problem, we give an algorithm that efficiently computes a Nash equilibrium whose cost is no larger than any given prize-collecting Multiway Cut. As explained earlier, we can easily obtain an $\alpha-$approximate P-MC using a polynomial-time $\alpha-$approximation algorithm for Multiway Cut [8,24]. We can then obtain a Nash equilibrium that is no more expensive than this solution. Currently, the best known approximation algorithm for Multiway Cut is given by [24] and returns a 1.34-approximate solution.

Theorem 3 *Given a P-MC X, we can find a Nash equilibrium whose social cost is no more than the cost of X in polynomial time.*

Proof. We begin by proving the following lemma.

Lemma 5 *For any P-MC X for the problem instance (G, T, β), we can form a NE payment strategy for X given that the following two conditions hold:*

(i.) *Capacity of the minimum $s-t$ cut in flow graph $F(G, T, X)$ is the same as the size of X.*
(ii.) *If $S(X)$ is the payment strategy devised by Algorithm 1 on X, then for every terminal i there exists a cut $M \in M(G - S_{-i}(X), i, N_i(X))$ such that M lies completely within the component $C_i(X) \cup S_i(X)$.*

Proof. If the first condition holds then we know from the proof of Theorem 2 that Algorithm 1 will be able to devise a payment strategy for all edges of X. The size of the best response strategy of terminal i for strategy $S(X)$ is $m(G - S_{-i}(X), i, N_i(X))$. But because there exists a flow of size $|S_i(X)|$ from terminal i to $S_i(X)$ within the component $C_i(X) \cup S_i(X)$ and given the fact that the second condition holds, it follows that $|S_i(X)| \leq m(G - S_{-i}(X), i, N_i(X))$. Therefore the payment strategy will form a Nash Equilibrium. ■

The following algorithm can now be used to form a cheap Nash equilibrium solution in polynomial time.

1. Check for condition $(i.)$
   1.1 If $(i.)$ does not hold then find a cheaper P-MC X and go to step 1.
2. Check for condition $(ii.)$
   2.1 If $(ii.)$ does not hold then find a cheaper P-MC X and go to step 1.
3. Since both conditions of Lemma 5 hold, use Algorithm 1 on P-MC X to form a Nash equilibrium

To finish the algorithm, we must show how to efficiently create a cheaper P-MC when one of the conditions in Lemma 5 does not hold. If condition $(i.)$ does not hold true for X, then we can use Lemma 2 to construct a P-MC which is cheaper than X. Condition $(ii.)$ can be checked in polynomial time using standard flow methods. If condition $(ii.)$ does not hold then we can use the same arguments as in Observation 2 to construct another P-MC which is cheaper than X. When both conditions are satisfied, we know from Lemma 5 that a NE strategy can be obtained.

Each time a new P-MC is constructed by the algorithm, Observation 2 and Lemma 2 tell us that the number of terminals isolated by the cut never increases. This means that every new P-MC has either smaller number of edges or it isolates fewer terminals or both. Since the number of terminals and edges are finite the algorithm can consider at most $|E| + k$ P-MCs. Also, for every P-MC it takes polynomial time to compute a new cheaper P-MC or to find the final assignment, therefore the algorithm terminates in polynomial time. This proves the theorem. ■

4 Network Multicut Game

In the *Network Multicut Game* each player i wants to be cut from a particular node t_i of G provided that the cost of the strategy S_i of player i does not exceed β_i. Recall that Network Multicut Game is a special case of the Network Cutting Game, where $T_i = \{t_i\}$ is singleton for every player. As pointed out before, the socially optimal solution is the minimum cost multicut for the pairs $\{(1, t_1), (2, t_2), \ldots, (k, t_k)\}$ if β_i values are large enough. In general, OPT is a solution that minimizes the total cost of all the players.

For the Network Multicut game, we don't know whether there always exists a Nash equilibrium or not. In this section, we instead prove the following theorem.

Theorem 4 *There always exists a 2-approximate Nash equilibrium for the Network Multicut Game that is as cheap as the socially optimal solution.*

We prove this theorem by giving an algorithm that takes the edges of OPT as the input, and returns an assignment of edges of OPT to players that is a 2-approximate Nash equilibrium. First, we will give some properties of Nash equilibria that are as cheap as OPT for the Network Multicut Game, which enables us to give a simple algorithm that constructs a 2-approximate Nash equilibrium on the edges of OPT. Specifically, we first form stable payments locally for every pair of neighboring connected components in OPT, and then combine these payments to form a global 2-approximate Nash equilibrium.

Let C_j and C_k be two arbitrary components of OPT and let δ_{jk} denote the set of edges of G that are between C_j and C_k. Since C_j and C_k are cut apart in OPT, all the edges of δ_{jk} are cut in OPT. We call that C_j and C_k are neighbor components if $\delta_{jk} \neq \emptyset$. With the help of this notation, we can write the following simple observations.

Lemma 6 *For any two neighbor components C_j and C_k of OPT, there exists a player i such that either $i \in C_j$ and $t_i \in C_k$ or vice versa.*

Proof. For the purpose of contradiction, assume the contrary of the statement, i.e., there exists neighbor components C_j and C_k of OPT such that there is no player i for which either $i \in C_j$ and $t_i \in C_k$ or vice versa. But then one can obtain a new solution that cuts less edges than OPT by simply merging the components C_j and C_k. Notice that any player l that is cut from t_l in OPT is also cut from t_l in the new solution. Therefore, the new solution is cheaper than OPT. ■

Lemma 7 *Let i be a player such that $i \in C_j$ and $t_i \in C_k$ for 2 arbitrary neighbor components C_j and C_k. Then player i cannot cut any edge that is not in δ_{jk} in any Nash equilibrium that is as cheap as OPT.*

Proof. For the purpose of contradiction, assume player i cuts an edge $e \notin \delta_{jk}$ as part of her strategy S_i in a Nash equilibrium solution (S_i, S_{-i}) that is as cheap as OPT. Assume player i unilaterally deviates from her strategy S_i to a different strategy $S'_i = S_i - e$ of her, i.e., she cuts exactly the same set of edges in S_i but e. Notice that in the solution (S'_i, S_{-i}), i and t_i are still cut apart and player i cuts less edges than she does in (S_i, S_{-i}) and therefore, (S_i, S_{-i}) is not a Nash equilibrium solution. ■

Lemma 8 *In a solution where the edges of OPT are cut, and a player i is not assigned any edge, i.e., $S_i = \emptyset$, then i cannot reduce her cost by unilateral deviation (even in the case that β_i is finite).*

Proof. The statement is trivially true if i and t_i are in different connected components of OPT, since $cost(i) = 0$. Therefore, assume both i and t_i are in the same connected component C_j of OPT and $cost(i) = \beta_i$. If $m(C_j, i, t_i) \geq \beta_i$ player i cannot reduce her cost by unilateral deviation since she has to cut at least $m(C_j, i, t_i)$ to satisfy her cut requirement. If $m(C_j, i, t_i) < \beta_i$, player i can reduce her cost by playing the strategy S'_i that cuts the edges of a minimum-size $i - t_i$ cut instead of strategy $S_i = \emptyset$. Observe that (S'_i, S_{-i}) satisfies the cut requirements of all the players whose cut requirement is satisfied in (S_i, S_{-i}). Therefore, no player's cost will increase if player i deviates to S'_i, which contradicts the fact that OPT is the socially optimal solution. ■

To prove the existence of a 2-approximate Nash equilibrium as cheap as OPT, we give an algorithm that assigns all the edges of OPT to the players and proves that no player can reduce her cost by more than half by unilaterally deviating from the strategy where she cuts the edges assigned to her by the algorithm. If a player i is not cut from t_i in OPT, then our algorithm does not assign any edge to player i, i.e., $S_i = \emptyset$. If i and t_i are in different connected components of OPT, say C_j and C_k respectively, then our algorithm assigns a subset of δ_{jk} to player i. Notice that if C_j and C_k are not neighbor components then $S_i = \emptyset$.

A player i may have an incentive for unilateral deviation from her strategy S_i only if i and t_i are in different neighboring connected components of OPT because of Lemma 8. Let $G - S_{-i}$ denote the subgraph of G where the edges cut by other players are removed and let i be a player such that $0 < |S_i| \leq \beta_i$. Observe that a best deviation of player i from her strategy S_i, which is denoted by $\chi_i(S_i)$, is a cheapest strategy that cuts i from t_i in $G - S_{-i}$. The cost of the best deviation of player i from strategy S_i, i.e., $|\chi_i(S_i)|$, is as much as $m(G - S_{-i}, i, t_i)$. Let G_{jk} be the subgraph of G, which is composed of the connected components C_j and C_k of OPT and edges δ_{jk}. Since player i does not cut any edge of OPT that is not an element of δ_{jk}, then $OPT - \delta_{jk} \subset S_{-i}$ and therefore, $m(G - S_{-i}, i, t_i) = m(G_{jk}, i, t_i) = |\chi_i(S_i)|$.

Algorithm Let L_{jk} be the set of players i such that either $i \in C_j$ and $t_i \in C_k$ or vice versa. Without loss of generality let $L_{jk} = \{1, 2, ..., |L_{jk}|\}$. Recall that $L_{jk} \neq \emptyset$ by Lemma 6. Notice that the socially optimal solution of the Network Multicut Game for players L_{jk} on G_{jk} (which we denote by $OPT(G_{jk}, L_{jk})$) is δ_{jk}. This is by the same argument as in Observation 3. For every δ_{jk}, we make one pass over the players L_{jk}. For player 1 of L_{jk}, we send a max-flow from node 1 to t_1 on G_{jk} (If the size of the max-flow is more than β_1 then we just send an arbitrary flow of size β_1). Let m_1 denote the subset of edges of δ_{jk} that are used by that flow. Notice that $|m_1| = \min\{m(G_{jk}, 1, t_1), \beta_1\}$. The algorithm asks player 1 to cut the edges of m_1, i.e., sets $S_1 = m_1$ and proceeds with player 2. We send a max-flow from 2 to t_2 on $G_{jk} - m_1$ (Similarly, if the size of the max-flow is more than β_2 then we just send an arbitrary flow of size β_2) and ask player 2 to cut the edges m_2, the subset of $\delta_{jk} - m_1$ that are used by that flow and so on.

Let M denote the subset of the edges of δ_{jk} that are cut by the players at the end of the one pass described in detail above, i.e., $M = \bigcup_{i \in L_{jk}} m_i$. Notice that if $M = \delta_{jk}$ for all neighbor components C_j and C_k then the above algorithm will return a Nash equilibrium. This is because $|m_i| = m(G - OPT + m_i, i, t_i)$ for all i since the flow we used to construct m_i does not use any edges of OPT except m_i, and thus $|m_i| \leq \min\{m(G - S_{-i}, i, t_i), \beta_i\}$ which by Proposition 1 implies that i is stable.

However, it may be that $M \neq \delta_{jk}$ and so the greedy algorithm given above does not always return a Nash equilibrium. Fortunately, we will show that $|M| \geq |\delta_{jk}|/2$ in Lemma 9. We next assign the remaining edges of δ_{jk} to the players L_{jk} proportionally to the number of edges they are assigned so far. Since $|M| \geq |\delta_{jk}|/2$, then any player $i \in L_{jk}$ which has been assigned $|m_i|$ edges is now assigned at most $|m_i|$ extra edges. The assignment given by the algorithm is a 2-approximate Nash equilibrium, since the cost of the best deviation of each player $i \in L_{jk}$ is $\min\{m(G - OPT + S_i, i, t_i), \beta_i\}$, which is at least $|m_i|$ by the above argument.

Lemma 9 $|M| \geq |\delta_{jk}|/2$.

Proof. The critical observation is that $|OPT(G_{jk} - M, L_{jk})| - |OPT(G_{jk} - M, L_{jk} - \{1\})| \leq m_1$. For the purpose of contradiction, assume the inequality does not hold. If the max-flow between 1 and t_1 on G_{jk} is less than β_1, then $m(G_{jk}, 1, t_1) = |m_1|$. Observe $m(G_{jk} - M, 1, t_1) \leq |m_1|$ since player 1 cannot send a bigger flow in a smaller graph. Then one can obtain a cheaper solution for the Network Multicut Game for the set of players L_{jk} on the graph $G_{jk} - M$ than $OPT(G_{jk} - M, L_{jk})$ by first cutting the edges of $OPT(G_{jk} - M, L_{jk} - \{1\})$ and then cutting a min-cut between 1 and t_1 on the remaining edges. If the max-flow between 1 and t_1 on G_{jk} is at least β_1, then $|m_1| = \beta_1$. Then one can obtain a cheaper solution for the Network Multicut Game for the set of players L_{jk} on the graph $G_{jk} - M$ than $OPT(G_{jk} - M, L_{jk})$ by only cutting the edges of $OPT(G_{jk} - M, L_{jk} - \{1\})$. Note that in this solution player 1 does not cut any edges and faces a cost of β_1.

With the same argument one can show $|OPT(G_{jk} - M, L_{jk} - \{1\})| - |OPT(G_{jk} - M, L_{jk} - \{1, 2\})| \leq m_2$ and so on. Finally, we have $|OPT(G_{jk} - M, L_{jk} - \{1, 2, ..., (|L_{jk}| - 1)\})| - |OPT(G_{jk} - M, \emptyset)| \leq m_{|L_{jk}|}$. Summing up all the telescoping inequalities, we obtain $|OPT(G_{jk} - M, L_{jk})| \leq |M|$. It is also clear that $|OPT(G_{jk}, L_{jk})| - |OPT(G_{jk} - M, L_{jk})| \leq |M|$. Therefore, $|\delta_{jk}| = |OPT(G_{jk}, L_{jk})| \leq 2|M|$. As desired, this proves that at least half of the edges of δ_{jk} are cut after one pass over the players of L_{jk}. ∎

5 Edges with Non-Uniform Costs

In the previous sections, the cost to player i for cutting edges S_i was just the number of edges $|S_i|$. We now consider a generalized version of our games where the edges have positive edge weights/costs $w(e)$. In order for an edge e to be cut, players will have to pay the weight $w(e)$ of the edge. That is, the cost to player i is no longer just the number of edges $|S_i|$, but their total weight which we represent by $w(S_i)$. Similarly, the cost of a strategy vector S is now $\text{cost}(S) = w(S) + \sum_{j \in Q(S)} \beta_j$, where $Q(S)$ is defined as in the Introduction.

For this more general model with non-uniform edge costs, we first show that if the weight of an edge cannot be split between players, then the price of stability can be very high. By allowing players to share the cost of edges, however, we are able to extend most of our results to this general case.

No Cost Sharing The game defined above does not allow players to share the cost of an edge, since if an edge $e \in S_i$, then the cost of player i increases by the full weight $w(e)$ of the edge. We can show that in this case the price of stability can be as high as k, the number of players. Consider an example of the Single-Source Cutting game in Figure 2. The children represent the player nodes and the root t is the node that all players want to disconnect from. We assume that $\beta_i > 1$ for all i. In this game, no player would cut the edge costing $1 + \epsilon$ since cutting the adjacent unit weight edge would satisfy the player's cut requirement at a lower cost. Thus the game has only one Nash Equilibrium, the cost of which is k, whereas the socially optimal solution costs $1 + \epsilon$. The price of stability is clearly k. This result also holds for the Network Multi-cut Game since it is a generalization of the Single-Source Cutting Game. Hence the following proposition holds:

Proposition 2 *In weighted Single-Source Cutting games with no cost sharing allowed, the price of stability can be as large as k, the number of players.*

Notice that even if the edges did not have weights, this example shows that the *price of anarchy* can be as high as k. Therefore:

Proposition 3 *The price of anarchy for unweighted Single-Source Cutting games can be as large as k, the number of players.*

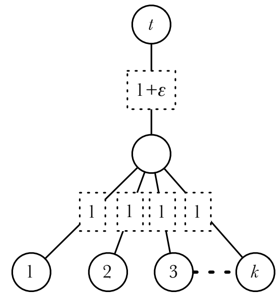

Fig. 2 Example showing lower bounds on the Price of Anarchy for uniform edge weights, and Price of Stability for non-uniform edge weights.

If we allow players to share costs of cutting edges, then the results become much nicer. Specifically, we consider fair sharing and arbitrary sharing schemes.

Fair Sharing In this model, the players that cut an edge e split the cost of this edge equally among themselves. Specifically, for a given strategy vector S, define k_e to be the number of players i such that $e \in S_i$. Then, player i only pays $w(e)/k_e$ for cutting edge e, i.e., player i's cost is $\sum_{e \in S_i} w(e)/k_e$ when i's cut requirements are satisfied. All the games that we considered in Section 2, 3, and 4 are congestion games [5] under the fair sharing scheme. Using standard techniques [5], it can be shown that the price of stability is $O(\log k)$. Unfortunately, it can also be as high as $\Omega(\log k)$: just consider the example in Figure 2 with weights $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}$ on the bottom edges instead of 1. Thus the following proposition holds.

Proposition 4 *In weighted Single-Source Cutting games under fair sharing scheme, the price of stability can be as large as $\Omega(\log k)$.*

Arbitrary Cost Sharing In this model players can choose to pay for arbitrary fractions of edge weights. Specifically, the strategy of each player i is a payment function S_i where $S_i(e)$ is the amount player i pays for the cost of edge e. An edge e is considered cut if the total payment for e exceeds its weight, i.e., $\sum_i S_i(e) \geq w(e)$. The cost of each player i is the total amount of payment shemakes for cutting the edges, i.e., $\sum_e S_i(e)$. Observe that the arbitrary sharing model gives the players much larger freedom in selecting their strategies since the strategy space of each player is the positive orthant of the m-dimensional Euclidean space where $m = |E|$.

The algorithms presented in previous sections work for edges with unit cost. In order to extend those results to non-uniform weighted edges in the arbitrary cost sharing scheme, we make the following changes: Scale up the weights of edges so that all weights are integers. Simultaneously increase the value of β_i by the same factor. Now replace any edge having weight $w(e)$ with $w(e)$ parallel edges of unit cost. We can now use the algorithms in Section 2, 3, and 4 to assign these edges with unit cost to players. This assignment can be easily mapped to the original graph with weighted edges where the players pay for fractions of edge weights. This gives us a Nash equilibrium solution where the edge weights are arbitrarily shared.

Poly-time computable NE for Network Multiway Cut Game (NMWG) under Arbitrary Cost Sharing Since the algorithm for poly-time computable NE for the NMWG considered in Section 3.1 may take $O(E)$ steps, for non-uniform edge weights this may result in exponential running time. So the same algorithm does not work for this case. However we show that it is possible to form an approximate NE for NMWG in polynomial time.

Theorem 5 *Suppose we have a weighted NMCG and an P-MC S^α that is within a factor α of OPT. Then for any $\epsilon > 0$, there is a poly-time algorithm which returns a $(1+\epsilon)$-approximate NE for a P-MC S', where $w(S') \leq w(S^\alpha)$*

Proof. Here we use the standard technique from [6]. To find a $(1+\epsilon)$-approximate NE, we start by defining $\lambda = \frac{\epsilon w(S^\alpha)}{\alpha(1+\epsilon)(|E|+k)}$. We now use the algorithm of section 3.1 to pay for all but λ for each edge in S^α. In this algorithm, every time a condition cannot be satisfied, we construct a new P-MC with smaller weight. But now we are trying to pay for λ amount less for each edge in S^α. So every time a smaller weighted P-MC is found, its weight must be at least λ smaller than the previously considered P-MC.

Each step in the algorithm can be performed in polynomial time. Every time either of the conditions (i) and (ii) of Lemma 5 fail, the cost of the new P-MC reduces at least by λ amount. This means that the algorithm can consider at most $\alpha\frac{(1+\epsilon)}{\epsilon}(|E|+k)$ P-MCs. Therefore, in time polynomial in $|E|+k$, α and ϵ^{-1}, we have formed a P-MC S' with $w(S') \leq w(S^\alpha)$ such that the players are willing to buy S' if its edges have costs decreased by λ.

For all players and for each edge e in S', we now increase $S'_i(e)$ in proportion to S'_i so that e is fully paid for. Now S' is clearly paid for. To see that this is a $(1+\epsilon)$-approximate NE, note that player i was stable before its payments were increased. Player i's payments were increased by:

$$\begin{aligned}\lambda\frac{w(S'_i)}{w(S')-|S'|\lambda}|S'| &= \frac{\epsilon w(S^\alpha)w(S'_i)|S'|}{\alpha(1+\epsilon)(|E|+k)(w(S')-|S'|\lambda)}\\ &\leq \frac{\epsilon w(S')w(S'_i)}{(1+\epsilon)(w(S')-|S'|\lambda)} \leq \epsilon w(S'_i).\end{aligned}$$

Thus any best response yields at most an ϵ factor improvement for any player i, and so this is a $1+\epsilon$ approximate Nash equilibrium. ■

6 Discussion and Open Problems

In this paper, we proved that in Network Multiway Cut Games, the price of stability is 1, and a cheap Nash equilibrium can be computed efficiently. The same techniques do not extend to Network Multi-Cut Games; we are able to show, however, that there always exists a 2-approximate Nash equilibrium as cheap as OPT in such games. We also consider cutting games with non-uniform edge costs, and show that our results extend to those games if players are allowed to share the costs of the edges in arbitrary ways.

The most immediate open problem is to quantify the price of stability in Network Multi-Cut Games. While we prove that OPT is always a 2-approximate Nash equilibrium, it is unknown if the price of stability for these games is 1, since we are not aware of any examples where OPT is not an exact Nash equilibrium in Multi-Cut Games. [23] gives examples for *directed* Multi-Cut Games with no Nash equilibrium, and those examples can be extended to show high price of stability as well. However, the price of stability, as well as existence of Nash equilibrium, in *undirected* Network Multi-Cut Games (the ones considered in our paper) are unknown. For general Network Cutting Games defined in the Introduction, [23] gives examples where no Nash equilibrium exists, even for undirected graphs.

The price of stability in undirected Multi-Cut Games is closely related to the integrality gap of certain special cases of Multi-Cut. For example, consider the special case of Multi-Cut where the optimal cut divides the graph into exactly two connected components. If the integrality gap of all such instances is 1, then the price of stability for Multi-Cut Games can be shown to be 1 as well, using techniques similar to Section 4.

Many of our results about Nash equilibrium extend to strong equilibrium as well, as pointed out by [23]. It would be interesting to extend these results to directed graphs, as well as node cuts. Since our results for Single-Source Games and Multiway Cut Games use flow techniques, they will most likely extend to directed graphs and node cuts without much extra work. Other future directions include considering dynamics in cutting games, and convergence of best-response dynamics to equilibrium, as well as considering other cost-sharing schemes for edges with non-uniform edge costs.

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